Part IV

Discrete-Time Control Theory (wpls's)

Chapter 13

Discrete-Time Maps and Systems (ti & wpls)

At any given moment, an arrow must be either where it is or where it is not. But obviously it cannot be where it is not. And if it is where it is, that is equivalent to saying that it is at rest.

— Zeno's (335–262 B.C.) paradox of the moving (still?) arrow

In this chapter, we present here briefly some facts on the discrete counterparts of WPLSs, which we call *discrete-time well-posed linear systems (wpls's).* They are the systems governed by difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \quad y_j = Cx_j + Du_j, \quad j \in \mathbf{Z}, \end{cases} \tag{13.1}$$

for $A, B, C, D \in \mathcal{B}$; see Definition 13.3.1 and Lemma 13.3.3 for definitions.

We show that almost all our continuous-time results have discrete-time analogies (see Theorem 13.3.13), and also many further results hold due to the boundedness of the generating operators (A, B, C, D). Roughly speaking, we write continuous-time results (and definitions) in lower case (e.g., $\mathrm{L}^2 \mapsto \ell^2$), as in (13.63).

In Section 13.1, we study bounded linear time-invariant maps $\ell^2_r(\mathbf{Z}; U) \to \ell^2_r(\mathbf{Z}; Y)$ ("$\mathrm{ti}_r(U, Y)$", where $\|u\|^2_{\ell^2_r(\mathbf{Z};U)} := \sum_k \|r^{-k} u_k\|^2_U$), and the corresponding transfer functions (this corresponds to Chapters 2 and 3). The Cayley transform is treated in Section 13.2.

In Section 13.3, we study wpls's (this corresponds to Chapter 6, also Chapters 4, 7 and 8 are treated in Theorem 13.3.13). The I/O maps of wpls's are exactly the causal maps in $\cup_{r>0} \mathrm{ti}_r$ (see (13.46)).

In Section 13.4, we show how to obtain wpls's from WPLSs, by discretization. This allows us to reduce several WPLS problems to wpls problems, which are often substantially simpler due to bounded input and output operators. (This differs from the Cayley transform of Lemma 13.2.1 and from the method of Lemma 13.1.4.)

Discrete-time Riccati equations (DAREs) and spectral factorization are treated in Chapter 14 (this corresponds to Chapters 9 and 5) and minimization problems in Chapter 15 (this corresponds to Chapter 10). Discrete-time H^∞ (and Nehari) problems are treated in Sections 11.5 and 12.2.

Also in this chapter, U, W, H, Y and Z denote Hilbert spaces of arbitrary dimensions and B denotes a Banach space.

13.1 Discrete-time I/O maps (tic)

The Priest's grey nimbus in a niche where he dressed discreetly. I will not sleep here tonight. Home also I cannot go. A voice, sweetened and sustained, called to him from the sea. Turning the curve he waved his hand. A sleek brown head, a seal's, far out on the water, round. Usurper.

— James Joyce (1882–1941), "Ulysses"

In this and the following section, we present results corresponding to Chapters 2 and 3; in particular, we extend the discrete-time Fourier multiplier and H^∞ boundary function theorems to I/O maps over unseparable Hilbert spaces, in Lemmas 13.1.5 and 13.1.6. Our third main result is Lemma lticConvol(d), which treats time-invariant causal operators ($\mathrm{tic}_{\mathrm{loc}}$) that are "almost r-stable" (that map functions with finite support into ℓ^2_r). We also define ti and tic and treate their basic properties including adjoints, inverses, convolution forms and $\mathcal{Z}$-transforms. Further results are obtained through Theorem 13.3.13.

We start by presenting our notation. Let $S \subset \mathbf{Z}$, $p \in [1, \infty)$ and $r > 0$. Recall that $x : S \to B$ (equivalently, $x \in B^S$) means that x is a function from S to B, i.e., a B-valued sequence on S. We set $\|\{x_j\}_{j\in S}\|_{\ell^\infty_r(S;B)} := \sup_{j\in S} \|r^{-j}x_j\|_B$, $\ell^\infty := \ell^\infty_1$. We also define

$$\ell^p_r(S;B) := \{x : S \to B \,\big|\, \|x\|^p_{\ell^p_r} := \sum_{j\in S} \|r^{-j}x_j\|^p_B < \infty\} \tag{13.2}$$

and $\ell^p := \ell^p_1$. We have $\|x\|_{\ell^q_r} \le \|x\|_{\ell^p_r} \le \infty$ ($x : S \to B$, $\infty \ge q \ge p \ge 1$, $r > 0$) (proof: assume w.l.o.g. that $\|x\|_{\ell^p_r} = 1$, ...). For $S \subset \mathbf{N}$ we have (use Lemma B.3.13 for $p < q$)

$$\|x\|_{\ell^p_s(S;B)} \le M_{r/s,p,q}\|x\|_{\ell^q_r(S;B)} \quad (\infty > s > r > 0,\ p, q \in [1, \infty]). \tag{13.3}$$

By $\mathcal{BC}(U,Y)$ we denote compact linear operators $U \to Y$, and

$$\ell^1_{\mathcal{BC}}(S;\mathcal{B}(U,Y)) := \{T \in \ell^1(S;\mathcal{B}(U,Y)) \,\big|\, T_j \in \mathcal{BC}(U,Y) \ \text{ for all } \ j \ne 0\}, \tag{13.4}$$

$$\ell^1_\pm := \{T \in \ell^1 \,\big|\, T_j = 0 \ \text{ for all } \ \pm j < 0\}, \tag{13.5}$$

$$\ell^1_{\mathcal{BC},\pm} := \{T \in \ell^1_{\mathcal{BC}} \,\big|\, T_j = 0 \ \text{ for all } \ \pm j < 0\}. \tag{13.6}$$

Note that the vectors $e_k := \chi_{\{k\}}$ ($k \in S$) form the standard orthonormal base of the Hilbert space $\ell^2(S)$, and $\{x_\alpha e_k \,\big|\, \alpha \in \mathcal{A},\ k \in S\}$ is an orthonormal base for $\ell^2(S;H)$ whenever, $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is an orthonormal base for H. Obviously,

$$\mathrm{c_c}(S;B) := \{(x_j)_{j\in S} \,\big|\, x_j = 0 \text{ for } j \text{ not in some finite subset of } S\} \tag{13.7}$$

is dense in $\ell^p_r(S;B)$. By Lemma B.4.15, $\ell^p_r(S;H)^* = \ell^q_{1/r}(S;H)$ when $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$.

The *convolution*

$$(a_j)_{j\in\mathbf{Z}} * (b_k)_{k\in\mathbf{Z}} := \Big(\sum_j a_j b_{n-j}\Big)_{n\in\mathbf{Z}} \tag{13.8}$$

is bilinear and bounded $\ell^1_r(\mathbf{Z};B)\times\ell^p_r(\mathbf{Z};B')\to\ell^p_r(\mathbf{Z};B'')$ whenever $B\times B'\to B''$ is bilinear and bounded (e.g., $B=\mathcal{B}(U,Y)$, $B'=\mathcal{B}(Y,Z)$ and $B''=\mathcal{B}(U,Z)$); see Lemma D.1.7. By the Fubini theorem, we have $(a*)^*=(\mathbf{Я}a^*)*$ in the sense that $\langle a*b,c\rangle=\langle b,(\mathbf{Я}a^*)*c\rangle$ when, e.g., $a\in\ell^1$, $b\in\ell^p$ and $c\in\ell^q$. We define the isometric isomorphism (multiplication operator) $r^{\cdot}\in\mathcal{B}(\ell^p_s,\ell^p_{rs})$ by

$$r^{\cdot}=((x_j)_{j\in S}\mapsto(r^jx_j)_{j\in S}) \tag{13.9}$$

Obviously, $(r^{\cdot}a)*b=r^{\cdot}(a*r^{-\cdot}b)$, hence (cf. Remark 13.3.9)

$$\mathcal{T}_r(a*):=r^{\cdot}(a*)r^{-\cdot}=(r^{\cdot}a)* \tag{13.10}$$

defines an isometric isomorphism $\mathcal{T}_r:\ell^1_s*\mapsto\ell^1_{rs}*$ (we identify ℓ^1_s* with ℓ^1_s as a Banach space). We identify $S\to B$ with $\{x:\mathbf{Z}\to B\,\big|\,x_j=0\text{ for all }j\in\mathbf{Z}\setminus S\}$.

The *left shift* $\tau=\tau^1$ is defined by $(\tau x)_i:=x_{i+1}$ for $x:\mathbf{Z}\to B$. We set $\mathbf{N}:=\{0,1,2,\ldots\}$, $\mathbf{Z}_-:=\mathbf{Z}\setminus\mathbf{N}=\{-1,-2,-3,\ldots\}$, so that $\pi^+:=\pi_{\mathbf{N}}$ maps $(\mathbf{Z}\to U)\to(\mathbf{N}\to U)$ and $\pi^++\pi^-=I$, where $\pi^-:=\pi_{\mathbf{Z}_-}$ (recall that $\pi_Nu:=\chi_Nu$ for all sequences u and sets N). We set $P_ku:=u_k$ ($k\in\mathbf{Z}$). The reflection $\mathbf{Я}$ is defined as in continuous time: $(\mathbf{Я}x)_i:=x_{-i}$, hence

$$\tau\mathbf{Я}=\mathbf{Я}\tau^{-1},\ \ \mathbf{Я}\tau=\tau^{-1}\mathbf{Я},\ \ \mathbf{Я}\pi^+\mathbf{Я}=\tau^{-1}\pi^-\tau,\ \ \mathbf{Я}\pi^-\mathbf{Я}=\tau^{-1}\pi^+\tau. \tag{13.11}$$

However, the canonical discrete-time *reflection* is the one satisfying $\mathbf{Я}_{-1}\pi^+=\pi^-\mathbf{Я}_{-1}$, namely the one defined by $(\mathbf{Я}_{-1}x)_i:=x_{-1-i}$ (cf. Proposition 13.3.5). We have

$$\tau^{-1}\mathbf{Я}_{-1}=\mathbf{Я}=\mathbf{Я}_{-1}\tau,\ \ \mathbf{Я}_{-1}\pi^+=\pi^-\mathbf{Я}_{-1},\ \ \tau\mathbf{Я}_{-1}=\mathbf{Я}_{-1}\tau^{-1}. \tag{13.12}$$

Moreover, $\mathbf{Я}^*=\mathbf{Я}$ and $\mathbf{Я}_{-1}^*=\mathbf{Я}_{-1}$ on $\langle\cdot,\cdot\rangle_{\ell^2_r,\ell^2_{1/r}}$, $\mathbf{Я}^{-1}=\mathbf{Я}$, $\mathbf{Я}_{-1}^{-1}=\mathbf{Я}_{-1}$, and

$$\|\tau x\|_{\ell^p_r}=r\|x\|_{\ell^p_r},\ \ \|\mathbf{Я}x\|_{\ell^p_r}=\|x\|_{\ell^p_{1/r}},\ \ \|\mathbf{Я}_{-1}x\|_{\ell^p_r}=r\|x\|_{\ell^p_{1/r}}(x:\mathbf{Z}\to B,\ r>0). \tag{13.13}$$

We define the *Z-transform* $\widehat{u}:=\mathcal{Z}u$ of $u:\mathbf{Z}\to U$ by $\widehat{u}(z):=\sum_{j\in\mathbf{Z}}z^ju_j$ for those z for which the sum converges (one often uses z^{-1} instead of z to make the formulae more akin to their continuous-time counterparts at the cost of having to study functions holomorphic at infinity).

One easily verifies that the Z-transform maps $\ell^2_r(\mathbf{N};U)$ onto the Hardy space $\mathrm{H}^2_{r^{-1}}:=\mathrm{H}^2(r^{-1}\mathbf{D};U)$ through an isometric times $\sqrt{2\pi}$ isomorphism (i.e., $\|\widehat{u}\|_{\mathrm{H}^2_{1/r}}=\sqrt{2\pi}\|u\|_{\ell^2_r}$; use Lemma D.1.15 and scaling), and $\ell^1_r(\mathbf{N};U)$ into $\mathrm{H}^\infty_{1/r}:=\mathrm{H}^\infty(\mathbf{D}_{1/r};U)$ linearly and 1-1, with $\|\widehat{u}\|_{\mathrm{H}^\infty_{1/r}}\le\|u\|_{\ell^1_r}$ (note the exceptional meaning of H^∞_r (instead of $\mathrm{H}^\infty(\mathbf{C}^+_r;U)$) in this section; recall that $\mathbf{D}_r:=r\mathbf{D}=\{z\in\mathbf{C}\,\big|\,|z|<r\}$ and $\|\widehat{u}\|_{\mathrm{L}^2(r\partial\mathbf{D};U)}:=\int_0^{2\pi}\|\widehat{u}(re^{it})\|_U^2\,dt$, hence $\|1\|_2=\sqrt{2\pi}$). It follows that

$$\widehat{\mathbf{Я}u}(z)=\widehat{u}(1/z),\quad\widehat{\tau u}=z^{-1}\widehat{u}\quad(u:\mathbf{Z}\to U). \tag{13.14}$$

We start by defining the discrete-time counterparts of TI and TIC (cf. Definitions 2.1.1 and 2.1.4):

Definition 13.1.1 (ti and tic) *Let $r > 0$. We define* $\mathrm{ti}_r(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{E} \in \mathcal{B}(\ell^2_r(\mathbf{Z};U), \ell^2_r(\mathbf{Z};Y))$ *that are* time-invariant*, i.e.,* $\tau^1\mathbb{E} = \mathbb{E}\tau^1$.

