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Integral Equations and Operator Theory

Characterization of transfer functions of Pritchard–Salamon or other realizations with a bounded input or output operator

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Abstract. We show that the transfer functions that have a (continuous-time) well-posed realization with a bounded input operator are exactly those that are strong-H^2 (plus constant feedthrough) over some right half-plane. The dual condition holds iff the transfer function has a realization with a bounded output operator. Both conditions hold iff the transfer function has a Pritchard–Salamon (PS) realization.

A state-space variant of the PS result was proved already in [3], under the additional assumption that the weighting pattern (or impulse response) is a function (whose values are bounded operators). We illustrate by an example that this does not cover all PS systems, not even if the input and output spaces are separable.

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1. The definitions and results

In this section we first explain what a transfer function, a realization, a Pritchard–Salamon (PS) system and a bounded input or output operator mean. Then we present our main results in Theorem 1.2, followed by a discussion on the results and historical remarks.

We state many well-known or straightforward facts without proof. Various subsets of those facts can be found, e.g., [7], [4] and [10] in a general setting and

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in, e.g., [8], [1], [3] and [4, Section 6.9] in the PS setting (including alternative, equivalent definitions).

In the simplest case, a linear time-invariant control system is governed by the equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\\ x(0) &= x_0\end{aligned} \tag{1.1}$$

(for $t \geq 0$), where the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ are bounded linear operators. In this article $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ stand for complex Hilbert spaces of arbitrary dimensions. We call u is the *input* (function), x the *state trajectory* and y the *output* (function) of the system.

However, in this section we shall assume that $A : \mathrm{Dom}(A) \to \mathtt{X}$ is the (infinitesimal) generator of a strongly continuous semigroup of bounded linear operators on $\mathtt{X}$, and that $B^* : \mathrm{Dom}(A^*) \to \mathtt{X}$, $C : \mathrm{Dom}(A) \to \mathtt{X}$ and $D : \mathtt{U} \to \mathtt{Y}$ are linear and continuous. We equip $\mathrm{Dom}(A)$ with the graph norm $(\|x\|_{\mathtt{X}}^2 + \|Ax\|_{\mathtt{X}}^2)^{1/2}$. Different additional assumptions will be presented in Definitions 1.1 and 2.2 below.

We call $B : \mathtt{U} \to \mathrm{Dom}(A^*)^*$ (the adjoint of B^* with pivot space $\mathtt{X}$) the (control or) *input operator*, C the (observation or) *output operator*, and D the *feedthrough operator* of the system $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. We call C *bounded* if it has an extension in $\mathcal{B}(\mathtt{X}, \mathtt{Y})$; in this case we identify C with this extension. Similarly, B is *bounded* if $B \in \mathcal{B}(\mathtt{U}, \mathtt{X})$ (i.e., if B^* extends to $\mathcal{B}(\mathtt{X}, \mathtt{U})$).

Definition 1.1.

- **(a):** We call C *admissible* (for A) if for some (hence any) $T > 0$ there exists $\gamma > 0$ such that

$$\|C\mathscr{A}^{\cdot}x_0\|_{\mathrm{L}^2([0,T];\mathtt{Y})} \leq \gamma \|x_0\|_{\mathtt{X}} \quad (x_0 \in \mathrm{Dom}(A)). \tag{1.2}$$

- **(b):** We call $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ a *WPLS (on* $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$*) with a bounded input operator* if $B \in \mathcal{B}(\mathtt{U}, \mathtt{X})$ and C is admissible.
- **(c):** We call $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ a *WPLS (on* $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$*) with a bounded output operator* if $C \in \mathcal{B}(\mathtt{X}, \mathtt{Y})$ and B^* is admissible for A^*.
- **(d):** We call $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ a *PS-system* on $(\mathtt{U}, \mathcal{W}, \mathtt{X}, \mathtt{Y})$ if
  - **1.:** $\mathcal{W}$ is a Hilbert space and $\mathcal{W} \subset \mathtt{X}$ densely and continuously,
  - **2.:** $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a WPLS with a bounded input operator, and
  - **3.:** $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ restricts to a WPLS on $(\mathtt{U}, \mathcal{W}, \mathtt{Y})$ with a bounded output operator.
- **(e):** We call $\left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$ the *dual system* of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

Further details on the above type of systems are given in Section 2.

If B and C are bounded, then B^* and C are admissible and hence then $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is of the types (b), (c) and (d).

The dual system of a WPLS with a bounded input operator is a WPLS with a bounded input operator, and vice versa.

The semigroup $\mathscr{A}$ extends to a strongly continuous semigroup on $\mathtt{X}_{-1} := \mathrm{Dom}(A^*)^*$, isomorphic to the original one, and its generator is an extension of A and belongs to $\mathcal{B}(\mathtt{X},\mathtt{X}_{-1})$. We denote these extensions by the same letters ($\mathscr{A}$ or A). (These extensions and $\mathtt{X}_{-1}$ are not needed for systems of type (c) or (d); cf. (1.3) and (1.5).)

The operator B^* is admissible for A^* iff for some (hence any) $T>0$ there exists $\beta<\infty$ such that for all $u\in \mathrm{L}^2([0,T];\mathtt{U})$ we have

$$\mathscr{B}^T u := \int_0^T \mathscr{A}^{T-s}Bu(s)\,ds \in \mathtt{X} \tag{1.3}$$

and $\|\mathscr{B}^T u\|_{\mathtt{X}} \le \beta\|u\|_{\mathrm{L}^2([0,T);\mathtt{U})}$.

Therefore, the condition (d)3. holds iff $\mathscr{A}|_{\mathcal{W}}$ is a strongly continuous semigroup on $\mathcal{W}$ (by $A_{\mathcal{W}}$ we denote its generator), $C|_{\mathrm{Dom}(A_{\mathcal{W}})}$ extends to $\mathcal{B}(\mathcal{W},\mathtt{Y})$, and for some $T>0$ and $\beta<\infty$ we have

$$\mathscr{B}^T u \in \mathcal{W} \quad \text{and} \quad \|\mathscr{B}^T u\|_{\mathcal{W}} \le \beta\|u\|_2 \quad (u\in \mathrm{L}^2([0,t];\mathtt{U})). \tag{1.4}$$

(The alternative condition $\|B^*_{\mathcal{W}}(\mathscr{A}^{\cdot}_{\mathcal{W}})^*x_0\|_{\mathrm{L}^2([0,T];\mathtt{U})} \le \beta\|x_0\|_{\mathcal{W}}$ would be uncomfortable, since it would require us to take the adjoints with the pivot space $\mathcal{W}$ instead of $\mathtt{X}$.)

The *transfer function* $\widehat{\mathscr{D}}$ of a system of type (b), (c) or (d) is defined by

$$\widehat{\mathscr{D}}(s) := C(s-A)^{-1}B + D \qquad (s\in\sigma(A)^c). \tag{1.5}$$

We call the system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ a *realization* of $\widehat{\mathscr{D}}$. We identify holomorphic functions that coincide on some right half-plane, hence it suffices that (1.5) holds on some right half-plane contained in $\sigma(A)^c$.

