Solvability Complexity Index (= SCI) and Towers of Algorithms

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In this talk I shall first review shortly some of the approaches on computability and then present some of the main results of joint work with J. Ben-Artzi, A. C. Hansen and M. Seidel:

Can everything be computed? - On the Solvability Complexity Index and Towers of Algorithms

The paper has not been submitted yet but circulated among specialists within complexity theory. I include the abstract of the circulated version:

Abstract

This paper addresses and establishes some of the fundamental barriers in the theory of computations and finally settles the long standing computational spectral problem.

Due to the barriers presented in this paper, there are many problems, some of them at the heart of computational theory, that do not fit into the classical frameworks of complexity theory. Hence, we are in need for a new extended theory of complexity, capable of handling these new issues. Such a theory is presented in this paper. Many computational problems can be solved as follows: a sequence of approximations is created by an algorithm, and the solution to the problem is the limit of this sequence (think about computing eigenvalues of a matrix for example). However, as we demonstrate, for several basic problems in computations (computing spectra of infinite dimensional operators, solutions to linear equations or roots of polynomials using rational maps) such a procedure based on one limit is impossible. Yet, one can compute solutions to these problems, but only by using several limits. This may come as a surprise, however, this touches onto the definite boundaries of computational mathematics. To analyze this phenomenon we use the Solvability Complexity Index (SCI). The SCI is the smallest number of limits needed in order to compute a desired quantity. In several cases (spectral problems, inverse problems) we provide sharp results on the SCI, thus we establish the absolute barriers for what can be achieved computationally. For example, we show that the SCI of spectra and essential spectra of infinite matrices is equal to three, and that the SCI of spectra of self-adjoint infinite matrices is equal to two, thus providing the lower bound barriers and the first algorithms to compute such spectra in two and three limits. This finally settles the long standing computational spectral problem.

Moreover, we establish barriers on error control. We prove that no algorithm can provide error control on the computational spectral problem or solutions to infinite-dimensional linear systems. In particular, one can get arbitrarily close to the solution, but never knowing when one is "epsilon" away. In addition, we provide bounds for the SCI of spectra of classes of Schrödinger operators, thus we affirmatively answer the long standing question on whether or not these spectra can actually be computed. Finally, we show how the SCI provides a natural framework for understanding barriers in computations. It has a direct link to the Arithmetical Hierarchy, and in particular, we demonstrate how the impossibility result of McMullen on polynomial root finding with rational maps in one limit, and the framework of Doyle and McMullen on solving the quintic in several limits, can be put in the SCI framework.