We define $\mathrm{tic}_r(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{D} \in \mathrm{ti}_r(U,Y)$ *that are* causal*, i.e.,* $\pi^-\mathbb{D}\pi^+ = 0$.

Finally, $\mathrm{tic}_{\mathrm{loc}}(U,Y)$ *is the set of linear maps* $\mathbb{D}^+ : U^{\mathbf{N}} \to Y^{\mathbf{N}}$ *that are time-invariant* ($\tau^{-1}\mathbb{D}^+ = \mathbb{D}^+\tau^{-1}$) *and causal* ($\pi_{\{0\}}\mathbb{D}^+\pi_{\{1,2,3,\ldots\}} = 0$).

Maps in $\mathrm{ti} := \mathrm{ti}_1$ *are called* stable*; maps in* $\mathrm{ti}_{\exp} := \cup_{r<1}\mathrm{ti}_r$ *are called* exponentially stable*, and maps in* $\mathrm{ti}_\infty \setminus \mathrm{ti}$ *are called unstable, where* $\mathrm{ti}_\infty := \cup_{r>0}\mathrm{ti}_r$. *We set* $\mathrm{tic} := \mathrm{tic}_1$, $\mathrm{tic}_{\exp} := \mathrm{tic} \cap \mathrm{ti}_{\exp}$, $\mathrm{tic}_\infty := \mathrm{tic} \cap \mathrm{ti}_\infty$.

If $\mathbb{E} \in \mathrm{ti}_r(U,Y)$*, then its* (noncausal) adjoint $\mathbb{E}^*$ *is the* $\mathrm{ti}_{1/r}(Y,U)$ *map that satisfies*

$$\sum_{n\in\mathbf{Z}} \langle (\mathbb{E}u)(n), y(n)\rangle\, dt = \sum_{n\in\mathbf{Z}} \langle u(n), (\mathbb{E}^* y)(n)\rangle\, dt \quad (u \in \ell^2_r(\mathbf{Z};U),\ y \in \ell^2_{1/r}(\mathbf{Z};Y)), \tag{13.15}$$

and its causal adjoint *is* $\mathbb{E}^{\mathrm{d}} := \text{Я}\mathbb{E}^*\text{Я} = \text{Я}_{-1}\mathbb{E}^*\text{Я}_{-1} \in \mathrm{ti}_r(Y,U)$*, where* $(\text{Я}x)_i := x_{-i}$.

(In the literature, "exponentially stable" is often called "power stable", but we prefer this analogy to continuous time.)

By Lemma 2.1.10 (see Theorem 13.3.13), we have $\mathrm{tic} = \mathrm{tic}_\infty \cap \mathrm{ti}$, $\mathrm{tic}_{\exp} = \cup_{r<1}\mathrm{tic}_r$, $\mathrm{tic}_\infty = \cup_{r>0}\mathrm{tic}_r$.

Let $\mathbb{E} \in \mathrm{ti}_r$, $r > 0$. One easily verifies that $\mathbb{E}^{\mathrm{d}} \in \mathrm{ti}_r$ is causal ($\in \mathrm{tic}_r$) iff $\mathbb{E}$ is. Obviously, $\tau^n\mathbb{E} = \mathbb{E}\tau^n$. for all $n \in \mathbf{Z}$ (and $\pi_{\{\ldots,n-2,n-1,n\}}\mathbb{E}\pi_{\{n+1,n+2,n+3,\ldots\}} = 0$ iff $\mathbb{E} \in \mathrm{tic}_\infty$).

If a map is causal and anti-causal, then it takes the form of a multiplication operator:

Lemma 13.1.2 (Static $\mathbb{D}$) *Let* $\mathbb{D}, \mathbb{D}^* \in \mathrm{tic}_\infty$*. Then* $\mathbb{D} \in \mathcal{B}$*. Moreover, the imbedding* $\mathcal{B} \mapsto \mathrm{TIC}$ *is isometric, preserves norms, and commutes with algebraic operations.* □

(The proof of Lemma 2.1.7 applies here too; see Remark 13.3.9 for a stability shift.)

If $\mathbb{D} \in \mathrm{tic}_\infty(U,Y)$, then, obviously, $\pi^+\mathbb{D}\pi^+ \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$ and the map $\mathrm{tic}_\infty \mapsto \mathrm{tic}_{\mathrm{loc}}$ is injective, hence we can and will identify $\mathbb{D}$ and $\pi^+\mathbb{D}\pi^+$. Thus, $\mathrm{tic}_r \subset \mathrm{tic}_s \subset \mathrm{tic}_\infty \subset \mathrm{tic}_{\mathrm{loc}}$ when $0 < r < s < \infty$.

On the other hand, if $\mathbb{D}^+ \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$, then, obviously, $\tau^{-n}\mathbb{D}^+ = \mathbb{D}^+\tau^{-n}$ and $\pi_{\{0,1,\ldots,n\}}\mathbb{D}^+\pi_{\{n+1,n+2,n+3,\ldots\}} = 0$ for all $n \in \mathbf{N}$. One easily verifies that $\mathrm{tic}_{\mathrm{loc}}$ maps correspond one-to-one to linear, causal ($\pi^+\mathbb{D}\pi^- = 0$), time-invariant ($\tau^n\mathbb{D} = \mathbb{D}\tau^n$ for all $n \in \mathbf{Z}$) maps between sequences $\mathbf{Z} \to U$ and $\mathbf{Z} \to Y$ whose supports are bounded from the left. Obviously, such a map belongs to tic_∞ iff it is bounded under some ℓ^2_r norm; we give another necessary and sufficient condition in (b) below:

Lemma 13.1.3 (tic maps are convolutions)

(a) The set $\mathrm{tic}_{\mathrm{loc}}(U,Y)$ *is exactly the set maps* $\mathbb{D} : U^{\mathbf{N}} \to Y^{\mathbf{N}}$ *that have a (necessarily unique) representation of the form*

$$\mathbb{D} = \sum_{j\in\mathbf{N}} T_j \tau^{-j}, \quad \text{i.e.,} \quad (\mathbb{D}u)_k = \sum_{j\in\mathbf{N}} T_j u_{k-j} \quad (u : \mathbf{N} \to U,\ k \in \mathbf{Z}), \tag{13.16}$$

equivalently, $\mathbb{D} = (T_j)_{j\in\mathbf{N}}*$, *where* $T_j \in \mathcal{B}(U,Y)$ *for all* $j \in \mathbf{N}$.

(b) Assume (13.16). Then $T_j = P_{\{i\}}\mathbb{D}P^*_{\{0\}}$ *for all* j, *and the following are equivalent:*

(i) $\mathbb{D} \in \mathrm{tic}_r(U,Y)$ *for some* $r > 0$;
(ii) $\|T_j\| \le Ms^j$ *for all* $j \in \mathbf{N}$ *and some* $s > 0$.

(c1) If (i) holds, then $\widehat{\mathbb{D}}(z) = \sum_{j\in\mathbf{N}} T_j z^j \in \mathrm{H}^\infty(\mathbf{D}_r;\mathcal{B}(U,Y))$ *and (ii) holds for* $s = r$ *and* $M = \|\mathbb{D}\|_{\mathrm{tic}_r}$.

(c2) Conversely, if (ii) holds, then (i) holds for any $r > s$ *(and* $\|\mathbb{D}\|_{\mathrm{tic}_r} \le M'_{r/s}M$*).*

(c3) If $\mathbb{D} \in \mathrm{tic}_r$, *then* $\mathbb{D} \in \ell^1_s(\mathbf{N};\mathcal{B}(U,Y))*$ *for all* $s > r$.

(d) Assume that $\mathbb{D} \in \mathrm{tic}_{\mathrm{loc}}$ *and* $r > 0$ *are s.t.* $\mathbb{D}u_0e_0 \in \ell^2_r$ *for all* $u_0 \in U$ *(equivalently,* $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}_{1/r};\mathcal{B}(U,Y))$*). Let* $0 < s < r < t < \infty$. *Then* $\mathbb{D} \in \mathrm{tic}_t$, $\mathbb{D}[\ell^2_s(\mathbf{N};U) + \mathrm{c_c}] \subset \ell^2_r$, $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(r^{-1}\mathbf{D};\mathcal{B}(U,Y))$ *and*

$$\|\mathbb{D}\|_{\mathrm{tic}_t} \le M'_{t/r}\|\mathbb{D}\pi_{\{0\}}\| < \infty, \tag{13.17}$$

$$\|\mathbb{D}u\|_{\ell^2_r} \le \|\mathbb{D}\pi_{\{0\}}\|\|u\|_{\ell^1_r} \le M''_{r/s}\|\mathbb{D}\pi_{\{0\}}\|\|u\|_{\ell^2_s} \quad (u \in \ell^2_s(\mathbf{N};U)). \tag{13.18}$$

In particular, $\mathbb{D} \in \mathcal{B}(\ell^1_r(\mathbf{Z};U),\ell^2_r(\mathbf{Z};U))$, $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z};U),\ell^\infty_{1/r}(\mathbf{Z};U))$, *and* $\mathbb{D}\pi_{[0,t)}u \to \mathbb{D}u$ *and* $\mathbb{D}^t u \to \mathbb{D}u$ *in* ℓ^2_r *for all* $u \in \ell^2_s(\mathbf{N};U) + \mathrm{c_c}$.

Thus, $\mathrm{tic}_{\mathrm{loc}}$ is the set of convolution operators having $\mathbf{N} \to \mathcal{B}(U,Y)$ kernels, and tic_r is its subset of maps that are bounded $\ell^2_r \to \ell^2_r$. If $\mathbb{D} \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$ satisfies $\mathbb{D}[\mathrm{c_c}] \subset \ell^2_r$, then $\mathbb{D} \in \mathrm{tic}_{r'}$ for all $r' > r$, by (d).

As we will see from Definitions 13.3.1 and 13.3.4, (i) holds iff $\mathbb{D}$ has a wpls realization; hence all (linear, causal and time-invariant) maps satisfying (ii) have a wpls realization.

Proof of Lemma 13.1.3: (a) For all $u \in \mathbf{N} \to U$, we have

$$(\mathbb{D}u)_k = \sum_{j=0}^{k} (\mathbb{D}\pi_{k-j}u)_k = \sum_{j=0}^{k} (\pi_k\mathbb{D}\pi_{k-j}u)_k \tag{13.19}$$

$$= \sum_{j=0}^{k} (\tau^{-(k-j)}\pi_j\mathbb{D}\pi_0\tau^{k-j}u)_k = \sum_{j=0}^{k} P_j\mathbb{D}P_0^* u_{k-j}, \tag{13.20}$$

i.e., (13.16) holds for this u. The converse is obvious.

(b) This follows from (c1)&(c2).

(c1) Conversely, assume (i). The claim on $\widehat{\mathbb{D}}$ is obviously true. Let $\|u\|_U = 1$, so that $\|ue_0\|_{\ell^2_r} = 1$, where $e_0 := \chi_0$. Then, for any $j \in \mathbf{N}$,

$$\|\mathbb{D}\|^p_{\mathrm{tic}_r} \geq \|\mathbb{D}ue_0\|^p_{\ell^p_r} := \sum_k \|r^{-k}(\mathbb{D}ue_0)_k\|^p_Y \geq \|r^{-j}(\mathbb{D}ue_0)_j\|^p_Y = \|r^{-j}T_j u\|^p_Y, \tag{13.21}$$

hence $\|T_j u\|_Y \leq r^j \|\mathbb{D}\|_{\mathrm{tic}_r}$. Because u was an arbitrary unit vector, (ii) holds for $s = r$.

(c2) If (ii) holds and $r > s$, then (see Lemma D.1.7)

$$\|(T_j)_{j\in\mathbf{Z}} *\|_{\mathrm{tic}_r} \leq \|(T_j)_{j\in\mathbf{Z}}\|_{\ell^1_r} \leq M'_{r/s} M, \tag{13.22}$$

where $M'_{r/s} := \sum_{k\in\mathbf{N}} (r/s)^{-k} < \infty$. (Note that (i) does not have to hold $r = s$ (e.g., take $T_j = 1$ for all j).)

(c3) By (c1) (and (ii)) and (13.3), $\mathbb{D} \in \ell^\infty_r(\mathbf{N}; \mathcal{B}(U,Y))* \subset \ell^1_s(\mathbf{N}; \mathcal{B}(U,Y))*$ for any $s > r$.

(d) 1° $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}$: Obviously, $\mathbb{D}P_0^*[U] \subset \ell^2_r$ iff $(T_j u_0)_{j\in\mathbf{N}} \in \mathrm{L}^2$ for all $u_0 \in U$, i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}_{1/r}; \mathcal{B}(U,Y))$. Thus, we have the equivalence.