For any $\alpha\in\mathbb{R}$, we set $\mathbb{C}^+_\alpha := \{s\in\mathbb{C} \mid \operatorname{Re} s>\alpha\}$. By $\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{Y})$ we denote the Hilbert space of holomorphic functions $F:\mathbb{C}^+_\alpha\to\mathtt{Y}$ such that

$$\|F\|_{\mathrm{H}^2_\alpha} := \frac{1}{\sqrt{2\pi}}\sup_{r>\alpha}\|F(r+i\cdot)\|_{\mathrm{L}^2} < \infty. \tag{1.6}$$

The operator C is admissible for A iff there exists $\alpha\in\mathbb{R}$ such that

$$C(\cdot - A)^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{Y}) \quad \text{for every } x_0\in\mathtt{X}. \tag{1.7}$$

Now we are ready to state our main results:

Theorem 1.2. *Let $\beta\in\mathbb{R}$ and let $\widehat{\mathscr{D}}$ be a holomorphic function $\mathbb{C}^+_\beta\to\mathcal{B}(\mathtt{U},\mathtt{Y})$. Then the following hold.*

- **(b):** *$\widehat{\mathscr{D}}$ has a realization with a bounded input operator and $D=0$ iff there exists $\alpha\in\mathbb{R}$ such that $\widehat{\mathscr{D}}u_0\in\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{Y})$ for all $u_0\in\mathtt{U}$.*
- **(c):** *$\widehat{\mathscr{D}}$ has a realization with a bounded output operator and $D=0$ iff there exists $\alpha\in\mathbb{R}$ such that $\widehat{\mathscr{D}}(\bar{\cdot})^*y_0\in\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{U})$ for all $y_0\in\mathtt{Y}$.*
- **(d):** *$\widehat{\mathscr{D}}$ has a realization as a PS-system with $D=0$ iff the conditions in (b) and (c) hold.*

(The proof is given in Section 3.) Thus, $\widehat{\mathscr{D}}$ has a realization with a bounded input operator iff $\widehat{\mathscr{D}} - D$ satisfies the condition in (b) for some $D \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$ (in which case $D = \lim_{s\to+\infty} \widehat{\mathscr{D}}(s)$). Analogous comments apply to (c) and (d). Before we go on to explain Theorem 1.2 and its history, we define a few more concepts.

The (bilateral) *Laplace transform* of u is defined by $\widehat{u}(s) := \int_{-\infty}^{\infty} \mathrm{e}^{-st} u(t)\, dt$ for those $s \in \mathbb{C}$ for which the integral converges strongly. With initial state 0 and input $u \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{U})$, the *output* y can be defined through $\widehat{y} = \widehat{\mathscr{D}}\widehat{u}$. This determines the *I/O map* $\mathscr{D} : u \mapsto y$ of the system and corresponds to (1.1) in a weak sense, as explained in Section 2.

Set $\mathrm{L}^2_\alpha := \mathrm{e}^{\alpha\cdot}\mathrm{L}^2 = \{f \,|\, \mathrm{e}^{-\alpha\cdot} f \in \mathrm{L}^2\}$. Finally, recall that the Paley–Wiener (and Plancherel) Theorem holds also in the infinite-dimensional case:

Lemma 1.3. *The Laplace transform is an isometric isomorphism of* $\mathrm{L}^2_\alpha(\mathbb{R}_+; \mathtt{U})$ *onto* $\mathrm{H}^2(\mathbb{C}^+_\alpha; \mathtt{U})$. *Moreover, every element of* $\mathrm{H}^2(\mathbb{C}^+_\alpha; \mathtt{U})$ *has a radial limit function in* $\mathrm{L}^2(\alpha + i\mathbb{R}; \mathtt{U})$ *with the same norm.*

(See, e.g., [4, Lemma D.1.15] for the proof. We use the measure $m/2\pi$ on $\alpha + i\mathbb{R}$.)

The conditions in (b), (c) and (d) of Theorem 1.2 become equivalent if $\dim \mathtt{U}, \dim \mathtt{Y} < \infty$ (and in that case a fourth equivalent condition is that $\widehat{\mathscr{D}} \in \mathrm{H}^2(\mathbb{C}^+_\alpha; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$, a fifth one that $\mathscr{D}u = f * u$ where $f \in \mathrm{L}^2_\alpha(\mathbb{R}_+; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ for some $\alpha \in \mathbb{R}$). Except for (d), this case is mostly covered by Theorem 5.2 of [5] (there the scalar field is real but the same proof still applies).

The *impulse response* (or *weighting pattern*) R of a finite-dimensional system means the output of the system when the input equals the unit impulse δ (with initial state $x_0 = 0$). Thus, $\hat{R} = \widehat{\mathscr{D}}\hat{\delta} = \widehat{\mathscr{D}}$, where $\widehat{\mathscr{D}}$ is the transfer function of the system. Therefore, with input u the output formally becomes $R * u$. All of the above can be extended to very general systems by defining R through $\hat{R} = \widehat{\mathscr{D}}$.

If, e.g., B and C are bounded, then R can be identified with the function $f : \mathbb{R}_+ \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ given by

$$f(t) := C\mathscr{A}^t B + D\delta. \tag{1.8}$$

For more general systems, we only know that R is a causal distribution, being the inverse Laplace transform of the bounded analytic function (on some right half-plane) $\widehat{\mathscr{D}}$. For any WPLS with a bounded input operator B and zero feedthrough $D = 0$, we can identify the impulse response R with the operator $R := \mathscr{C}B \in \mathcal{B}(\mathtt{U}, \mathrm{L}^2_\alpha(\mathbb{R}_+; \mathtt{Y}))$ (for some $\alpha \in \mathbb{R}$), where $\mathscr{C} \in \mathcal{B}(\mathtt{X}, \mathrm{L}^2_\alpha(\mathbb{R}_+; \mathtt{Y}))$ is the unique extension of $C\mathscr{A}$ (cf. (1.2) and (1.7); see Sections 2 and 4 for further details).

In [3], Kaashoek, Ran and van der Mee essentially assumed that ($D = 0$ and) the impulse response is given by a function, i.e., that there exist $f : \mathbb{R}_+ \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ and $\alpha \in \mathbb{R}$ such that $\mathscr{D}u = f * u$ for all $u \in \mathrm{L}^2$ (or equivalently, such that $\widehat{f} = \widehat{\mathscr{D}}$). Then they showed that $\mathscr{D}$ has a PS-realization iff $fu_0, f^*y_0 \in \mathrm{L}^2_\alpha$ for each $u_0 \in \mathtt{U}$, $y_0 \in \mathtt{Y}$. Their condition is obviously equivalent to ours in the case that

f really is a function $\mathbb{R}_+ \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ (see Section 4 for details). To show that our result is a strict generalization of theirs, we construct in Section 4 a PS-system whose impulse response $\mathscr{C}B$ cannot be represented by a ($\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued) function; in particular, no $f : \mathbb{R}_+ \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ satisfies $fu_0 = \mathscr{C}Bu_0$ a.e. for every $u_0 \in \mathtt{U}$.

If $n := \dim \mathtt{U} < \infty$, then $\mathcal{B}(\mathtt{U}, \mathrm{L}^2_\alpha(\mathbb{R}_+;\mathtt{Y})) = \mathrm{L}^2_\alpha(\mathbb{R}_+;\mathtt{Y})^n = \mathrm{L}^2_\alpha(\mathbb{R}_+;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ (with equivalent norms), and the characterization of [3] applies. Similarly, it also applies whenever $\dim \mathtt{Y} < \infty$. See Corollaries 6.9.7 and 6.9.8 of [4] for more on these cases.

The existence parts of the results of [5], [3] and ours all use the shift-semigroup systems of Proposition 2.3. The converses are rather straightforward.

In the literature preceding [1], usually only smooth PS-systems were studied (i.e., those satisfying $\mathrm{Dom}(A_{\mathcal{V}}) \subset \mathcal{W}$). The solutions of LQR and four-block H^∞ problems for smooth PS-systems can be found in [8] and for general PS-systems (and for more general WPLSs) in [4]. Variants of Theorem 1.2 under additional stability or other constraints can be found in [4, Section 6.9]. Any of them can easily be observed from the proof in Section 3.

In Section 2 general WPLSs and their relation to PS-systems will be further explored. In Section 3 contains the proof of Theorem 1.2. In Section 4 we shall construct the counter-example mentioned above.