2° Now $M := \|\mathbb{D}P_0^*\| = \|\mathbb{D}\pi_{\{0\}}\| < \infty$, by Lemma A.3.6 (with, e.g., $X_3 := \ell^2_{\mathrm{loc}} = (\mathbf{Z} \to B)$). Therefore, $\|T_j\| \leq Mr^j$ for all $j \in \mathbf{N}$, by (13.21). The first inequality follows from this.

3° Given $u \in \ell^1_r(\mathbf{N}; U)$, we have

$$\|\mathbb{D}u_k e_k\|_{\ell^2_r} = \|\tau^{-k}\mathbb{D}u_k e_0\|_{\ell^2_r} \leq Mr^{-k}\|u_k\|_U, \tag{13.23}$$

by (13.13), hence $\|\mathbb{D}u\|_{\ell^2_r} \leq M\sum_k r^{-k}\|u_k\|_U = M\|u\|_{\ell^1_r} \leq M''\|u\|_{\ell^2_s}$, by (13.3). If $u \in \mathrm{c_c}$, then $\tau^{-n}u \in \ell^1_r(\mathbf{N}; U)$ for some $n \in \mathbf{N}$, hence then $\mathbb{D}u \in \ell^2_r$.

4° Since $\mathbb{D} \in \mathcal{B}(\ell^1_r(\mathbf{Z}; U), \ell^2_r(\mathbf{Z}; U))$, by 3°, we have $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z}; U), \ell^\infty_{1/r}(\mathbf{Z}; U))$, by Lemma B.4.15 (and Lemma A.3.24).

5° The last claim holds, because

$$\|\mathbb{D}u - \mathbb{D}^t u\|_{\ell^2_r} \leq \|\mathbb{D}u - \pi_{[0,t)}\mathbb{D}u\|_{\ell^2_r} + \|\pi_{[0,t)}(\mathbb{D}u - \mathbb{D}\pi_{[0,t)}u)\|_{\ell^2_r} \to 0, \tag{13.24}$$

as $t \to \infty$, for any $u \in \ell^2_s(\mathbf{N}; U)$, since $\mathbb{D} \in \mathcal{B}(\ell^2_s(\mathbf{N}; U), \ell^2_r)$. (For $u \in \mathrm{c_c}$, this is even easier.) □

We sometimes use the following lemma to derive discrete-time frequency-domain results from continuous-time ones:

Lemma 13.1.4 *Let $1 \leq p \leq \infty$. Define $\widehat{T}$ and $\widehat{T}^{-1}$ by*

$$(\widehat{T}\widehat{f})(\mathrm{e}^{-s}) := \widehat{f}(s) \qquad (s \in [0, +\infty) + i[-\pi, \pi)), \tag{13.25}$$

$$(\widehat{T}^{-1}\widehat{g})(s) := \widehat{g}(\mathrm{e}^{-s}) \qquad (s \in \overline{\mathbf{C}^+}). \tag{13.26}$$

Then $\widehat{T}$ maps $\mathrm{L}^p(i\mathbf{R}; B)$ onto $\mathrm{L}^p(\partial\mathbf{D}; B)$, $\mathrm{L}^p_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U,Y))$ onto $\mathrm{L}^p_{\mathrm{strong}}(\partial\mathbf{D}; \mathcal{B}(U,Y))$, and $\mathrm{H}^p(\mathbf{C}^+; B)$ onto $\mathrm{H}^p(\mathbf{D}; B)$ with norm ≤ 1, but none of these maps is one-to-one.

Moreover, $\widehat{T}\widehat{T}^{-1} = I$, and $\widehat{T}^{-1}$ maps $\mathrm{L}^\infty(\partial\mathbf{D}; \mathcal{B}(U,Y))$ into $\mathrm{L}^\infty(i\mathbf{R}; \mathcal{B}(U,Y))$ and $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}; \mathcal{B}(U,Y))$ into $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U,Y))$ and $\mathrm{H}^\infty(\mathbf{C}^+; B)$ into $\mathrm{H}^\infty(\mathbf{D}; B)$, *isometrically.*

Furthermore, $\widehat{T}(f\cdot g) = (\widehat{T}f)\cdot(\widehat{T}g)$ *and* $\widehat{T}^{-1}(f\cdot g) = (\widehat{T}^{-1}f)\cdot(\widehat{T}^{-1}g)$ *for all* f *and* g.

Finally, Lemmas 13.1.5 and 13.1.6 will show that $T \in \mathcal{B}(\mathrm{L}^2(J;U), \ell^2(N;U))$, $T \in \mathcal{B}(\mathrm{TI}(U,Y), \mathrm{ti}(U,Y))$ *and* $T \in \mathcal{B}(\mathrm{TIC}(U,Y), \mathrm{tic}(U,Y))$ *are onto, with norm* ≤ 1, *and* $T^{-1} \in \mathcal{B}(\ell^2(N;U), \mathrm{L}^2(J;U))$, $T^{-1} \in \mathcal{B}(\mathrm{ti}(U,Y), \mathrm{TI}(U,Y))$ *and* $T^{-1} \in \mathcal{B}(\mathrm{tic}(U,Y), \mathrm{TIC}(U,Y))$ *are isometries; where* Tu *is defined by* $\widehat{Tu} = \widehat{T}\widehat{u}$, *etc. and either* $J = \mathbf{R}_+$ & $N = \mathbf{N}$ *or* $J = \mathbf{R}$ & $N = \mathbf{Z}$. *Note that* $T(\mathbb{E}T^{-1}u) = (T\mathbb{E})u$ *etc.*

We also use two other methods to establish connections between discrete- and continuous-time maps; see Lemma 13.2.1, Theorem 13.2.3 and Section 13.4 for details.

Proof: Note that $\|\widehat{T}\widehat{f}\|_{\mathrm{L}^p(\partial\mathbf{D};B)} = \|\widehat{f}\|_{\mathrm{L}^p(i[-\pi,\pi);B)}$ and $\|\widehat{T}\widehat{f}\|_{\mathrm{L}^p(\partial\mathbf{D}_r;B)} = \|\widehat{f}\|_{\mathrm{L}^p(i[-\pi,\pi)-\log r;B)}$; the H^p claim follows from these, and the rest is obvious, because $s \mapsto \mathrm{e}^{-s}$ maps $[-\pi,\pi) \to \partial\mathbf{D}$ and $(0,+\infty)+i[-\pi,\pi) \to \mathbf{D}$, one-to-one and onto. $\square$

The ti maps have $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};*)$ transfer functions in the same way as the TI maps have $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};*)$ transfer functions:

Lemma 13.1.5 ($\widehat{\mathbf{ti}}_r = \mathbf{L}^\infty_{\mathbf{strong}}(r^{-1}\partial\mathbf{D},*)$) *Let* $\mathbb{E} \in \mathrm{ti}_r(U,Y)$. *Then there is a unique* transfer function $\widehat{\mathbb{E}}(z) \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ *s.t.* $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ *for all* $u \in \ell^2_r(\mathbf{Z};U)$.

Moreover, the mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ *is an isometric isomorphism of* ti_r *onto* $\mathrm{L}^\infty_{\mathrm{strong}}$ *and it commutes with adjoints and compositions; in particular, it is an isometric* B^**-algebra isomorphism when* $U = Y$ *and* $r = 1$.

See Section 3.1 or Section F.1 for $\mathrm{L}^\infty_{\mathrm{strong}}$.

Proof: We take $r = 1$ to simplify the notation. Let $\widehat{\mathbb{F}} \in \mathcal{B}(\ell^2(\partial\mathbf{D};U), \ell^2(\partial\mathbf{D};Y))$ be the operator defined by $\widehat{\mathbb{F}}\widehat{u} := \widehat{\mathbb{E}u}$.

From Theorem F.1.7(b) we obtain easily the lemma except for the fact that each $\mathrm{ti}(U,Y)$ map has a transfer function; hence we study this claim only.

This claim is known in the case of separable U and $Y = U$ (e.g., Theorem 1 of [FS]), hence in the case of separable U and an arbitrary Y (because there is a separable subspace $Y_0 \subset Y$ s.t. $\mathbb{E} \in \mathrm{ti}(U,Y_0)$, i.e., $\mathbb{E}f \in \mathrm{L}^2(\mathbf{Z},Y_0)$ for all $f \in \mathrm{L}^2(\mathbf{Z};U)$, by Lemmas A.3.1(a3) and B.3.16). We could prove the unseparable result by the methods of Theorem 3.1.3(a1), but we have chosen to combine the latter with the separable case to obtain a shorter proof.

By the results mentioned above, for each closed, separable subspace $V \subset U$ there is a transfer function $\widehat{\mathbb{E}}_V \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$, and for each $\mathbb{G} \in \mathrm{TI}(U,Y)$ a transfer function $\widehat{\mathbb{G}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

Let $\mathbb{E} \in \mathrm{ti}(U,Y)$. Then $\mathbb{G}_\mathbb{E} : f \mapsto \mathcal{L}^{-1}\widehat{T}^{-1}\mathcal{Z}\mathbb{E}\mathcal{Z}^{-1}\widehat{T}\mathcal{L}f$ is obviously in linear and $\|\mathbb{G}_\mathbb{E}\|_{\mathcal{B}(\mathrm{L}^2)} \le 1$.

To prove that $\mathbb{G}_\mathbb{E} \in \mathrm{TI}(U,Y)$, take arbitrary $f \in \mathrm{L}^2(\mathbf{R};U)$ and $t \in \mathbf{R}$. Choose a closed separable subspace $V \subset U$ s.t. $f \in \mathrm{L}^2(\mathbf{R};V)$. Let $F_V :=$

$\widehat{T}^{-1}\widehat{\mathbb{E}}_V \in \mathrm{L}^\infty_{\mathrm{strong}}(i[-\pi,\pi);\mathcal{B}(V,Y))$ (here we used the separable result), so that $\mathcal{L}(\mathbb{G}_{\mathbb{E}} f) = \widehat{T}^{-1}(\widehat{\mathbb{E}}_V \widehat{T}\widehat{f}) = F_V\widehat{f}$.

But $F_V \mathrm{e}^{-t\cdot}\widehat{f} = \mathrm{e}^{-t\cdot}F_V\widehat{f}$, hence $\mathbb{G}_{\mathbb{E}}\tau(t)f = \tau(t)\mathbb{G}_{\mathbb{E}}f$, for all $t \in \mathbf{R}$. Therefore, $\mathbb{G}_{\mathbb{E}} \in \mathrm{TI}(U,Y)$, hence $\widehat{\mathbb{G}}_{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. Equation $\widehat{\mathbb{G}}_{\mathbb{E}}\widehat{f} = \widehat{T}^{-1}(\widehat{\mathbb{E}}_V\widehat{T}\widehat{f})$ implies that

$$(\widehat{T}\widehat{\mathbb{G}}_{\mathbb{E}})(\widehat{T}\widehat{f}) = \widehat{\mathbb{E}}_V\widehat{T}\widehat{f} = \mathcal{Z}(\mathbb{E}\mathcal{Z}^{-1}\widehat{T}\widehat{f}) \quad \text{for all } f, V. \tag{13.27}$$

Consequently, $\widehat{\mathbb{E}} := \widehat{T}\widehat{\mathbb{G}}_{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$ is the transfer function of $\mathbb{E}$.

The "Moreover" claims are easy to prove, cf. the end of the proof of Theorem 3.1.3(a1). □

Lemma 13.1.6 ($\widehat{\mathbf{tic}_r} = \mathbf{H}^\infty_{1/r}$) *Let $\mathbb{D} \in \mathrm{tic}_r(U,Y)$. Then there is a unique $\widehat{\mathbb{D}}(z) \in \mathrm{H}^\infty(\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ s.t. $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ for all $u \in \ell^2_r(\mathbf{N};U)$, namely the one defined in Lemma 13.1.3(c1). Moreover, the mapping $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ is an isometric Banach algebra isomorphism of* tic_r *onto* $\mathrm{H}^\infty_{1/r}$.

In particular, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ *for* $0 < r < r'$.

Furthermore, $\widehat{\mathbb{D}}$ has the (nontangential) boundary function $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ in the sense of Theorem 3.3.1(c)&(e)&(f)

We call $D := \widehat{\mathbb{D}}(0) \in \mathcal{B}(U,Y)$ the *feedthrough operator* of $\mathbb{D}$.

Because of the last claim of the lemma, we may safely identify $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ and $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ (via an isometric Banach algebra isomorphism) and call both of them the transfer function of $\mathbb{D}$.

Analogously to the continuous case, we also identify $\mathbb{D} \in \mathrm{tic}_r$ and $\mathbb{D} \in \mathrm{tic}_{r'}$ if they (equivalently, their transfer functions) are extensions of a single tic (H^∞) map. Consequently, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ for $r < r'$.

Proof of Lemma 13.1.6: Obviously, $\widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ on $\mathbf{D}_{1/r}$ for all $\widehat{u} \in U$, hence for all $u \in \ell^2_r$, by time-invariance and continuity (alternatively, by [RR, Theorem 1.15B] and scaling). For $r = 1$, the isomorphism from Lemma D.1.15; the general case follows by scaling.

Thus, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ follows from $\mathrm{H}^\infty_{1/r} \subset \mathrm{H}^\infty_{1/r'}$.

By (c) and (e) of Theorem 3.3.1, $\widehat{\mathbb{D}}$ has a boundary function ($\mathrm{L}^\infty_{\mathrm{strong}}$ equivalence class, to be exact); that function is obviously equal to the one given in Lemma 13.1.5. □

Invertibility in tic_∞ is equivalent to invertibility of the feedthrough operator:

Lemma 13.1.7 ($\mathbb{X}^{-1}$) *Let $\mathbb{X} \in \mathrm{tic}_\infty := \cup_{r>0}\mathrm{tic}_r$. Then $\mathbb{X} \in \mathcal{G}\mathrm{tic}_\infty \Leftrightarrow X := \widehat{\mathbb{X}}(0) \in \mathcal{GB}$.* □

This follows from the fact $\widehat{\mathbb{X}}$ is (boundedly) invertible on a neighborhood of 0 iff $X \in \mathcal{GB}$ (see Lemma A.3.3(A2)). Obviously, $\widehat{\mathbb{X}}^{-1}(0) = X^{-1}$, because $(\widehat{\mathbb{X}}\widehat{\mathbb{Z}})(0) = XZ$ for any $\mathbb{X} \in \mathrm{tic}_\infty(U,Y)$, $\mathbb{Z} \in \mathrm{tic}_\infty(Y,Z)$.

Lemma 13.1.8 ($\mathbb{D}^{\mathrm{d}}$) *Let $u \in \ell^2(\mathbf{Z};U)$ and $\mathbb{E} \in \mathrm{ti}(U,Y)$. Then $\widehat{\boldsymbol{Я}u}(z) = \widehat{u}(\bar{z})$ for $z \in \partial\mathbf{D}$. Moreover, $\widehat{\mathbb{E}^*}(z) = \widehat{\mathbb{E}}(z)^*$, $\widehat{\boldsymbol{Я}\mathbb{E}\boldsymbol{Я}}(z) = \widehat{\mathbb{E}}(\bar{z})$ and $\widehat{\mathbb{E}^{\mathrm{d}}}(z) = \widehat{\mathbb{E}}(\bar{z})^*$ for $z \in \partial\mathbf{D}$.*

If $\mathbb{D} \in \mathrm{tic}_r(U,Y)$, $r > 0$, then $\widehat{\mathbb{D}^{\mathrm{d}}}(z) = \widehat{\mathbb{D}}(\bar{z})^*$ for $z \in \mathbf{D}_r$, i.e., $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \sum_{n=0}^{\infty} D_n^* z^n$, where $\widehat{\mathbb{D}}(s) = \sum_{n=0}^{\infty} D_n z^n$ is the Taylor series of $\widehat{\mathbb{D}}$ (with $D_n \in \mathcal{B}(U,Y)$ for $n \in \mathbf{N}$).
□

(The proof is almost identical to that of Lemma 3.3.8 (with replacements (13.63)) and hence omitted.) See Definition 3.1.1 for $\widehat{\mathbb{E}}^*$ on $\partial\mathbf{D}$ (and Theorem 3.1.3(d), which is applicable on $\partial\mathbf{D}$ too, by Theorem 13.2.3). Note that $\widehat{\mathbb{E}}^*$ need *not* be the pointwise adjoint of an arbitrary representative of $\widehat{\mathbb{E}}$, which might be unbounded and nonmeasurable, by Example 3.1.4. Note also that $\bar{z} = 1/z$ on $\partial\mathbf{D}$.

Notes

Much of the convolution and Z-transform theory at the beginning of this section is probably well known, and so is Lemma 13.1.6 except the boundary function claim (in the unseparable case). Also Lemma 13.1.2 and Lemma 13.1.3(a) are well known (see, e.g., [Mal00] or [Sbook]). Further results on ti_∞ maps are given in Theorem 13.3.13; see the corresponding continuous-time chapters for further notes.

13.2 The Cayley transform (♢, ♡)

And thus in anguish Beren paid
for that great doom upon him laid,
the deathless love of Lúthien,
too fair for love of mortal Men;
and in his doom was Lúthien snared,
the deathless in his dying shared;
and Fate them forged a binding chain
of living love and mortal pain.
— J.R.R. Tolkien (1892–1973): "The Lay of Leithian"

In this section, we present standard and further results on the Cayley transform of functions and particularly on that of (stable) tic operators.

We will often use composition with the *Cayley function* to map $\mathrm{H}^\infty(\mathbf{D};B)$ one-to-one onto $\mathrm{H}^\infty(\mathbf{C}^+;B)$, $\mathcal{C}(\partial\mathbf{D};B)$ one-to-one onto $\mathcal{C}(i\mathbf{R}\cup\{\infty\};B)$, or $\mathrm{L}^\infty_{\text{strong}}(\partial\mathbf{D};B)$ one-to-one onto $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};B)$:

Lemma 13.2.1 (Cayley function) *We define the* Cayley function *by*

$$\phi_{\text{Cayley}} : s \mapsto \frac{1-s}{1+s} \tag{13.28}$$

(a) We have $\phi_{\text{Cayley}}^{-1} = \phi_{\text{Cayley}}$ and $\phi_{\text{Cayley}}(s) = -\phi_{\text{Cayley}}(1/s)$ $(s \neq 0,-1)$.

(b) ϕ_{Cayley} maps $\mathbf{C}^+ \to \mathbf{D}$, $i\mathbf{R}\cup\{\infty\} \to \partial\mathbf{D}$, and $\overline{\mathbf{C}^+}\cup\{\infty\} \to \overline{\mathbf{D}}$ one-to-one and onto (and continuously in both directions, i.e., it is a homeomorphism) (but $\mathbf{C}^+_\omega \not\to \mathbf{D}_r$ for any $\omega \neq 0$ or $r \neq 1$). Moreover, the positive direction on $i\mathbf{R}$ $(+i\infty$ to $-i\infty)$ is mapped to the negative direction on $\partial\mathbf{D}$, and $f(\pm\infty) = -1 = f(\pm i\infty)$.

(c) If $it = \phi_{\text{Cayley}}(\mathrm{e}^{i\theta})$ (i.e., $\mathrm{e}^{i\theta} = \phi_{\text{Cayley}}(it)$), then $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -2(1+t^2)^{-1}$.

(d) We have $\int_0^{2\pi}(f\circ\phi_{\text{Cayley}}^{-1})(\mathrm{e}^{i\cdot})\,dm = \int_{\mathbf{R}} 2(1+t^2)^{-1}f(it)\,dt$ for measurable $f : i\mathbf{R} \to [0,+\infty]$ and for $f \in \mathrm{L}^1(i\mathbf{R};B)$, where B is a Banach space.

(e1) $f \mapsto f\circ\phi_{\text{Cayley}}$ maps $\mathrm{H}(\mathbf{D};B)$ one-to-one onto $\mathrm{H}(\mathbf{C}^+;B)$, $\mathrm{H}^\infty(\mathbf{D};B)$ one-to-one onto $\mathrm{H}^\infty(\mathbf{C}^+;B)$, $\mathcal{C}(\partial\mathbf{D};B)$ one-to-one onto $\mathcal{C}(i\mathbf{R}\cup\{\infty\};B)$, $\mathrm{L}^p(i\mathbf{R};B)$ one-to-one into $\mathrm{L}^p(\partial\mathbf{D};B)$, and $X(\partial\mathbf{D};B)$ one-to-one onto $X(i\mathbf{R};B)$, where $X = \mathrm{L}^\infty$, $X = \mathrm{L}^\infty_{\text{strong}}$, $X = \mathrm{L}^\infty_{\text{weak}}$ or $X = \mathrm{L}^p_{\text{loc}}$ $(1 \le p \le \infty)$.

Moreover, this map preserves the $\|\cdot\|_\infty$ norm on the boundary, the supremum norm, and nontangential angles (except at $-1 \in \partial\mathbf{D}$).

(e2) Let $f \in \mathrm{H}(\mathbf{C}^+;B)$ and $g \in \mathrm{H}(\mathbf{D};B)$. Then $f \in \mathrm{H}^p(\mathbf{C}^+;B)$ iff $(z \mapsto (1+z)^{2/p}f(\frac{1-z}{1+z})) \in \mathrm{H}^p(\mathbf{D};B)$. Analogously, $g \in \mathrm{H}^p(\mathbf{D};B)$ iff $(s \mapsto (1+s)^{-2/p}g(\frac{1-s}{1+s})) \in \mathrm{H}^p(\mathbf{C}^+;B)$.

(f) If $f \in \mathrm{L}^1_{\text{loc}}(i\mathbf{R};B)$, then $ir \in \mathrm{Leb}(f)$ iff $\phi_{\text{Cayley}}^{-1}(ir) \in \mathrm{Leb}(f\circ\phi_{\text{Cayley}})$.

(There are several different Cayley functions in the literature, but they only differ by some additional constants in the above formulae. The advantage of our function is that it is the inverse of itself. See also Lemma 13.2.6.)

Proof: (a)–(c) These are obvious.

(d) (Note that if either side converges absolutely, then so does the other (replace f by $\|f\|_B$).) This follows from (c) and Lemma B.4.10.

(e1) 1° Because ϕ_{Cayley} and $\phi_{\text{Cayley}}^{-1}$ are holomorphic, they preserve continuous and holomorphic functions, by Lemma D.1.2(b4). Trivially, the supremum norm is also preserved ($\partial\mathbf{D} \to i\mathbf{R} \cup \{\infty\}$ or $\mathbf{D} \to \mathbf{C}_+$), so $\mathrm{H}, \mathrm{H}^\infty, \mathcal{C}$ are already covered (note that $\partial\mathbf{D}$ and $i\mathbf{R} \cup \{\infty\}$ are compact).

2° By Lemma B.4.10 (applied to $\phi_{\text{Cayley}}(-\cdot)$; cf. (c)), also the $\|\cdot\|_\infty$-norm is preserved and cases $X = \mathrm{L}^\infty$ and $X = \mathrm{L}^p_{\text{loc}}$ are covered. If $B = \mathcal{B}(U,Y)$, then the cases $X = \mathrm{L}^\infty_{\text{strong}}$, $X = \mathrm{L}^\infty_{\text{weak}}$ follow from the case $X = \mathrm{L}^\infty$.

3° L^p*:* This follows from the theorem on p. 130 of [Hoffman] (whose proof applies also to vector-valued functions).

4° *Nontangential angles:* (See p. 967 for nontangential limits.) Because ϕ_{Cayley} is conformal $\mathbf{C} \setminus \{-1\} \mapsto \mathbf{C} \setminus \{-1\}$, the images of (small) nontangential cones are contained in nontangential cones, in both directions.

(e2) The (scalar case, with a different Cayley transform) proof the Theorem on p. 130 of [Hoffman] applies mutatis mutandis.

(f) This follows from Lemma B.5.5. (Here we have identified $\partial\mathbf{D}$ with $[-\pi,\pi)$ (via $e^{it} \mapsto t$); identification with $[0,2\pi)$ would affect the point $\phi_{\text{Cayley}}^{-1}(i0) = 1$, unless we would use the periodic extension of $f \circ \phi_{\text{Cayley}}^{-1} \circ e^{i\cdot}$ on $\mathbf{R}$.) □

The references of Theorem 5.1.6 will use the following fact (as the definition of $\widehat{\pi^+}$): The operator $\widehat{\pi^+}$ has the standard singular integral presentation

$$(\widehat{\pi^+}\widehat{f})(z) = \frac{\widehat{f}(z)}{2} + \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\widehat{f}(s)}{s-z}\,ds, \tag{13.29}$$

for $\widehat{f} \in \mathrm{L}^2(\partial\mathbf{D};H)$; this follows by applying scalar case (from, e.g., [Garnett]) to Λf for each $\Lambda \in H^*$ (because $\Lambda\widehat{\pi^+} = \widehat{\pi^+}\Lambda$). One gets the corresponding presentation for $\widehat{\pi^-}$ analogously.

Next we shall construct an isomorphism $\heartsuit : \mathrm{TI} \to \mathrm{ti}$ that can be used to transform results from continuous time to discrete time and vice versa.

Definition 13.2.2 (TI ↔ ti) *We define the* (signal) Cayley transform $\diamondsuit$: $\mathrm{L}^2(\mathbf{R};U) \to \ell^2(\mathbf{Z};U)$ *by* $\widehat{\diamondsuit}\widehat{f} := \gamma\cdot(\widehat{f}\circ\phi_{\text{Cayley}}^{-1})$*, i.e.,*

$$(\widehat{\diamondsuit}\widehat{f})(z) := \gamma(z)(\widehat{f}\circ\phi_{\text{Cayley}}^{-1})(z) \quad (f \in \mathrm{L}^2(\mathbf{R};U)), \tag{13.30}$$

where $\gamma(z) = \sqrt{2}/(1+z)$*.*

The (map) Cayley transform $\heartsuit$ *is defined by* $\heartsuit\mathbb{E} := \diamondsuit_Y\mathbb{E}\diamondsuit_U^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ *for* $\mathbb{E} : \mathrm{L}^2(\mathbf{R};U) \to \mathrm{L}^2(\mathbf{R};Y)$*.*

(See the proof of Theorem 13.2.3(a) for details.)