2. Well-posed linear systems

We list here some definitions and properties concerning well-posed linear systems, see [7] (or, e.g., [4] or [10]) for further details. All corresponding definitions in Section 1 are special cases of these.

When A, B and C are as explained above Definition 1.1 and the vector $\widehat{\mathscr{D}}(z) \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ is given for some $z \in \sigma(A)^c$, the quadruple $(A, B, C, \widehat{\mathscr{D}})$ is called a *system node*. The *transfer function* of the system node is defined by

$$\widehat{\mathscr{D}}(s) := \widehat{\mathscr{D}}(z) + (s - z)C(s - A)^{-1}(z - A)^{-1}B \quad (s \in \sigma(A)^c). \tag{2.1}$$

(In fact, the correct name would be "characteristic function", but the two coincide on a right half-plane, and the difference elsewhere is insignificant in this article.) If B or C is bounded, then $D := \lim_{s\to+\infty} \widehat{\mathscr{D}}(s)$ exists and $\widehat{\mathscr{D}} = D + C(\cdot - A)^{-1}B$, as in (1.5).

If C is admissible for A, B^* is admissible for A^*, and $\widehat{\mathscr{D}}$ is bounded on some right half-plane, then $(A, B, C, \widehat{\mathscr{D}})$ is called a *well-posed linear system (WPLS)* (or abstract linear system or Salamon–Weiss system). Obviously, then also its *dual system* $(A^*, C^*, B^*, \widehat{\mathscr{D}}(\bar{\cdot})^*)$ is a WPLS. These definitions are in accordance with Definition 1.1; e.g., $(A, B, C, \widehat{\mathscr{D}})$ "a WPLS with a bounded input operator" iff it is a WPLS and B is bounded (similarly for C).

For any $x_0 \in \mathtt{X}$ and $\widehat{u} \in \mathrm{H}^2(\mathbb{C}^+_\omega;\mathtt{U})$, $\omega \in \mathbb{R}$, the *state* x and *output* y of a WPLS $(A, B, C, \widehat{\mathscr{D}})$ are defined through

$$\widehat{x} = (\cdot - A)^{-1}x_0 + (\cdot - A)^{-1}B\widehat{u} \tag{2.2}$$

$$\widehat{y} = C(\cdot - A)^{-1}x_0 + \widehat{\mathscr{D}}\widehat{u} \tag{2.3}$$

(on a right half-plane where the right-hand-side is well defined). Note that all above functions are holomorphic. It follows that $x : \mathbb{R}_+ \to \mathtt{X}$ becomes continuous and $y \in \mathrm{L}^2_\alpha(\mathbb{R}_+;\mathtt{Y})$ for some $\alpha \in \mathbb{R}$, as we shall explain later. First we introduce some more notation.

We set $(\tau^t u)(s) := u(t + s)$ and $\pi_\pm u := \chi_{\mathbb{R}_\pm} u$, where $\chi_E(t) := \begin{cases} 1, & t \in E; \\ 0, & t \notin E \end{cases}$, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. We identify any function with its zero extension (hence π_+ becomes a projection on $\mathrm{L}^2(\mathbb{R};\mathtt{U})$).

By $\mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ (or H^∞_ω) we denote the space of bounded holomorphic functions $\mathbb{C}^+_\omega \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ with norm $\|\widehat{\mathscr{D}}\|_{\mathrm{H}^\infty_\omega} := \sup_{s\in\mathbb{C}^+_\omega} \|\widehat{\mathscr{D}}(s)\|_{\mathcal{B}(\mathtt{U},\mathtt{Y})}$.

It is well known that to each H^∞_ω (transfer) function corresponds a unique I/O map $\mathscr{D} : u \mapsto y$ and conversely [9]:

Proposition 2.1. *Let $\omega \in \mathbb{R}$. Any $\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ determines uniquely a map $\mathscr{D} : \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ by*

$$(\widehat{\mathscr{D}u})(s) = \widehat{\mathscr{D}}(s)\widehat{u}(s) \quad (s \in \mathbb{C}^+_\omega,\ u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})). \tag{2.4}$$

The map $\widehat{\mathscr{D}} \mapsto \mathscr{D}$ is an isometry of $\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ onto the subspace of all $\mathscr{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}), \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y}))$ that satisfy $\mathscr{D}\tau^t = \tau^t\mathscr{D}$ $(t \in \mathbb{R})$ and $\pi_-\mathscr{D}\pi_+ = 0$.

In the literature, there are many equivalent definitions of "ω-stable WPLSs". We shall give below one of them (from [6]), whose algebraic formulations are very useful in the proof of Theorem 1.2. Then we shall explain the connection of this definition to the (above) definition of a WPLS.

Definition 2.2 (WPLS). Let $\omega \in \mathbb{R}$. An *ω-stable well-posed linear system on* $(\mathtt{U},\mathtt{X},\mathtt{Y})$ is a quadruple $\Sigma = \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right]$, where $\mathscr{A}^t$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are bounded linear operators of the following type:

1. $\mathscr{A}^t \colon \mathtt{X} \to \mathtt{X}$ $(t \geq 0)$ is a strongly continuous semigroup of bounded linear operators on $\mathtt{X}$ satisfying $\sup_{t\geq 0} \|\mathrm{e}^{-\omega t}\mathscr{A}^t\|_{\mathtt{X}} < \infty$;
2. $\mathscr{B} \colon \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathtt{X}$ satisfies $\mathscr{A}^t\mathscr{B}u = \mathscr{B}\tau^t\pi_- u$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ and $t \in \mathbb{R}_+$;
3. $\mathscr{C} \colon \mathtt{X} \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ satisfies $\mathscr{C}\mathscr{A}^t x = \pi_+\tau^t\mathscr{C}x$ for all $x \in \mathtt{X}$ and $t \in \mathbb{R}_+$;
4. $\mathscr{D} \colon \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ satisfies $\tau^t\mathscr{D}u = \mathscr{D}\tau^t u$, $\pi_-\mathscr{D}\pi_+ u = 0$ and $\pi_+\mathscr{D}\pi_- u = \mathscr{C}\mathscr{B}u$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ and $t \in \mathbb{R}$.

Given a WPLS $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$, we denote its semigroup by $\mathscr{A}$, its *growth bound* by $\omega_A := \inf\{t^{-1}\log\|\mathscr{A}^t\| \mid t > 0\} < \infty$, and its transfer function by $\widehat{\mathscr{D}}$. For any $\omega > \omega_A$, we define $\mathscr{D}$ as in Proposition 2.1 (indeed, $\widehat{\mathscr{D}} \in \mathrm{H}^\infty_\omega$) and $\mathscr{B}$ by

$$\mathscr{B}u := \lim_{t\to\infty} \int_0^t \mathscr{A}^r Bu(-r)\,dr \quad (u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})) \tag{2.5}$$

(the integral converges in $\mathtt{X}_{-1}$ and the limit in $\mathtt{X}$), and by $\mathscr{C} \in \mathcal{B}(\mathtt{X}, \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{Y}))$ we denote the unique extension of $C\mathscr{A} : \mathrm{Dom}(A) \to \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{Y})$. This quadruple $\left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$ is an ω-stable WPLS on $(\mathtt{U},\mathtt{X},\mathtt{Y})$.

Conversely, for any ω-stable WPLS $\Sigma = \left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$ there exist a unique WPLS $(A, B, C, \widehat{\mathscr{D}})$ that is related to Σ as above (necessarily $\omega \ge \omega_A$). In particular, an ω-stable WPLS is an ω'-stable WPLS for any $\omega' \ge \omega$ (we identify $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ with their unique extensions/restrictions obtained by changing ω).