Next we show that the map $\diamondsuit$ is unitary ((a)), that $\widehat{\heartsuit} = \cdot \circ \phi_{\mathrm{Cayley}}^{-1}$ ((b3)), and that $\widehat{\mathrm{ti}} \circ \phi_{\mathrm{Cayley}} = \widehat{\mathrm{TI}}$ ((b2)):

Theorem 13.2.3 ($\heartsuit$: TI $\leftrightarrow$ ti)

(a) The map $\diamondsuit$ is an isometric isomorphism of $\mathrm{L}^2(\mathbf{R};U)$ *onto* $\ell^2(\mathbf{Z};U)$ *and of* $\mathrm{L}^2(\mathbf{R}_+;U)$ *onto* $\ell^2(\mathbf{N};U)$).

Indeed, for $u : \mathbf{Z} \to U$ we have

$$\|u\|_2^2 := \sum_n \|u_n\|_U^2 = (2\pi)^{-1} \int_0^{2\pi} \|\widehat{u}(e^{i\theta})\|_U^2 \, d\theta \tag{13.31}$$

$$= (2\pi)^{-1} \int_{-\infty}^{+\infty} \|(\widehat{\diamondsuit}^{-1}\widehat{u})(it)\|_U^2 \, dt = \int_{-\infty}^{+\infty} \|(\diamondsuit^{-1}u)(t)\|_U^2 \, dt \tag{13.32}$$

Moreover, for $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ the formula (13.30) holds on $\mathbf{D}$ too.

(b1) The map $\heartsuit$ is an isometric isomorphism of $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ onto $\mathcal{B}(\ell^2(\mathbf{Z};U),\ell^2(\mathbf{Z};Y))$. Moreover, $\heartsuit$ commutes with adjoints and valid compositions of operators. Thus,

$$\heartsuit(\mathbb{E}\mathbb{F}) = (\heartsuit\mathbb{E})(\heartsuit\mathbb{F}), \quad \heartsuit\mathbb{E}^{-1} = (\heartsuit\mathbb{E})^{-1}, \quad \heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*, \tag{13.33}$$

for $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ and $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};Y),\mathrm{L}^2(\mathbf{R};H))$.

(b2) The map $\heartsuit$ is an isometric isomorphism of TI *onto* ti *and of* TIC *onto* tic.

(b3) Let $\mathbb{E} \in$ TI, $\mathbb{F} := \heartsuit\mathbb{E}$. Then $\widehat{\mathbb{F}} = \widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}}^{-1}$ in $\mathrm{L}^\infty_{\mathrm{strong}}$, i.e., on $\partial\mathbf{D}$ (and in H^∞, i.e., on $\mathbf{D}$, if $\mathbb{E} \in$ TIC).

(c1) $\diamondsuit\pi_\pm = \pi^\pm\diamondsuit$, $\heartsuit\pi_\pm = \pi^\pm\heartsuit$, $\diamondsuit\boldsymbol{Я} = \boldsymbol{Я}_{-1}\diamondsuit = \tau\boldsymbol{Я}\diamondsuit$, $\heartsuit\boldsymbol{Я} = \boldsymbol{Я}_{-1} = \tau\boldsymbol{Я}$.

(c2) $\widehat{\tau^k\diamondsuit} = \widehat{\diamondsuit}(\phi_{\mathrm{Cayley}})^k\cdot$, $\widehat{\tau^k} = \widehat{\heartsuit}(\phi_{\mathrm{Cayley}})^k\cdot$, $\widehat{\diamondsuit\tau(t)} = e^{t\frac{1-z}{1+z}} \cdot \widehat{\diamondsuit}$, $\widehat{\heartsuit\tau(t)} = e^{t\frac{1-z}{1+z}}\cdot$.

(c3) $\heartsuit L = L$ for $L \in \mathcal{B}(U,Y)$.

(c4) $\heartsuit\mathbb{E}^{\mathrm{d}} = (\heartsuit\mathbb{E})^{\mathrm{d}}$ for $\mathbb{E} \in$ TI.

(d) Let $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$. Then $\pi_\pm\mathbb{E}\pi_\pm$ is invertible on $\pi_\pm\mathrm{L}^2$ iff $\pi^\pm(\heartsuit\mathbb{E})\pi^\pm$ is invertible on $\pi^\pm\ell^2$.

(e) Let $\mathbb{E}, P \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$. Then $\heartsuit\mathbb{E} \geq 0$ [$\gg 0$] on $(\heartsuit P)\ell^2$ iff $\mathbb{E} \geq 0$ [$\gg 0$] on $P\mathrm{L}^2$.

Note that these results do not hold in the unstable case (see Lemma 13.2.1(b)). However, Section 13.4 treats another way to relate TI and ti, *discretization*, that handles also the unstable case.

Proof: (a) By Lemma 13.2.1(d), we have

$$\int_0^{2\pi} \|(\widehat{\diamondsuit}\widehat{u})(e^{i\theta})\|^2 \, d\theta = \int_{\mathbf{R}} 2^{-1}|1+it|^2 \|\widehat{u}(it)\|^2 \frac{d\theta}{dt} \, dt = \int_{\mathbf{R}} \|\widehat{u}(it)\|^2 \, dt \tag{13.34}$$

for measurable $f : i\mathbf{R} \to U$.

This proves the first "=" sign in (13.31); the two following "=" signs are from Lemma D.1.15.

It follows that $\diamondsuit$ is an isometric isomorphism of L^2 onto ℓ^2. The fact that $\pi^+\diamondsuit = \diamondsuit\pi_+$ follows from the scalar case (given on, e.g., pp. 104–106 of [Hoffman]). (Indeed, our ϕ_{Cayley} uses an extra $z \mapsto -z$ on $\partial\mathbf{D}$ compared to that of [Hoffman], hence this transforms the ascending order (positive orientation) of $\partial\mathbf{D}$ to the ascending order on $-\infty\ldots+\infty$), because, obviously, $\Lambda\diamondsuit_U = \diamondsuit_{\mathbf{C}}\Lambda$ for $\Lambda \in U^*$. Note that if $u \in \mathrm{L}^2(\mathbf{R};U)$, then $u \in \mathrm{L}^2(\mathbf{R}_+;U) \Leftrightarrow \Lambda u \in \mathrm{L}^2(\mathbf{R}_+;\mathbf{C})$ for all $\Lambda \in U^*$, and that an analogous claim holds for ℓ^2.)

(b1) This follows from (a) except for $\heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*$, which is obtained as follows:

$$\langle\mathbb{E}^* u, v\rangle = \langle u, \mathbb{E}v\rangle = \langle\diamondsuit u, \heartsuit\mathbb{E}\diamondsuit v\rangle = \langle\diamondsuit^{-1}(\heartsuit\mathbb{E})^*\diamondsuit u, v\rangle. \tag{13.35}$$

(b2) This follows from (b1) and (b3).

(b3) Let $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then

$$\widehat{\diamondsuit}\widehat{\mathbb{E}}\widehat{u} := \gamma((\widehat{\mathbb{E}}\widehat{u})\circ\phi_{\mathrm{Cayley}}^{-1}) = (\widehat{\mathbb{E}}\circ\phi_{\mathrm{Cayley}}^{-1})\gamma(\widehat{u}\circ\phi_{\mathrm{Cayley}}^{-1}) = (\widehat{\mathbb{E}}\circ\phi_{\mathrm{Cayley}}^{-1})\widehat{\diamondsuit}\widehat{u} \tag{13.36}$$

for all $u \in \mathrm{L}^2(\mathbf{R};U)$. Therefore, $\widehat{\diamondsuit}\widehat{\mathbb{E}}\widehat{\diamondsuit}^{-1} = \widehat{\mathbb{E}}\circ\phi_{\mathrm{Cayley}}^{-1}$ (cf. also Lemma 13.1.6).

(c1) The first identity on $\pi_\pm$ was proved in the proof of (a1); the second identity follows from the first.. By Lemma 13.1.8,

$$\widehat{\boldsymbol{Я}\diamondsuit u}(z) = \widehat{\diamondsuit u}(1/z) = \gamma(1/z)\widehat{u}(\frac{1-1/z}{1+1/z}) = z\gamma(z)\widehat{u}(-\frac{1-z}{1+z}) = z\widehat{\diamondsuit\boldsymbol{Я}u}(z) \tag{13.37}$$

for $z \in \partial\mathbf{D}$. Because $\widehat{\tau} = z$ and $\boldsymbol{Я}_{-1} = \tau\boldsymbol{Я}$, we obtain the third identity; the fourth identity follows from the third.

(c2) Because $(\phi_{\mathrm{Cayley}})^k\circ\phi_{\mathrm{Cayley}} = z^k = \widehat{\tau^k}$, we have $\widehat{\diamondsuit}((\phi_{\mathrm{Cayley}})^k\widehat{u}) = z^k\widehat{\diamondsuit}\widehat{u}$, i.e., the first (and hence the second) identity holds. Moreover, $\widehat{\tau(t)u} = e^{st}\widehat{u}$, hence $\widehat{\diamondsuit\tau(t)} = e^{t\phi_{\mathrm{Cayley}}(z)}\widehat{\diamondsuit}$.

(c3) This is obvious.

(c4) This follows from (c1) and the formula $\heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*$ from (b1).

(d) By (b), $\mathbb{G}\pi_+\mathbb{E}\pi_+ = \pi_+ = \pi_+\mathbb{E}\pi_+\mathbb{G}$ for some $\mathbb{G} \in \mathcal{B}(\pi_\pm\mathrm{L}^2)$ (we may identify $\mathbb{G}$ with $\pi_+\mathbb{G}\pi_+ \in \mathcal{B}(\mathrm{L}^2)$) iff $(\heartsuit\mathbb{G})\pi^+(\heartsuit\mathbb{E})\pi^+ = \pi^+ = \pi^+(\heartsuit\mathbb{E})\pi^+(\heartsuit\mathbb{G})$.

(e) Now $\langle\heartsuit\mathbb{E}\heartsuit P\diamondsuit f, \heartsuit P\diamondsuit f\rangle \geq 0$ for all $\diamondsuit f \in \ell^2$ iff $\langle\mathbb{E}Pf, Pf\rangle \geq 0$ for all $f \in \mathrm{L}^2$. By replacing $\mathbb{E}$ by $\mathbb{E}-\varepsilon I$ we get the "$\gg 0$" claim. □

We remark that losslessness (Definition 2.5.1) is $\heartsuit$-invariant:

Corollary 13.2.4 *Let $J = J^* \in \mathcal{B}(Y)$ and $S = S^* \in \mathcal{B}(U)$. Let $\mathbb{E} \in \mathrm{TIC}$, $\mathbb{F} := \heartsuit\mathbb{E} \in$* tic.

Then $\mathbb{F}$ is (J,S)-lossless *iff $\mathbb{E}$ is (J,S)*-lossless*.*

Proof: This follows from (13.33) and Theorem 13.2.3(e). □

It has been shown in Chapter 11 of [Sbook] that for every $\Sigma = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_1$ with $\mathbb{A}$ contractive and $\mathbb{A}+I$ one-to-one, there is $\Sigma' = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]' \in \mathrm{WPLS}_0$ s.t. $\mathbb{D}' = \heartsuit^{-1}\mathbb{D}$ (and conversely).

We now part extend Definition 13.2.2 for unstable $\mathbb{D}$ for later use (also much more is true):

Proposition 13.2.5 ($\heartsuit$: unstable $\mathbb{D}$) *If $\Omega \subset \mathbf{C}^+$ is open and $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega; \mathcal{B}(U,Y))$, then we set $\mathbb{D}u := \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$, for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ s.t. $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}^+;U)$. Moreover, we set $\heartsuit\mathbb{D} := \diamondsuit_Y \mathbb{D} \diamondsuit_U^{-1}$.*

Analogously, if $\Omega' \subset \mathbf{D}$ is open and $\widehat{\mathbb{F}} \in \mathrm{H}(\Omega'; \mathcal{B}(U,Y))$, then we set $\mathbb{F}u := \mathcal{Z}^{-1}\widehat{\mathbb{F}}\widehat{u}$ for all $u \in \ell^2(\mathbf{N};U)$ s.t. $\widehat{\mathbb{F}}\widehat{u} \in \mathrm{H}^2(\mathbf{D};U)$.

The following holds:

(b1) If $\mathbb{D}$ is as above, then $\widehat{\heartsuit\mathbb{D}} = \widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1}$.

(b2) $\heartsuit(\mathbb{D}\widetilde{\mathbb{D}}) = (\heartsuit\mathbb{D})(\heartsuit\widetilde{\mathbb{D}})$ for $\widetilde{\mathbb{D}} \in \mathrm{H}(\Omega; \mathcal{B}(H,U))$.

(b3) **($\widehat{\mathbb{D}}[\mathrm{H}^2] \subset \mathrm{H}^2 \Rightarrow \widehat{\mathbb{D}} \in \mathrm{H}^\infty$)** *If $\widehat{\mathbb{D}}$ is defined on whole $\mathrm{L}^2(\mathbf{R}_+;U)$, then $\widehat{\mathbb{D}} \in \mathrm{TIC}(U,Y)$ and $\heartsuit\mathbb{D}$ coincides with that of Definition 13.2.2.*

Proof: (Note that $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2$ means that $\widehat{\mathbb{D}}\widehat{u}$ has a (unique, by Lemma D.1.2(e)) extension to $\mathbf{C}^+$, and that this extension is in H^2.)

(b1) If $u, \mathbb{D}u \in \mathrm{L}^2$, then $\widehat{(\heartsuit\mathbb{D})\diamondsuit u} = (\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1})\widehat{\diamondsuit u}$, by (13.36). Conversely, if $(\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1})\widehat{\diamondsuit u} \in \mathrm{H}^2$ for some $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, then $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2$, by (13.36) and Theorem 13.2.3(a).

(b2) This is obvious from the definition.

(b3) Fix and open $\Omega' \subset \Omega$ s.t. $\emptyset \neq \overline{\Omega'} \subset \Omega$, so that $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U,Y))$. Now $\widehat{\mathbb{D}} \in \mathcal{B}(\mathrm{H}^2, \mathrm{H}^\infty(\Omega';Y))$, $\mathrm{H}^2 \underset{c}{\subset} \mathrm{H}^\infty(\Omega';Y)$ (by Lemma F.3.2(a)&(b)) and $\widehat{\mathbb{D}}[\mathrm{H}^2] \subset \mathrm{H}^2$, hence $\widehat{\mathbb{D}} \in \mathcal{B}(\mathrm{H}^2,\mathrm{H}^2)$, i.e., $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U), \mathrm{L}^2(\mathbf{R}_+;Y))$.

Since $\widehat{\mathbb{D}}e^{\cdot t} = e^{\cdot t}\widehat{\mathbb{D}}$ for all $t \in \mathbf{R}$, $\mathbb{D}$ commutes with translations, i.e., $\mathbb{D} \in \mathrm{TIC}(U,Y)$.

Obviously, the map $\heartsuit\mathbb{D}$ of this definition is equal to the restriction to $\ell^2(\mathbf{N};U)$ of the map $\heartsuit\mathbb{D}$ of Definition 13.2.2. $\square$

Sometimes we wish to map ∞ to some other point of $\partial\mathbf{D}$ than to -1. Then we can combine the Cayley transform with a rotation:

Lemma 13.2.6 (Different Cayley) *Proposition 13.2.5 and Theorem 13.2.3 except possibly the claims on $\mathbf{Я}$, τ and $()^{\mathrm{d}}$ (in (c1), (c2) and (c4)) hold even if we replace ϕ_{Cayley} by $\phi_{\mathrm{Cayley}} \circ \widehat{R}_\alpha$ for some $\alpha \in \partial\mathbf{D}$, where $(\widehat{R}\widehat{u})(z) := \widehat{u}(\alpha z)$ (equivalently, $Ru_k := \alpha^k u_k$ ($k \in \mathbf{Z}$)) for all z and u.*

Proof: Obviously, $\widehat{Ru}(z) = \sum_k z^k \alpha^k u_k = (\widehat{R}\widehat{u})(z)$ for all z and u, R is an isometric isomorphism on ℓ^2, $R\pi^+ = \pi^+ R$ etc. Part of the proposition and of the theorem follows directly from this and the rest (with the above exceptions) can easily be verified. $\square$

Notes

Theorem 13.2.3(a) is essentially given in [RR], where one can find further information on this transform (alternatively, see Section 11.4 of [Sbook]).

13.3 Discrete-time systems ($wpls(U,H,Y)$)

Do you think when two representatives holding diametrically opposing views get together and shake hands, the contradictions between our systems will simply melt away? What kind of a daydream is that?

— Nikita Khrushchev (1894–1971)

In this section we present wpls's, the discrete counterparts of WPLSs. We will present the main definitions and results of the continuous-time part of this monograph converted to the discrete-time form; this section covers mainly the theory of Chapters 2–7. Thus, generators, $\mathcal{Z}$-transforms of maps, stability and feedback. Less straight-forward results (on stabilizability) are presented at the end of this section, and most of main results are contained in Theorem 13.3.13, which covers the discrete counterparts of almost all continuous-time results in this monographs (see the other chapters for details).

We start with the definition:

Definition 13.3.1 (wpls) *Let* $r > 0$. *An* r-stable discrete-time well-posed linear system (r-stable wpls) *on* (U,H,Y) *is a quadruple* $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *of operators for which*

(1.) $A \in \mathcal{B}(H)$, *and* $\sup_{k\in\mathbf{N}} \|r^{-k}A^k\| < \infty$;

(2.) $\mathbb{B} \in \mathcal{B}(\ell^2_r(\mathbf{Z}_-;U),H)$ *satisfies* $\mathbb{B}\tau\pi^- = A\mathbb{B}$;

(3.) $\mathbb{C} \in \mathcal{B}(H,\ell^2_r(\mathbf{N};Y))$ *satisfies* $\mathbb{C}A = \pi^+\tau\mathbb{C}$;

(4.) $\mathbb{D} \in \mathrm{tic}_r(U,Y)$ *and* $\pi^+\mathbb{D}\pi^- = \mathbb{C}\mathbb{B}$;

we write $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_r(U,H,Y)$ *to express this, and we set* $\mathrm{wpls} := \cup_{r>0}\mathrm{wpls}_r$.

If $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}$, *and (3.) and (4.) hold for* $r = 1$, *then* $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *is a* stable-output system *(*$\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{sos}$*).*

If [$\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_r$ *and]* $r^{-j}A^jx \to 0$ *strongly (resp. weakly) as* $j \to \infty$ *for all* $x \in H$, *then [*$\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *and] A is* strongly (resp. weakly) r-stable *and* $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *is* strongly (resp. weakly) internally r-stable.

We call A the state map, $\mathbb{B}$ *the* reachability map, $\mathbb{C}$ *the* observability map *and* $\mathbb{D}$ *the* I/O map *of* $\left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$*; the map A (resp.* $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$*) is* r-stable *if (1.) (resp. (2.), (3.), (4.)) holds.*

The prefix "1-" is often omitted, e.g., systems in wpls_1 *are called* stable. *Systems in* wpls_r *for some* $r < 1$ *are called* exponentially stable*; similarly, if (1.) holds for some* $r < 1$, *we call A* exponentially stable.

Exponentially stable operators are often called power stable, but we wish to have our terminology compatible with the continuous-time notation; hence we sometimes also write $\mathbb{A}(t) := \mathbb{A}^t := A^t$.

We shall often use the basic identities $\mathbb{B}\tau^k\pi^- = A^k\mathbb{B}$, $\mathbb{C}A^k = \pi^+\tau^k\mathbb{C}$.

Lemma 13.3.2 ($\mathbf{wpls}_r \subset \mathbf{wpls}_{r'}$) *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_r$ *for some* $r > 0$. *Then* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_{r'}$ *for all* $r' > r$. □

(The proof is analogous to that of Lemma 6.1.2 and omitted.) In fact, if A is r-stable, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ is r' stable for all $r' > r$, by Lemma 13.3.8.

One usually defines discrete systems by (13.40) below. This is not a problem, because wpls's correspond 1-1 to the solutions of (13.40):

Lemma 13.3.3 (Generators of a wpls)

(a) For each $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}$, *there is a unique quadruple of operators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$, *called the* generators *of* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$, *s.t. for* $x \in H$ *and* $u \in \mathrm{c_c}(\mathbf{N}; U)$ *we have*

$$\begin{aligned} \mathbb{B}u &= \sum_{j=0}^{\infty} A^j B u_{-j-1} = \sum_{k=-\infty}^{-1} A^{-k-1} B u_k, \\ (\mathbb{C}x)_k &= CA^k x \quad (k \in \mathbf{N}), \\ (\mathbb{D}u)_k &= \sum_{j=0}^{\infty} CA^j B u_{k-j-1} + Du_k = \sum_{j=-\infty}^{k-1} (\tau^{-j-1} \mathbb{C} B u_j)(k) + Du_k \quad (k \in \mathbf{Z}). \end{aligned} \tag{13.38}$$

Moreover, (13.38) hold for any $u \in \ell^2_{\mathrm{loc}}([n, +\infty); U) + \ell^2_r(\mathbf{Z}; U)$, $n \in \mathbf{Z}$ *and* r *is s.t.* $\Sigma \in \mathrm{wpls}_r$. *We also have (here* $e_k := \chi_k$ $(k \in \mathbf{Z})$*)*

$$Bu = \mathbb{B}(ue_{-1})), \;\; Cx = (\mathbb{C}x)_0, \;\; Du = (\mathbb{D}(ue_0))_0 = \widehat{\mathbb{D}}(0) \text{ for } u \in U, \; x \in H. \tag{13.39}$$

Moreover, the unique solution (on $\mathbf{N}$*) of the difference equation pair*

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \quad\; y_j = Cx_j + Du_j, \end{cases} \tag{13.40}$$

with initial value $x_0 \in H$ *and input* $u \in \mathrm{c_c}(\mathbf{N}; U)$ *is given by*

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} A^j & \mathbb{B}\tau^j \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1, 2, \ldots) \tag{13.41}$$

(formula (13.41) determines $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *uniquely on* $H \times \mathrm{c_c}(\mathbf{N}; U)$, *hence as a wpls).*

(b) Conversely, for each $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$, *the operators defined by (13.38) are the unique solution of (13.40) (and (13.41)), and they extend to a (unique) wpls. The resulting wpls is r-stable (and* $\mathbb{D} \in \ell^1_r$*) for any* $r > \rho(A)$ *(and for no* $r < \rho(A)$*). We call this wpls the* wpls generated by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, *and we write* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$.

(c) Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_r(U, H, Y)$. *Then (13.41) is the solution of (13.40) for any* $x_0 \in H$ *and* $u \in \mathbf{N} \to U$ *(the* initial value setting*). Similarly,*

$x_j = \mathbb{B}\tau^j u$, $y = \mathbb{D}u$ is a solution of (13.40) for any $u \in \ell^2_r(\mathbf{Z};U)$ (the time-invariant setting *satisfying $x_j \to 0$ as $j \to -\infty$. Moreover, we have*

$$\mathbb{B}^t u := \mathbb{B}\tau^t \pi^+ u = \sum_{k=0}^{t-1} A^k B u_{t-1-k} = \sum_{k=0}^{t-1} A^{t-1-k} B u_k \quad (u : Z \to U), \tag{13.42}$$

$$\mathbb{D} = D + C\mathbb{B}\tau \tag{13.43}$$

$$(\mathbb{D}^* u)_k = \sum_{j=0}^{\infty} B^*(A^*)^j A^* u_{n+j+1} + D^* u_k \tag{13.44}$$

$$(u \in \ell^2_{-r}(\mathbf{Z};U) \text{ or } u : (-\infty,n) \to U,\ n \in \mathbf{Z}). \tag{13.45}$$

Note that (b) shows that the whole wpls is exponentially stable iff A is (cf. Lemma 6.1.10); equivalently, iff $\rho(A) < 1$, i.e., iff $\sigma(A) \subset \mathbf{D}$ (see Lemma 13.3.7).

As above, we will denote the generators of operators and feedthrough operators of tic$_\infty$ operators by corresponding (ordinary) letters.

Proof: (a)&(b) Except for the claims proved below, the stable case of this follows from Section 4 of [S99], see [Mal00] for proofs, and the general case follows by scaling (see Remark 13.3.9); also the reader can easily verify the results.

The equation $\widehat{\mathbb{D}}(0) = D$ follows easily from (13.38).

Equations (13.41) define a wpls uniquely, because $\mathrm{c_c}$ is dense in ℓ^2.

The condition $r > \rho(A)$ implies that $\|r^{-k}A^k\| \le 1$ for big k, hence $\|r^{-k}A^k\|$ is then bounded (similarly, it is unbounded for $r < \rho(A)$). Replacing r by $r' \in (\rho(A), r)$ above, we see from (13.38) that $\mathbb{B}, \mathbb{C}, \mathbb{D}$ are r-stable and $\mathbb{D} \in \ell^1_r$.