The operator $\mathscr{B}^t$ of (1.3) satisfies $\mathscr{B}^t := \mathscr{B}\tau^t\pi_-$, and the formulas (2.2) and (2.3) are equivalent to the equations

$$x(t) = \mathscr{A}^t x_0 + \mathscr{B}^t u \quad (t \ge 0), \qquad y = \mathscr{C} x_0 + \mathscr{D} u. \tag{2.6}$$

(Also (1.1) and (1.5) hold for whenever $D := \lim_{s\to+\infty} \widehat{\mathscr{D}}(s)$ exists; we just have to replace C by its *Weiss extension* if C is unbounded.)

Dietmar Salamon has shown that any H^∞_ω function has a realization [5]:

Proposition 2.3. *Let $\omega \in \mathbb{R}$, $\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$. Then $\widehat{\mathscr{D}}$ has the ω-stable realizations*

$$\Sigma_\omega := \left[\begin{array}{c|c}\pi_+\tau & \pi_+\mathscr{D}\pi_-\\ \hline I & \mathscr{D}\end{array}\right] \tag{2.7}$$

on $(\mathtt{U}, \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{Y}), \mathtt{Y})$ and

$$\Sigma^\omega := \left[\begin{array}{c|c}\tau\pi_- & \pi_-\\ \hline \pi_+\mathscr{D}\pi_- & \mathscr{D}\end{array}\right] \tag{2.8}$$

on $(\mathtt{U}, \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U}), \mathtt{Y})$, where $\mathscr{D}$ is defined by (2.4).

Now we have presented general WPLSs and the machinery needed in the proof of the main result. In which sense WPLSs are more general than PS-systems?

Given a WPLS $\left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$, the system $\left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline 0 & 0\end{array}\right]$ is a PS-system (on $(\mathtt{U},\mathtt{X},\mathrm{Dom}(A^*)^*,\mathtt{Y})$) and $\left[\begin{array}{c|c}\mathscr{A} & 0\\ \hline \mathscr{C} & 0\end{array}\right]$ is a PS-system (on $(\mathtt{U},\mathrm{Dom}(A),\mathtt{X},\mathtt{Y})$). Thus, PS-systems allow for as much unboundedness as WPLSs for B and C *but not simultaneously*, thus posing also much stronger conditions on the I/O map $\mathscr{D}$. E.g., in the parabolic case with A invertible, a WPLS typically has the operators $A^{-1/2}B : \mathtt{U} \to \mathtt{X}$ and $CA^{-1/2} : \mathtt{X} \to \mathtt{Y}$ bounded [7, Theorem 5.7.3], hence CA^{-1} is bounded on $\mathrm{Ran}(B)$; in the case of a PS-system we see that $CA^{-1/2}$ is bounded on $\mathrm{Ran}(B)$ (take $\mathtt{X} := \mathcal{V}$). Thus, in certain sense, the distance of $\mathrm{Ran}(B)$ and $\mathrm{Dom}(C)$ (the sum of the unboundednesses of B and C) can be twice as much in a WPLS as in a PS-system.

3. The proof of Theorem 1.2

In this section we present three auxiliary results and then we use them to prove Theorem 1.2.

First we note that strong H^2 functions are H^∞ on any smaller half-plane:

Lemma 3.1 ("$\mathrm{H}^2_{\mathrm{strong}}$"). *Let $\alpha \in \mathbb{R}$ and $\widehat{\mathscr{D}} : \mathbb{C}^+_\alpha \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ be such that $\widehat{\mathscr{D}}u_0 \in \mathrm{H}^2(\mathbb{C}^+_\alpha;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ for every $u_0 \in \mathtt{U}$. Then*

$$M := \sup_{\|u_0\|_{\mathtt{U}}\le 1} \|\widehat{\mathscr{D}}(\cdot)u_0\|_{\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{Y})} < \infty, \tag{3.1}$$

$\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ for any $\omega > \alpha$, and $D := \lim_{s\to+\infty}\widehat{\mathscr{D}}(s) = 0$.

Proof. By Theorem 3.10.1 of [2], $\widehat{\mathscr{D}}$ is holomorphic. By the closed-graph theorem, (3.1) holds (if $u_n \to 0$ in $\mathtt{U}$ and $\widehat{\mathscr{D}}u_n \to f$ in H^2_α, then $\Lambda\widehat{\mathscr{D}}u_n \to \Lambda f$ in $\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathbb{C})$ and hence $\Lambda\widehat{\mathscr{D}}(s)u_n \to \Lambda f(s)$ for each $s \in \mathbb{C}^+_\alpha$, $\Lambda \in \mathtt{Y}^*$, although $\widehat{\mathscr{D}}(s)u_n \to \widehat{\mathscr{D}}(s)0 = 0$ for each $s \in \mathbb{C}^+_\alpha$, hence then $f \equiv 0$ on $\mathbb{C}^+_\alpha$).

If $\mathtt{U} = \mathbb{C} = \mathtt{Y}$, then we have $\|\widehat{\mathscr{D}}\|_{\mathrm{H}^\infty_\omega} \le ((\omega-\alpha)/2)^{-1/2}M$, by, e.g., (6.4.3) of [2]. In general, $\|\Lambda\widehat{\mathscr{D}}u_0\|_{\mathrm{H}^\infty_\omega} \le ((\omega-\alpha)/2)^{-1/2}M$ when $\|u_0\|_{\mathtt{U}} \le 1$ and $\|\Lambda\|_{\mathtt{Y}^*} \le 1$, hence $\|\widehat{\mathscr{D}}\|_{\mathrm{H}^\infty_\omega} \le ((\omega-\alpha)/2)^{-1/2}M \to 0$. In particular, $D = 0$. □

If $\widehat{\mathscr{D}}$ is "dual-$\mathrm{H}^2_{\mathrm{strong}}$", then $\mathscr{D}$ maps L^2 to the set of continuous functions:

Lemma 3.2. *Let $\alpha \in \mathbb{R}$ and $\widehat{\mathscr{D}} : \mathbb{C}^+_\alpha \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ be such that $\widehat{\mathscr{D}}(\bar{\cdot})^* y_0 \in \mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{U})$ for all $y_0 \in \mathtt{Y}$. Then*

$$M := \sup_{\|y_0\|_{\mathtt{U}}\le 1} \|\widehat{\mathscr{D}}(\bar{\cdot})^* y_0\|_{\mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{U})} < \infty, \tag{3.2}$$

$\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ for any $\omega > \alpha$, $D := \lim_{s\to+\infty}\widehat{\mathscr{D}}(s) = 0$, and $\mathscr{D}u$ is continuous and $\|\mathrm{e}^{-\alpha t}(\mathscr{D}u)(t)\|_{\mathtt{Y}} \le M\|u\|_{\mathrm{L}^2_\alpha}$ for all $t \in \mathbb{R}$ and $u \in \mathrm{L}^2_\alpha(\mathbb{R};\mathtt{U})$.

The proof is based on the fact that when $\widehat{u} \in \mathrm{H}^2$, we have $\widehat{\mathscr{D}}\widehat{u} \in$"weak-$\mathrm{H}^1$" (see the proof), hence its Fourier transform is continuous (hence so is $\mathscr{D}u$).

Proof. Obviously, $\widehat{\mathscr{D}}(\bar{\cdot})^*$ is holomorphic iff $\widehat{\mathscr{D}}$ is, so the first three claims follow from Lemma 3.1 and Proposition 2.1.

1° Assume first that $u \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R}_+;\mathtt{U})$ (i.e., that u is continuously differentiable and has a compact support). Then $\mathscr{D}u$ is continuous, by [7, Corollary 4.6.13(i)] (and Proposition 2.3).