(c) Obviously, (13.41) solves (13.40) in both cases. On the other hand, $\mathbb{B}\tau^j u = \mathbb{B}\pi^-\tau^j u$, and $\pi^-\tau^j u \to 0$ as $j \to -\infty$. The formulae for $\mathbb{B}^t$, $\mathbb{D}$ and $\mathbb{D}^*$ are straightforward. □

Any tic$_\infty$ map has a realization:

Definition 13.3.4 (Realization) *Let $\mathbb{D} \in \mathrm{tic}_r(U,Y)$. If $\left[\begin{smallmatrix} A & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ for some Hilbert space H, then we call $\left[\begin{smallmatrix} A & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ (together with H) a* realization *of $\mathbb{D}$.*

We call the (strongly r-stable) system

$$\left[\begin{array}{c|c} \pi^+\tau & \pi^+\mathbb{D}\pi^- \\ \hline I & \mathbb{D} \end{array}\right] = \left(\begin{array}{c|c} \pi^+\tau^1 & \pi^+(\mathbb{D}\cdot e_{-1}) \\ \hline \pi_{\{0\}} & D \end{array}\right) \in \mathrm{wpls}_r(U, \ell^2_r(\mathbf{N};Y), Y) \tag{13.46}$$

the exactly observable realization *of $\mathbb{D}$.*

We now state the discrete version of dual systems. This requires (13.12), hence we have to use $\boldsymbol{\mathcal{A}}_{-1}$ instead of $\boldsymbol{\mathcal{A}}$ (recall that $(\boldsymbol{\mathcal{A}}_{-1}x)_k := x_{-1-k}$). Fortunately, $\boldsymbol{\mathcal{A}}_{-1}\mathbb{D}^*\boldsymbol{\mathcal{A}}_{-1} = \boldsymbol{\mathcal{A}}\tau^{-1}\mathbb{D}^*\tau\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}\mathbb{D}^*\boldsymbol{\mathcal{A}} =: \mathbb{D}^{\mathrm{d}}$; but for the duals of $\mathbb{B}$ and $\mathbb{C}$ the difference between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}_{-1}$ is meaningful:

Proposition 13.3.5 (Dual system) *Let* $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{wpls}_r$*,* $r > 0$*. Then its* (causal) dual system *(or (causal) adjoint system)*

$$\left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]^{\mathrm{d}} := \left[\begin{array}{c|c} (A^{\mathrm{d}})^{\cdot} & \mathbb{C}^{\mathrm{d}} \\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}}\end{array}\right] := \left[\begin{array}{c|c} (A^*)^{\cdot} & \mathbb{C}^* \boldsymbol{\text{Я}}_{-1} \\ \hline \boldsymbol{\text{Я}}_{-1}\mathbb{B}^* & \boldsymbol{\text{Я}}_{-1}\mathbb{D}^* \boldsymbol{\text{Я}}_{-1}\end{array}\right] \tag{13.47}$$

is also in wpls_r*. Moreover,* $\left(\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]^{\mathrm{d}}\right)^{\mathrm{d}} = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]^{\mathrm{d}} = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^*\end{smallmatrix}\right)$.

Here the adjoints are taken with respect to the ℓ^2 inner product (i.e., without a weight function), e.g., for $\mathbb{C} \in \mathcal{B}(H, \ell^2_r(\mathbf{N};Y))$ we have that $\mathbb{C}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{N};Y), H)$ and $\langle \mathbb{C}x, y\rangle = \langle x, \mathbb{C}^* y\rangle$ for $x \in H,\ y \in \ell^2_{1/r}(\mathbf{N};Y)$.

Note that $\ell^2_{1/r}$ is the dual of ℓ^2_r with respect to the (weightless) ℓ^2 inner product.

Proof of Proposition 13.3.5: Using (13.12) and (13.11) one can verify that (1.)–(4.) of Definition 13.3.1 hold (e.g., $\mathbb{C}^* \boldsymbol{\text{Я}}_{-1} \in \mathcal{B}(\ell^2_r(\mathbf{Z}_-;Y), H)$, and equation $A^*\mathbb{C}^* \boldsymbol{\text{Я}}_{-1} = (\pi^+\tau\mathbb{C})^* \boldsymbol{\text{Я}}_{-1} = \cdots = \mathbb{C}^* \boldsymbol{\text{Я}}_{-1}\tau\pi^-$ is easily verified).

The claim on generators follows easily from (13.38). □

Next we write out the symbols ("Z-transforms") of the components of a wpls:

Lemma 13.3.6 *Let* $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{wpls}_r(U,H,Y)$*,* $r > 0$*. Then, for* $|z| < 1/\rho(A)$ *and* $u_0 \in U$*, we have*

$$\widehat{\mathbb{A}}(z) = (I - zA)^{-1} = \sum_{k=0}^{\infty} A^k z^k, \tag{13.48}$$

$$\widehat{\mathbb{B}}(z) = z(I - zA)^{-1}B = (z^{-1} - A)^{-1}B = \sum_{k=0}^{\infty} A^k z^{k+1} B, \tag{13.49}$$

$$\widehat{\mathbb{C}}(z) = C(I - zA)^{-1} = \sum_{k=0}^{\infty} CA^k z^k, \tag{13.50}$$

$$\widehat{\mathbb{D}}(z) = D + Cz(I - zA)^{-1}B = D + C(z^{-1} - A)^{-1}B \tag{13.51}$$

$$= D + \sum_{k=0}^{\infty} CA^k B z^{k+1} = D + \widehat{\mathbb{C}}(z)Bz = D + C\widehat{\mathbb{B}}(z) \tag{13.52}$$

$$\widehat{\mathbb{B}}(z)u_0 = \mathbb{B}(z^{-\cdot}u_0), \quad \mathbb{D}z^{-\cdot}u_0 = z^{-\cdot}\widehat{\mathbb{D}}(z)u_0. \tag{13.53}$$

in the sense that $\widehat{\mathbb{A}x_0} = \widehat{\mathbb{A}}x_0$*,* $\widehat{\mathbb{B}\tau u} = \widehat{\mathbb{B}}\widehat{u}$*,* $\widehat{\mathbb{C}x_0} = \widehat{\mathbb{C}}x_0$ *and* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ *on* $\mathbf{D}_{1/r}$ *for all* $x_0 \in H$ *and* $u \in \ell^2_r(\mathbf{N};U)$*.*

Thus, $\mathbb{C}$ is stable iff $\widehat{\mathbb{C}} \in \mathrm{H}^2_{\mathrm{strong}}$, i.e., iff $C(I - zA)^{-1}x_0 \in \mathrm{H}^2(\mathbf{D};Y)$ for all $x_0 \in H$. Analogously, $\mathbb{B}$ is stable iff $B^*(I - zA^*)^{-1}x_0 \in \mathrm{H}^2(\mathbf{D};U)$ for all $x_0 \in H$.

Proof: The equations for $\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}, \widehat{\mathbb{D}}$ are straightforward.

Set then $u := z^{-\cdot}u_0$, so that $u \in \ell^2_r(\mathbf{Z};U) + \ell^2_{\mathrm{loc}}(\mathbf{N};U)$. Obviously, $\widehat{\mathbb{B}}(z)u_0 = \mathbb{B}(z^{-\cdot}u_0)$, hence

$$\widehat{\mathbb{D}}(z)u_0 = Du_0 + C\mathbb{B}u = (\mathbb{D}u)_0. \tag{13.54}$$

Since $\tau^k u = z^{-k}u$, we have $(\mathbb{D}u)_k = (\mathbb{D}\tau^k u)(0) = z^{-k}\widehat{\mathbb{D}}(z)u_0$ ($k \in \mathbf{Z}$). □

Lemma 13.3.7 (Exp. stable) *The following are equivalent for $A \in \mathcal{B}(H)$:*

(i) A is exponentially stable, i.e., $\sup \|r^{-k}A^k\| < \infty$ *for some $r < 1$.*

(ii) $A^{\cdot}x_0 \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$;

(ii') $(s \mapsto (I - sA)^{-1}x_0) \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(H))$ for all $x_0 \in H$;

(iii) $\|\sum_0^\infty A^k\phi_k\|_H \le M\|\phi\|_2$ for all $\phi \in \mathrm{c_c}(\mathbf{N};H)$;

(iv) $\rho(A) < 1$, where $\rho(A) := \lim_{k\to\infty}\|A^k\|^{1/k} = \inf_{k\to\infty}\|A^k\|^{1/k} = \max|\sigma(A)| \le \|A\|$;

(v) $\sigma(A) \subset \mathbf{D}$.

The value $\rho(A)$ is called the *spectral radius* of A (see Lemma A.3.3).

Proof: By Lemma A.3.3(r1)&(s1), we have (iv)⇔(v). Equivalence "(i)⇔(iv)" is almost trivial. We obtain "(ii)⇔(ii')" from (13.48) and "(ii)⇔(i)" from [W89d] (which shows that the weak form of (ii) is sufficient). Implication "(i)⇔(iii)" follows as in the proof of Lemma A.4.5. □

Lemma 13.3.8 (Stability) *Let $\Sigma = \left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}(U,H,Y)$ and $0 < r < r' < \infty$. Then*

(a1) Σ is exponentially stable iff A is exponentially stable.

(a2) If A is r-stable (or $\rho(A) \le r$), then $\Sigma \in \mathrm{wpls}_{r'}$, $\mathbb{B}\tau \in \mathrm{tic}_{r'}(U,H)$, and $\mathbb{D} \in \ell^1_{r'}(\mathbf{N};\mathcal{B}(U,Y))*$.

(b1) If $\mathbb{B}$ is r-stable, then $\mathbb{B}\tau$ and $\mathbb{D}$ is r'-stable.

(b2) If $\mathbb{B}\tau$ is r-stable, then $\mathbb{B}$ and $\mathbb{D}$ are r-stable.

(b3) If $\mathbb{C}$ is r-stable, then $\mathbb{D} \in \mathrm{tic}_{r'} \cap \mathcal{B}(\ell^1_r, \ell^2_r)$, $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z};U), \ell^\infty_{1/r}(\mathbf{Z};U))$, $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(r^{-1}\mathbf{D};\mathcal{B}(U,Y))$ and Lemma 13.1.3(d) applies.

Thus, Σ is r-stable for all $r > \rho(A)$.

Proof: (a1) If Σ is exponentially stable, then so is A, by definition. Assume that A is exponentially stable, i.e., that $\|A^k\| \le Mr^k$ for all $k \in \mathbf{N}$ for some $M < \infty$, $r < 1$. By using Lemma 13.3.3(b), one easily verifies that $\|\mathbb{A}x_0\|_2 \le M(1-r^2)^{-1}\|x_0\|^2$ and $\left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in$ wpls is exponentially stable.

(a2) By Remark 13.3.9, we can w.l.o.g. assume that $r' = 1$, hence we obtain this from (a1) for Σ (see Lemma 13.3.7(iv)&(i) for $\rho(A)$). By applying (a1) with $C = I$ and $D = 0$, we get $\mathbb{B}\tau =$"$\mathbb{D}$"$\in \mathrm{tic}(U,H)$.

Finally, choose $s \in (r,1)$ to obtain $\mathbb{D} \in \mathrm{tic}_{s'}$ (by (a1)). Then $\mathbb{D} \in \ell^1_t(\mathbf{N};\mathcal{B}(U,Y))$ for any $t > s$, particularly for $t = 1$, by Lemma 13.1.3(c3).

(b2) This follows from (13.43) for $\mathbb{D}$; take $C = I, D = 0$ to get $\mathbb{B}\tau = \mathbb{D}$.

(b3) Assume that $\mathbb{C}$ is r-stable. By (13.38), $\mathbb{D}u_0e_0 = Du_0e_0 + \tau^{-1}\mathbb{C}Bu_0e_0 \in \ell^2_r$ for each $u_0 \in U$, hence we get the claims from Lemma 13.1.3(d).

(b1) If $\mathbb{B}$ is r-stable, then so is $\mathbb{B}^{\mathrm{d}}$ and hence then $\mathbb{D}^{\mathrm{d}}$ and $\mathbb{D}$ are r'-stable, by (b3); application with $C = I$ and $D = 0$ shows that also $\mathbb{B}\tau$ is r'-stable. □

Now we are able to present the discrete counterpart of Remark 6.1.9 (see (13.9) for $r^{\cdot} : (x_j) \mapsto (r^jx_j)$):

Remark 13.3.9 (Stability shift) *Let* $\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) = \left[\begin{array}{c|c}A^{\cdot} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_s(U,H,Y)$. *Then the* stability shift *(or scaling operator)* $\mathcal{T}_r : \left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) \mapsto \left(\begin{array}{c|c}rA & rB\\ \hline C & D\end{array}\right)$ *satisfies*

$$\mathcal{T}_r\left[\begin{array}{c|c}A^{\cdot} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] = \left(\begin{array}{c|c}rA & rB\\ \hline C & D\end{array}\right) = \left[\begin{array}{c|c}(rA)^{\cdot} & \mathbb{B}r^{-\cdot}\\ \hline r^{\cdot}\mathbb{C} & r^{\cdot}\mathbb{D}r^{-\cdot}\end{array}\right] \in \mathrm{wpls}_{rs}(U,H,Y). \quad (13.55)$$

Thus, $\mathcal{T}_r : \mathrm{wpls}_s \mapsto \mathrm{wpls}_{rs}$ *is a bijection.*

Moreover, $\mathcal{T}_r : \mathbb{E} \mapsto r^{\cdot}\mathbb{E}r^{-\cdot}$ *is an isometric isomorphism* $\mathrm{ti}_r(U,Y) \to \mathrm{ti}_{rs}(U,Y)$ *as well as* $\ell^1_s(\mathbf{Z};\mathcal{B}(U,Y))* \to \ell^1_{rs}(\mathbf{Z};\mathcal{B}(U,Y))*$. □

(We leave the simple proof to the reader (cf. (13.10)).)

We let $\mathcal{T}_r$ also denote its components (note that this is in accordance with $\mathcal{T}_r\mathbb{E} := r^{\cdot}\mathbb{E}r^{-\cdot}$ for $\mathbb{E} = \mathbb{D} \in \mathrm{tic}$).