2° By the (Hölder–)Schwarz Inequality and Lemma 1.3, the $\mathrm{L}^1(\alpha+i\mathbb{R};\mathtt{U})$ norm of $\langle\widehat{\mathscr{D}}\widehat{u},y_0\rangle_{\mathtt{Y}} = \langle\widehat{u},\widehat{\mathscr{D}}^*y_0\rangle_{\mathtt{U}}$, when $\|y_0\|_{\mathtt{Y}} \le 1$, is at most

$$\|\widehat{u}\|_{\mathrm{L}^2(\alpha+i\mathbb{R};\mathtt{U})}\|\widehat{\mathscr{D}}^*y_0\|_{\mathrm{L}^2(\alpha+i\mathbb{R};\mathtt{U})} = M\|\widehat{u}\|_{\mathrm{L}^2(\alpha+i\mathbb{R};\mathtt{U})} = M\|u\|_{\mathrm{L}^2_\alpha}. \tag{3.3}$$

We have $f = \hat{\hat{f}}(-\cdot)/2\pi$ and $\|\hat{f}\|_\infty \le \|f\|_1$, hence $\|f\|_\infty \le \|\hat{f}\|_{\mathrm{L}^1(i\mathbb{R};\mathtt{U})}$, for any $f \in \mathrm{L}^1(\mathbb{R}_+;\mathtt{U})$ such that $\hat{f} \in \mathrm{L}^1(i\mathbb{R};\mathtt{U})$. From this and (3.3) we conclude that $\|\mathrm{e}^{-\alpha\cdot}\langle\mathscr{D}u,y_0\rangle_{\mathtt{Y}}\|_\infty = M\|u\|_{\mathrm{L}^2_\alpha}$ when $\|y_0\|_{\mathtt{Y}} \le 1$. Thus, $\|\mathrm{e}^{-\alpha t}(\mathscr{D}u)(t)\|_{\mathtt{Y}} \le M\|u\|_{\mathrm{L}^2_\alpha}$ for all t. By time-invariance, the claim holds for any $u \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R};\mathtt{U})$.

3° Let now $u \in \mathrm{L}^2_\alpha(\mathbb{R};\mathtt{U})$ be arbitrary. Choose $\{u_n\} \subset \mathcal{C}^1_{\mathrm{c}}(\mathbb{R};\mathtt{U})$ such that $u_n \to u$ in L^2_α as $n \to +\infty$ (Proposition 3.3). By the above, $\mathrm{e}^{-\alpha\cdot}\mathscr{D}u_n$ converges uniformly, hence the limit, say $\mathrm{e}^{-\alpha\cdot}y$, is continuous and $\|\mathrm{e}^{-\alpha\cdot}y\|_\infty \le M\|u\|_{\mathrm{L}^2_\alpha}$. But $\mathscr{D}u_n \to \mathscr{D}u$ in L^2_ω, hence a subsequence converges a.e., hence $\mathscr{D}u = y$. □

We used above the following well-known facts:

Proposition 3.3 **(L^2_α).** *Let $J \subset \mathbb{R}$ be an interval and $\alpha \in \mathbb{R}$. The space $\mathrm{L}^2_\alpha(J;\mathtt{U})$ is a Banach space, and the space $\mathcal{C}^1_c(\mathbb{R};\mathtt{U})$ is dense in $\mathrm{L}^2_\alpha(J;\mathtt{U})$. If $f_n \to f$ in $\mathrm{L}^2_\alpha(J;\mathtt{U})$, as $n \to \infty$, then a subsequence converges pointwise a.e. on J.*

(For L^2 the proofs are the same as in the scalar case [4, Section B.3], and $\mathrm{L}^2_\alpha = \mathrm{e}^{\alpha\cdot}\mathrm{L}^2$.)

Now we can prove the main result. Part "if" of (c) will be deduced from Lemma 3.2, the rest of (b) and (c) follow easily from (1.7) and duality. "Only if" of (d) follows, and "if" of (d) requires a longer construction.

Proof of Theorem 1.2*: "Only if" of (b):* If $\widehat{\mathscr{D}}$ is the transfer function of a WPLS $(\frac{A \mid B}{C \mid D})$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$ (for some $\mathtt{X}$) with $D = 0$ and a bounded input operator $B \in \mathcal{B}(\mathtt{U},\mathtt{X})$, then, by (1.7), there exists $\alpha \in \mathbb{R}$ such that $C(\cdot - A)^{-1}Bu_0 \in \mathrm{H}^2(\mathbb{C}^+_\omega;\mathtt{Y})$ for all $u_0 \in \mathtt{U}$. By (1.5), $\widehat{\mathscr{D}} = C(\cdot - A)^{-1}B$.

"Only if" of (c): Apply the above to the dual system.

"If" of (c): Assume now that $\widehat{\mathscr{D}}(\bar{\cdot})^*y_0 \in \mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{U})$ for all $y_0 \in \mathtt{Y}$. Pick some $\omega > \alpha$ and set $\mathtt{X} := \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})$. By Lemma 3.2, $\widehat{\mathscr{D}} \in \mathrm{H}^\infty_\omega$ (and $D = 0$), hence (2.8) defines a WPLS $[\frac{\mathscr{A} \mid \mathscr{B}}{\mathscr{C} \mid \mathscr{D}}]$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$. By Lemma 3.2, we have $\tilde{C} \in \mathcal{B}(\mathtt{X},\mathtt{Y})$, where $\tilde{C}u := (\mathscr{D}u)(0)$. But $\mathscr{C}u = \pi_+\mathscr{D}u$, hence

$$\tilde{C}u = (\mathscr{C}u)(0) = C\mathscr{A}^0u = Cu \quad \text{for all} \quad u \in \mathrm{Dom}(A). \tag{3.4}$$

Thus, C is "bounded".

"If" of (b): Assume that $\widehat{\mathscr{D}}u_0 \in \mathrm{H}^2(\mathbb{C}^+_\alpha;\mathtt{Y})$ for all $u_0 \in \mathtt{U}$. Apply the proof of (c) above to $\widehat{\mathscr{D}}(\bar{\cdot})^*$ and take the dual of the resulting system to complete the proof of (b).

Remark: by replacing $\mathtt{X}$ by $\mathcal{R}\mathtt{X}\mathcal{R} = \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})$ (where $(\mathcal{R}f)(t) := f(-t)$; thus $\mathscr{A} \mapsto \mathcal{R}\mathscr{A}\mathcal{R}$, $\mathscr{B} \mapsto \mathcal{R}\mathscr{B}$ and $\mathscr{C} \mapsto \mathscr{C}\mathcal{R}$), we observe that, for this $\widehat{\mathscr{D}}$, the WPLS (2.7) has a bounded input operator and $D = 0$.

(d) "Only if" follows from (b), (c) and Definition 1.1(d), so we assume that the conditions in (b) and (c) hold and construct a PS-realization of $\widehat{\mathscr{D}}$. Fix some $\omega > \alpha$ and define $[\frac{\mathscr{A} \mid \mathscr{B}}{\mathscr{C} \mid \mathscr{D}}]$ by (2.7). Set $\mathcal{W} := \{\pi_+\mathscr{D}\pi_-u \mid u \in \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U})\}$ (the Hankel range), and

$$\|x_0\|^2_{\mathcal{W}} := \|x_0\|^2_{\mathrm{L}^2_\omega} + \inf_{u \in \mathrm{L}^2_\omega,\ \pi_+\mathscr{D}\pi_-u = x_0} \|u\|^2_{\mathrm{L}^2_\omega} \quad (x_0 \in \mathtt{X}). \tag{3.5}$$

(One easily verifies that this makes $\mathcal{W}$ an inner product space.)

1° $[\frac{\mathscr{A} \mid \mathscr{B}}{\mathscr{C} \mid \mathscr{D}}]$ *is an ω-stable WPLS on* $(\mathtt{U},\mathcal{W},\mathtt{Y})$*:* Define

$$\mathcal{X} := \mathrm{Ker}(\mathscr{B})^\perp \subset \mathrm{L}^2_\omega(\mathbb{R}_-;\mathtt{U}). \tag{3.6}$$

Then $\|\mathscr{B}u\|^2_{\mathcal{W}} = \|\mathscr{B}u\|^2_{\mathrm{L}^2_\omega} + \|u\|^2_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{X}$. Consequently, the restriction $T : \mathcal{X} \to \mathcal{W}$ of $\mathscr{B} := \pi_+\mathscr{D}\pi_-$ satisfies $\|Tu\| \geq \|u\|$; it is also onto, hence boundedly invertible. Consequently, $\mathcal{W}$ is complete (since $\mathcal{X}$ is), hence a Hilbert space. Moreover, $\mathcal{W} \subset \mathrm{L}^2_\omega$ continuously.

For every $t>0$ we have $\mathscr{B}\tau^t\pi_- = \pi_+\tau^t\mathscr{D}\pi_- = \pi_+\tau\mathscr{B} = \mathscr{A}^t\mathscr{B}$, hence

$$\|\mathscr{A}^t\mathscr{B}u\|_{\mathcal{W}}^2 = \|\pi_+\tau^t\mathscr{B}u\|_{\mathcal{W}}^2 + \|\tau^t u\|_{\mathrm{L}^2_\omega}^2 \le \mathrm{e}^{2\omega t}\|\mathscr{B}u\|_{\mathcal{W}}^2 \quad (u\in\mathcal{X}), \tag{3.7}$$

hence $\|\mathscr{A}^t\|_{\mathcal{B}(\mathcal{W})} \le \mathrm{e}^{\omega t}$. Thus, $\mathscr{A}|_{\mathcal{W}}$ is an ω-stable semigroup on $\mathcal{W}$, because its semigroup properties are inherited from $\mathscr{A}$. Similarly, $\|\mathscr{A}^t\mathscr{B}u - \mathscr{B}u\|_{\mathcal{W}}^2 \le \|\pi_+\tau^t\mathscr{B}u - \mathscr{B}u\|_{\mathrm{L}^2_\omega}^2 + \|\tau^t u - u\|_{\mathrm{L}^2_\omega}^2$, as $t\to 0+$, hence $\mathscr{A}|_{\mathcal{W}}$ is strongly continuous.

Because $\mathcal{X}\subset\mathrm{L}^2_\omega$ is closed, the orthogonal projection $P:\mathrm{L}^2_\omega\to\mathcal{X}$ is continuous, hence so is $\mathscr{B} = TP\in\mathcal{B}(\mathrm{L}^2_\omega,\mathcal{W})$. Obviously, $\mathscr{C}$ remains continuous with this stronger topology of $\mathcal{W}\subset\mathrm{L}^2_\omega$ and the other properties of the WPLS (2.7) are preserved. It follows that Σ is an ω-stable WPLS on $(\mathtt{U},\mathcal{W},\mathtt{Y})$.

2° *The output operator for* $(\mathtt{U},\mathcal{W},\mathtt{Y})$ *is "bounded":* Define $\tilde{C}:\mathcal{W}\to\mathtt{Y}$ by $\tilde{C}\mathscr{B}u := (\mathscr{D}u)(0)$ (i.e., $\tilde{C} := (\mathscr{D}T^{-1}\dot{)}(0))$ for any $u\in\mathcal{X}$. By the proof of (b) above, $\tilde{C}$ is bounded (since $T^{-1}\in\mathcal{B}(\mathcal{W},\mathcal{X})$ and $(\mathscr{D}\cdot)(0)\in\mathcal{B}(\mathrm{L}^2_\omega,\mathtt{Y})$). But, for any $u\in\mathcal{X}$ and $t\ge 0$, we have

$$\tilde{C}\mathscr{A}^t\mathscr{B}u = \tilde{C}\mathscr{B}\tau^t u = (\mathscr{D}\tau^t u)(0) = (\mathscr{D}u)(t) = (\pi_+\mathscr{D}\pi_- u)(t) = (\mathscr{C}\mathscr{B}u)(t), \tag{3.8}$$

i.e., $\tilde{C}\mathscr{A}x_0 = \mathscr{C}x_0$ $(= x_0\in\mathcal{W})$ for all $x_0\in\mathscr{B}[\mathcal{X}] = \mathcal{W}$, Thus, $\tilde{C}$ is an extension of C, hence C is "bounded".

3° $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ *is a PS-system on* $(\mathtt{U},\mathcal{W},\mathcal{V},\mathtt{Y})$ *when we define* $\mathcal{V}$ *to be the closure of* $\mathcal{W}$ *in* $\mathtt{X} = \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{Y})$*:* Indeed, 1. of Definition 1.1(d) follows because $\mathcal{W}\subset\mathtt{X}$ continuously (by (3.5)) and 3. was established in 1° and 2°, so only 2. remains to be shown.

By 1°, $\mathscr{A}^t x_0\in\mathcal{W}$ for all $x_0\in\mathcal{W}$. Since $\mathscr{A}^t\in\mathcal{B}(\mathtt{X})$, it follows that $\mathscr{A}^t[\bar{\mathcal{W}}]\subset\bar{\mathcal{W}}$, i.e., that $\mathscr{A}^t x_0\in\mathcal{V}$ for all $x_0\in\mathcal{V}$, for any $t\ge 0$. Consequently, $\mathscr{A}$ is an ω-stable strongly continuous semigroup on $\mathcal{V}$ (since $\mathscr{A}$ is on $\mathtt{X}$, as noted in Proposition 2.3).

Moreover, $\mathrm{Ran}(\mathscr{B}) = \mathcal{W}\subset\mathcal{V}$. Therefore, $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ is a WPLS on $(\mathtt{U},\mathcal{V},\mathtt{Y})$, because the properties 1.–4. of Definition 2.2 are inherited from those of the WPLS $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ on $(\mathtt{U},\mathtt{X},\mathtt{Y})$. (In the sequel, we shall use the subindices $\mathcal{V}$ and $\mathtt{X}$, respectively, for these two systems and their components.) Thus, it only remains to be shown that $B_{\mathcal{V}}$ bounded.

By the remark in the proof of (b), the input operator $B_{\mathtt{X}}$ for $\mathtt{X}$ is bounded ($B_{\mathtt{X}}\in\mathcal{B}(\mathtt{U},\mathtt{X})$).

The map $\mathscr{C}$ is sometimes called the *state-to-output map* of Σ. The state-to-output map of the dual of $\Sigma_{\mathtt{X}}$ is given by $\mathscr{B}_{\mathtt{X}}^{\mathrm{d}} := \mathcal{R}\mathscr{B}_{\mathtt{X}}^*$ and that of $\Sigma_{\mathcal{V}}$ by $\mathscr{B}_{\mathcal{V}}^{\mathrm{d}} := \mathcal{R}\mathscr{B}_{\mathcal{V}}^*$. (In control theory, the adjoint $\mathscr{B}^*$ is taken with respect to the L^2 inner product regardless of stability, i.e., the dual of L^2_ω is identified with $\mathrm{L}^2_{-\omega}$, as in (3.9). See [7, Lemma 3.5.9(i) and Theorem 6.2.3] or [4, p. 157] for further details.)

Given any $x_0\in\mathrm{Dom}(A_{\mathcal{V}}^*)$, we have

$$\int_{\mathbb{R}}\langle\mathscr{B}_{\mathcal{V}}^{\mathrm{d}}x_0,\mathcal{R}u\rangle_{\mathtt{U}}\,dm = \int_{\mathbb{R}}\langle\mathscr{B}_{\mathcal{V}}^*x_0,u\rangle_{\mathtt{U}}\,dm = \langle x_0,\mathscr{B}u\rangle_{\mathcal{V}} = \langle x_0,\mathscr{B}u\rangle_{\mathtt{X}} \tag{3.9}$$

$$= \int_{\mathbb{R}}\langle\mathscr{B}_{\mathtt{X}}^*x_0,u\rangle_{\mathtt{U}}\,dm = \int_{\mathbb{R}}\langle\mathscr{B}_{\mathtt{X}}^{\mathrm{d}}x_0,\mathcal{R}u\rangle_{\mathtt{U}}\,dm \tag{3.10}$$

for all $u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})$, hence $B_{\mathcal{V}}^*(\mathscr{A}_{\mathcal{V}}^t)^*x_0 = (\mathscr{B}_{\mathcal{V}}^{\mathrm{d}}x_0)(t) = (\mathscr{B}_{\mathtt{X}}^{\mathrm{d}}x_0)(t) = B_{\mathtt{X}}^*(\mathscr{A}_{\mathtt{X}}^t)^*x_0$ for all $t \ge 0$. Set $t=0$ to observe that $B_{\mathcal{V}}^* = B_{\mathtt{X}}^*$ on $\mathrm{Dom}(A_{\mathcal{V}}^*)$, hence also $B_{\mathcal{V}}^*$ is bounded (having the extension $B_{\mathtt{X}}^*|_{\mathcal{V}} \in \mathcal{B}(\mathcal{V},\mathtt{U})$), hence so is $B_{\mathcal{V}}$. □

N.B. In Theorem 1.2, the WPLS (both WPLSs in (d)) can be made α-stable iff $\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\alpha;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ (just take $\omega=\alpha$ in the above proof to observe this). Moreover, in (c) the norm $\|C\|_{\mathcal{B}(\mathtt{X},\mathtt{Y})}$ can be made $\le M$, where M is given by (3.2) (see the proof of (c)); a dual claim holds for (b). See [4, Theorem 6.9.1] for further details.

The above choice of $\mathcal{W}$ in (d) is from [3]. It was recently pointed to us that the system on $(\mathtt{U},\mathcal{W},\mathtt{Y})$ in the above proof is called the input normalized realization and that the system on $(\mathtt{U},\mathcal{V},\mathtt{Y})$ the output normalized realization (or the reduced shift realization) of $\widehat{\mathscr{D}}$. Further details on such realizations are given in [7].

4. A PS-system whose impulse response is not a function

In this section we establish the counter-example mentioned at the end of Section 1. By Theorem 1.2(d), it suffices to construct a function $\widehat{\mathscr{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U}))$ such that $\widehat{\mathscr{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U}))$ but $\widehat{\mathscr{D}} \ne \widehat{f}$ for every function $f:\mathbb{R}_+ \to \mathcal{B}(\mathtt{U})$ (such that fu_0 is Laplace transformable for each $u_0 \in \mathtt{U}$). Such a function $\widehat{\mathscr{D}}$ will be achieved in Lemma 4.4, the input and output space $\mathtt{U}=\mathtt{Y}$ being separable.

Naturally, by $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ we mean that $F:\mathbb{C}^+ \to \mathtt{Y}$ is holomorphic and

$$\|F\|_{\mathrm{H}^2_{\mathrm{strong}}} := \sup_{\|u_0\|\le 1} \|Fu_0\|_{\mathrm{H}^2} < \infty, \tag{4.1}$$

where $\mathrm{H}^2 := \mathrm{H}^2_0$ and $\mathbb{C}^+ := \mathbb{C}^+_0$. See also Lemma 3.1.

When we identify a function $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U},\mathtt{Y}))$ with the corresponding multiplication map $u_0 \mapsto Fu_0$, the following result holds:

Lemma 4.1. *We have* $\mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U},\mathtt{Y})) = \mathcal{B}(\mathtt{U},\mathrm{H}^2(\mathbb{C}^+;\mathtt{Y}))$*, isometrically.*

Proof. If $V \in \mathcal{B}(\mathtt{U},\mathrm{H}^2(\mathbb{C}^+;\mathtt{Y}))$, then, obviously, $F : u_0 \mapsto (Vu_0)(s)$ is linear and bounded $\mathtt{U} \to \mathtt{Y}$ for any $s \in \mathbb{C}^+$, hence then $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U},\mathtt{Y}))$, by Lemma 3.1. This establishes "$\supset$"; obviously, also the converse holds, isometrically. □

For every $R \in \mathcal{B}(\mathtt{U},\mathrm{L}^2(\mathbb{R}_+;\mathtt{Y}))$ we can define the *Laplace transform* $\widehat{R} \in \mathcal{B}(\mathtt{U},\mathrm{H}^2(\mathbb{C}^+;\mathtt{Y}))$ by $\widehat{R}u_0 := \widehat{Ru_0}$. This map is an isometric isomorphism onto, by Lemma 1.3.

The space

$$\mathrm{L}^2_{\mathrm{strong}}(\mathbb{R}_+;\mathcal{B}(\mathtt{U},\mathtt{Y})) := \{f:\mathbb{R}_+ \to \mathcal{B}(\mathtt{U},\mathtt{Y}) \,\big|\, fu_0 \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{Y}) \text{ for all } u_0 \in \mathtt{U}\}. \tag{4.2}$$

is a subspace $\mathcal{B}(\mathtt{U}, \mathrm{L}^2(\mathbb{R}_+; \mathtt{Y}))$, by the closed-graph theorem [4, Lemma F.1.6]. Therefore, every $f \in \mathrm{L}^2_{\mathrm{strong}}(\mathbb{R}_+; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ has a Laplace transform $\widehat{f} \in \mathrm{H}^2_{\mathrm{strong}}$ satisfying $\widehat{fu_0} = \widehat{f} u_0$ ($u_0 \in \mathtt{U}$).

From this and Theorem 1.2(b) we conclude that an impulse response R can be realized as a WPLS with a bounded input operator iff $R \in \mathcal{B}(\mathtt{U}, \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{Y}))$ for some $\omega \in \mathbb{R}$. Example 4.3 below shows that not all such R can be identified with a function.

In Lemma 4.4 we shall show that the impulse response R of Example 4.3 also satisfies $\hat{R}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$. By Theorem 1.2(d), this implies that R is the impulse response (i.e., $\hat{R}$ is the transfer function) of a PS-system. Thus, by Theorem 1.2(d), we will establish the following:

Corollary 4.2. *There exists a PS-system with input space* $\mathtt{U} = \ell^2(\mathbb{N})$ *and output space* $\mathtt{Y} = \ell^2(\mathbb{N})$ *such that its impulse response does not correspond to any function* $\mathbb{R}_+ \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$.

Now it only remains to construct the function $\widehat{\mathscr{D}} = \widehat{R}$ with the properties promised above. We start with the incompleteness of $\mathrm{L}^2_{\mathrm{strong}}$:

Example 4.3. Let $\mathtt{U} := \ell^2(\mathbb{N})$. The continuous functions $f_n : [0,1] \to \mathcal{B}(\mathtt{U})$ constructed below form a nonconvergent Cauchy-sequence in $\mathrm{L}^2_{\mathrm{strong}}([0,1]; \mathcal{B}(\mathtt{U}))$. Naturally, the (corresponding multiplication operator) sequence converges in the Banach space $\mathcal{B}(\mathtt{U}; \mathrm{L}^2([0,1]; \mathtt{U}))$ to a map R. We also have $f_n(t) = f_n(t)^*$ for all $t \in [0,1]$ and $n \in \mathbb{N}$.

As above, we identify f_n with the multiplication operator $M_{f_n} : u_0 \mapsto f_n u_0$.

In the proof below we construct diagonal functions $f_n : [0,1] \to \mathcal{B}(\mathtt{U})$ such that $f_n u_0$ converges in L^2 for each $u_0 \in \mathtt{U}$ but any "limit function" f would be such that $f(t)$ is an unbounded operator for almost every $t \in [0,1]$. This is achieved by letting the diagonal elements of "f" to be suitable translates of an unbounded scalar function $g \in \mathrm{L}^2$, so that at each $t \in [0,1]$ arbitrarily high values are attained by some of the translates.

Proof. 1° *The construction of* $\{f_n\}$*:* Set $g(t) := |t|^{-1/3}$, $g_n(t) := (|t| + 1/n)^{-1/3}$. Observe that $g \in \mathrm{L}^2([-1,1])$. Furthermore, $g_n(t) \to g(t)$ monotonously for each t and $\|g - g_n\|_{\mathrm{L}^2([-1,1])} \to 0$.

Denote the natural base of $\mathtt{U}$ by $\{e_j \mid j \in \mathbb{N}\}$. By $P_k \in \mathcal{B}(\mathtt{U})$ we denote the natural (coordinate) projection $P_k : \sum_{j \in \mathbb{N}} x_j e_j \mapsto x_k e_k$ ($k \in \mathbb{N}$).

Let $\{q_k\} \subset [0,1]$ be dense. For every $t \in [0,1]$ and $n \in \mathbb{N}$, define $f_n(t) := \sum_{k \in \mathbb{N}} g_n(t - q_k) P_k$, i.e.,

$$f_n(t)x := \sum_{k \in \mathbb{N}} g_n(t - q_k) x_k e_k \quad (x \in \mathtt{U}). \tag{4.3}$$

Obviously, $f_n(t)^* = \sum_{k \in \mathbb{N}} \overline{g_n(t - q_k)} P_k^* = \sum_{k \in \mathbb{N}} g_n(t - q_k) P_k$ for all n and t.

2° *$f_n : [0,1] \to \mathcal{B}(\mathtt{U})$ is continuous:* Let $n \in \mathbb{N}$, $t \in [0,1]$ and $\epsilon > 0$. Because g_n is (uniformly) continuous, there exists $\delta > 0$ such that $|g_n(t') - g_n(t'')| < \epsilon$ for any $t', t'' \in [-1,1]$ such that $|t' - t''| < \delta$. If $t, t' \in [0,1]$, $|t' - t| < \delta$ and $x \in \mathtt{U}$, then

$$\|(f_n(t) - f_n(t'))x\|_{\mathtt{U}}^2 = \|\sum_k x_k(g_n(t-q_k) - g_n(t'-q_k))e_k\|_{\mathtt{U}}^2 \le \sum_k |x_k|^2\epsilon^2 = \epsilon^2\|x\|_{\mathtt{U}}^2. \tag{4.4}$$

Consequently, $\|f_n(t) - f_n(t')\| \le \epsilon$. Because $\epsilon > 0$ was arbitrary, f_n is continuous.

3° *$f_n \to R$ in $\mathcal{B}(\mathtt{U}, \mathrm{L}^2([0,1];\mathtt{U}))$:* For every $t \in [0,1]$, $n \in \mathbb{N}$, and $x \in \mathtt{U}$, we define the diagonal operator $R : \mathtt{U} \to \mathrm{L}^2$ by $R := \sum_{k\in\mathbb{N}} g(\cdot - q_k)P_k$, i.e.,

$$Rx := \sum_{k\in\mathbb{N}} g(\cdot - q_k)x_k e_k \quad (x \in \mathtt{U}). \tag{4.5}$$

Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\|g - g_n\|_{\mathrm{L}^2([-1,1))} < \epsilon$ and, consequently (because $\|h\|_2^2 = \sum_k \|P_k h\|_2^2$ for every $h : [0,1] \to \mathtt{U}$, and $P_k Rx = g(\cdot - q_k)x_k e_k$)

$$\|f_n x - Rx\|_{\mathrm{L}^2([0,1];\mathtt{U})}^2 \le \sum_k |x_k|^2 \|g_n(\cdot - q_k) - g(\cdot - q_k)\|_{\mathrm{L}^2([0,1])}^2 \le \epsilon^2\|x\|_{\mathtt{U}}^2 \quad (x \in \mathtt{U}). \tag{4.6}$$

Therefore, $M_{f_n} \to R$ in $\mathcal{B}(\mathtt{U}, \mathrm{L}^2([0,1];\mathtt{U}))$. In particular, $\{f_n\}$ is $\mathrm{L}^2_{\mathrm{strong}}$-Cauchy.

4° *$\{f_n\}$ does not converge in $\mathrm{L}^2_{\mathrm{strong}}$:* To obtain a contradiction, we assume that $R = M_f$ for some $f : [0,1] \to \mathcal{B}(\mathtt{U})$ and deduce that $\|f(t)\|_{\mathcal{B}(\mathtt{U})} = \infty$ a.e.

Indeed, if $f_n \to f$ in $\mathrm{L}^2_{\mathrm{strong}}([0,1];\mathcal{B}(\mathtt{U}))$, then $fx = \lim_n f_n x = Rx$ in L^2, hence $fx = Rx$ a.e. on $[0,1]$, for every $x \in X$. Consequently, there exists a null set N such that $fe_k = Re_k$ on $[0,1] \setminus N$ for any $k \in \mathbb{N}$.

Let $t \in [0,1] \setminus N$ and $\gamma < \infty$ be arbitrary. By the density of $\{q_k\}$ in $[0,1]$, there exists k such that $g(t - q_k) = |t - q_k|^{-1/3} > \gamma$ and hence $\|f(t)e_k\|_{\mathtt{U}} = \|R(t)e_k\| > \gamma$, by (4.5). Consequently, $\|f(t)\|_{\mathcal{B}(\mathtt{U})} > \gamma$. Because $\gamma < \infty$ was arbitrary, $f(t) \notin \mathcal{B}(\mathtt{U})$, a contradiction. □

(This shows that $\mathrm{L}^2_{\mathrm{strong}}$ is a proper, non-closed subspace of $\mathcal{B}([0,1];\mathrm{L}^2(\mathtt{U}))$.)

Now we establish the remaining required properties:

Lemma 4.4. *The function $\widehat{R}$ constructed in Example 4.3 satisfies $\widehat{R} = \widehat{R}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U}))$.*

Proof. Recall that we consider $\mathrm{L}^2([0,1];\mathtt{U})$ as a subspace of $\mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$. Thus, by Lemma 4.1, we have $\widehat{R} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbb{C}^+;\mathcal{B}(\mathtt{U}))$, where $\widehat{R}(s)x = \widehat{Rx}(s)$ for each $x \in \mathtt{U}$. Because R is real and diagonal, for any $x, y \in \mathtt{U}$ we obviously have $\langle Rx, y\rangle = \langle x, Ry\rangle$ as elements of $\mathrm{L}^2([0,1])$, hence

$$\langle \widehat{R}(s)x, y\rangle = \langle \widehat{Rx}(s), y\rangle = \int_0^1 \mathrm{e}^{-st}\langle (Rx)(t), y\rangle\,dt = \int_0^1 \mathrm{e}^{-st}\langle x, (Ry)(t)\rangle\,dt \tag{4.7}$$

$$= \langle x, \int_0^1 \mathrm{e}^{-\bar{s}t}(Ry)(t)\,dt\rangle = \langle x, (\widehat{R}y)(\bar{s})\rangle = \langle x, \widehat{R}(\bar{s})y\rangle. \tag{4.8}$$

□

N.B. in [4, Example F.3.6] it was shown that the boundary trace of $\widehat{R}$ belongs to $\mathrm{L}^2_{\mathrm{strong}}(i\mathbb{R}; \mathcal{B}(\mathtt{U}))$ (in fact, $\widehat{R}$ has a continuous extension to $\mathbb{C}$). Since R (hence nor $\hat{\hat{R}} = 2\pi R(-\cdot)$) is not $\mathrm{L}^2_{\mathrm{strong}}$, it follows that the Fourier transform does not map (all continuous elements of) $\mathrm{L}^2_{\mathrm{strong}}$ into $\mathrm{L}^2_{\mathrm{strong}}$.

More on $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{strong}}$ can be found in [4, Appendix F].

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