In Sections 6.6–6.7 and Chapter 7, we reduced all kinds of feedbacks to static output feedback for WPLSs. Next we shall do the same for wpls's. As in Section 6.6, we replace the input u by $u_L + Ly$, where u_L is an external input and $L \in \mathcal{B}(Y,U)$ is a static feedback operator (see Figure 6.2) to obtain equations

$$\begin{cases} x_{j+1} = Ax_j + B(Ly_j + (u_L)_j), \\ \quad y_j = Cx_j + D(Ly_j + (u_L)_j), \quad j \in \mathbf{Z}. \end{cases} \quad (13.56)$$

These are algebraically the same as (6.123)–(6.124), in particular, they have a unique solution (i.e., they are well-posed) iff $I - DL$ is invertible. If that is the case, we call the feedback admissible:

Definition 13.3.10 (Admissible static output feedback) *Let* $\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) = \left[\begin{array}{c|c}A^{\cdot} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}(U,H,Y)$. *An operator* $L \in \mathcal{B}(Y,U)$ *is called an* admissible (static) output feedback operator *for* $\left[\begin{array}{c|c}A^{\cdot} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ *if* $I - L\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty(U)$.

We call L r-stabilizing *if* $\Sigma_L \in \mathrm{wpls}_r$ *etc., as in Definition 6.6.4.*

By Lemmas A.1.1(f6) and 13.1.7, each of the conditions "$I - LD \in \mathcal{GB}(U)$", "$I - DL \in \mathcal{GB}(Y)$", and $I - \mathbb{D}L \in \mathcal{G}\mathrm{tic}_\infty(Y)$ is equivalent to $I - L\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty(U)$.

The corresponding closed-loop system is given below:

Lemma 13.3.11 *Let* $\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) = \left[\begin{array}{c|c}A^{\cdot} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_s(U,H,Y)$ *and* $I - LD \in \mathcal{GB}(U)$. *Then*

$$\left(\begin{array}{c|c}A_L & B_L\\ \hline C_L & D_L\end{array}\right) := \left(\begin{array}{c|c}A + BL(I-DL)^{-1}C & B(I-LD)^{-1}\\ \hline (I-DL)^{-1}C & (I-DL)^{-1}D\end{array}\right) \quad (13.57)$$

$$= \left[\begin{array}{c|c}A_L & \mathbb{B}_L\\ \hline \mathbb{C}_L & \mathbb{D}_L\end{array}\right] := \left[\begin{array}{c|c}A + BL(I-DL)^{-1}C & \mathbb{B}(I-L\mathbb{D})^{-1}\\ \hline (I-\mathbb{D}L)^{-1}\mathbb{C} & (I-\mathbb{D}L)^{-1}\mathbb{D}\end{array}\right] \in \mathrm{wpls}(U,H,Y). \quad (13.58)$$

Moreover, $A_L^j - A^j = \mathbb{B}\tau^j L\mathbb{C}_L = \mathbb{B}\tau^j L(I-\mathbb{D}L)^{-1}\mathbb{C} = \mathbb{B}_L L\tau^j\mathbb{C}$ *for* $j \geq 0$.

Proof: By solving (13.56), we obtain (13.57). By Lemma 13.3.3(b), the operators (13.57) generate a wpls, whose state map is necessarily A_L, and whose reachability, observability, and I/O maps $\mathbb{B}'_L, \mathbb{C}'_L, \mathbb{D}'_L$ can be found by solving

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} (A_L)^j & \mathbb{B}'_L\tau^j \\ \mathbb{C}'_L & \mathbb{D}'_L \end{bmatrix}\begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1,2,\ldots). \quad (13.59)$$

But from (13.59) and "$u = u_L + Ly$" we obtain

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} A^j + \mathbb{B}\tau^j L\mathbb{C}_L & \mathbb{B}_L\tau^j \\ \mathbb{C}_L & \mathbb{D}_L \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1, 2, \ldots). \tag{13.60}$$

Thus, $\mathbb{B}'_L = \mathbb{B}_L$, $\mathbb{C}'_L = \mathbb{C}_L$ and $\mathbb{D}'_L = \mathbb{D}$, and $A_L^j - A^j = \mathbb{B}\tau^j L\mathbb{C}_L$; the last equation follows Lemma A.1.1(f6). □

Thus, the formula for state feedback, defined as in Definition 6.6.10, takes the following form:

Lemma 13.3.12 *A state feedback pair* $\begin{pmatrix} K \mid F \end{pmatrix}$ *is admissible for* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in$ wpls *iff* $I - F \in \mathcal{GB}$. *If this is the case, then the resulting closed-loop system is given by (here* $M := (I-F)^{-1}$*)*

$$\Sigma_\flat := \left[\begin{array}{c|c} A^{\cdot} + \mathbb{B}\tau(\cdot)\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I \end{array}\right] = \left(\begin{array}{c|c} A + BMK & BM \\ \hline C + DMK & DM \\ MK & M - I \end{array}\right). \tag{13.61}$$

□

The pair $\begin{pmatrix} MK \mid 0 \end{pmatrix}$, where $M := (I-F)^{-1}$, is equivalent to $\begin{pmatrix} K \mid F \end{pmatrix}$ in the sense that it is admissible or stabilizing iff $\begin{pmatrix} K \mid F \end{pmatrix}$ is, and the resulting closed-loop system is

$$\left[\begin{array}{c|c} A + BMK & B \\ \hline C + DMK & D \\ MK & 0 \end{array}\right]. \tag{13.62}$$

We identify $K \in \mathcal{B}(H, U)$ as a *state feedback operator* to the (admissible) state feedback pair $\begin{bmatrix} K \mid 0 \end{bmatrix}$. Thus, $K \in \mathcal{B}(H, U)$ is *exponentially stabilizing* iff $A + BK$ is exponentially stable, etc. See also Lemma 13.3.16.

Also other definitions of Section 6.6 can be converted to the discrete time case analogously; the results can be converted in a similar way:

Theorem 13.3.13 (WPLS results hold for wpls's) *If we make the following replacements (here CT refers to continuous and DT to discrete time):*

$$\begin{aligned}
&\mathrm{WPLS} \mapsto \mathrm{wpls},\ \ \mathrm{SOS} \mapsto \mathrm{sos},\ \ \mathrm{TI} \mapsto \mathrm{ti},\ \ \mathrm{WR},\mathrm{SR},\mathrm{ULR},\mathrm{TIC} \mapsto \mathrm{tic}\\
&\mathrm{MTIC}^*_* \mapsto \heartsuit\mathrm{MTIC}^*_*,\ \ \mathrm{L}^2 \mapsto \ell^2,\ \ \mathcal{C}^\infty_\mathrm{c} \mapsto \mathrm{c_c},\ \ \pi_+ \mapsto \pi^+,\ \ \pi_- \mapsto \pi^-, \text{Я} \mapsto \text{Я}_{-1};\\
&\mathbf{R} \mapsto \mathbf{Z},\ \ \mathbf{R}_+ \mapsto \mathbf{N},\ \ \mathbf{R}_- \mapsto \mathbf{Z}_-,\ \ i\mathbf{R} \mapsto \partial\mathbf{D}\setminus\{-1\},\ \ i\mathbf{R}\cup\{\infty\} \mapsto \partial\mathbf{D},\ \ \mathbf{C}^+ \mapsto \mathbf{D},\ \ \overline{\mathbf{C}^+}\cup\{\infty\} \mapsto \overline{\mathbf{D}};\\
&[t_1,t_2] \mapsto [t_1,t_2-1],\ \ \int_{t_1}^{t_2} \mapsto \sum_{t_1}^{t_2-1};\\
&\mathbb{A} \mapsto A^{\cdot},\ \ \mathbb{A}(t) \mapsto A^t,\ \ \tau(t) \mapsto \tau^t;\ \ C_\mathrm{w},C_\mathrm{s},C_{\mathrm{L,w}},C_{\mathrm{L,s}},C_\mathrm{c} \mapsto C\ \ \textit{etc.};\\
&\textit{any regularity assumption/statement on a map or system} \ \mapsto\ \textit{a true assumption/statement}\\
&\textit{(the same applies to the boundedness of input and output operators)};\\
&\mathrm{Dom}(A), H_B, H^*_{C,K}, H_{\pm1}, H^*_{\pm1} \mapsto H;\\
&\textit{“[e]IARE”,”[e]CARE”,”[e]}B^*_\mathrm{w}\textit{-CARE”} \ \mapsto\ \textit{“[e]DARE”},\\
&\textit{[e]IARE,[e]CARE (the equations)} \ \mapsto\ \textit{[e]DARE (the corresponding DT equation)};\\
&S = D^*JD \mapsto S = D^*JD + B^*\mathcal{P}B \quad \textit{(similarly for anything based on equation } S = D^*JD\textit{)};\\
&(s-A)^{-1} \mapsto (I-sA)^{-1},\ \ (s-A)^{-1}B \mapsto s(I-sA)^{-1};\\
&(\mathcal{P},S,\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]) \mapsto (\mathcal{P},S,K) \quad \textit{(for solutions of the eIARE)},\\
&\text{Stability indices } \omega \text{ and Laplace/Z-transform arguments } s\text{:}\\
&\text{“}\omega \ge 0'' \mapsto \text{“}\omega \ge 1'',\ \ \text{“}\omega > 0 \mapsto \text{“}\omega > 1'',\\
&\text{“}\omega = 0'' \mapsto \text{“}\omega = 1'',\ \ \text{“}\omega \ne 0'' \mapsto \text{“}\omega \in (0,\infty)\setminus\{1\}'';\\
&\text{“}\mathrm{Re}\, s > \omega'' \mapsto'' s \in \mathbf{D}''_{1/\omega},\quad 0\textit{-stabilizing} \mapsto 1\textit{-stabilizing};\ \ s = +\infty \mapsto s = 0\\
&\mathrm{e}^{\pm\omega t} \mapsto \omega^{\pm t},\ \ \mathrm{e}^{\pm\omega\cdot} \mapsto \omega^{\pm\cdot},\ \ \omega+\alpha \mapsto \alpha\omega,\ \ s-\alpha \mapsto \alpha s,\ \ ir \mapsto \mathrm{e}^{ir}
\end{aligned} \tag{13.63}$$

(naturally, the above changes apply also any other stability index (resp. transform argument, element of $i\mathbf{R}$, time value) in place of ω (resp. s, ir, t), any other system in place of Σ etc.), then the following definitions are still applicable and the following results (among others) still hold:

Lemma A.4.2(h1), Proposition E.1.8; Sections 2.1 (note that Lemma 2.1.15 now says that $(\mathbb{D}(s^{\cdot}u_0))(k) = s^k\widehat{\mathbb{D}}(s)u_0$ for all $\mathbb{D} \in \mathrm{tic}_r(U,Y)$, $k \in \mathbf{Z}$, $s \in r\mathbf{D}$, $u_0 \in U$), 2.2, 2.4 and 2.5.

Chapter 4 except possibly Lemmas 4.1.3 and 4.1.5.

Sections 6.4 and 6.5 except Lemma 6.5.10(c) and possibly the claims on p.r.c. (probably also they are true); Section 6.6 and 6.7 except Proposition 6.6.18 (in fact, even 6.6.18 it is true except for its parts that are meaningless in the discrete-time case) and Example 6.6.23.

See Theorems 14.1.3, 15.1.1, 11.5.2 and 12.2.2 for Chapters 8–12; (mainly) Section 14.3 for Chapter 5 and Lemma 13.3.19 for Lemma 6.3.20.

Of course, also the (non-italic) text between subsections is almost completely applicable too (although the regularity problems disappear in this discrete-time case).

Moreover, most MTIC results can also be rewritten for discrete time for classes ℓ^1_+*, $\mathrm{tic}_{\exp}$ etc. (see Lemma 14.3.5) in place of MTIC classes (but the “$S \ne D^*JD$” requirement of Hypothesis 8.4.8 is not satisfied by these classes). There are some

CARE results that implicitly or explicitly have D^*JD in place of S. As explained above, the CAREs must be replaced by DAREs, hence this term must always be replaced by $S := D^*JD + B^*\mathcal{P}B$ while writing the results in their discrete-time forms (thus, most "D^*JD" terms and their simplified forms must be replaced by "$D^*JD + B^*\mathcal{P}B$", whereas any $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ terms may be removed; this makes results such as Theorem 10.2.9 and Proposition 9.9.12(c)(3.) much less useful in their discrete-time forms (since they are based on "$S = D^*JD$")). Most of the time the reader need not be concerned about this since this has been explicitly written into the results following the theorems listed above.

As noted around Example 14.2.9, there is no discrete equivalent for the B_{w}^*-CARE theory of Section 9.2 (in particular, we almost always have $S \neq D^*JD$); the same holds largely for the $\mathrm{Dom}(A_{\mathrm{crit}})$-CARE theory of Section 9.7 (since now $\mathrm{Dom}(A_{\mathrm{crit}}) = H = \mathrm{Dom}(A)$; note that most of the theory holds with S in place of D^*JD).

Proof of Theorem 13.3.13: All proofs hold in discrete time too, mutatis mutandis, usually the discrete time versions become simpler. Thus, the references from discrete time to continuous time are always non-essential. However, some results are proved in discrete time only, and the results are then transferred to continuous time by discretization. Therefore, if one wishes to verify the proofs linearly, one should verify the entire monograph in its discrete time form before verifying the continuous time forms (alternatively, one could read both settings simultaneously but go somewhat further in discrete time in such places).

There is a short cut: by using Theorem 13.2.3, one can convert the results corresponding to TIC maps only. Some other results are implied by Remark 6.5.11 (mainly the ones concerning TIC_∞ maps only). By discretization, one can convert uniqueness results from discrete time and existence results from continuous time.

For the rest, one must make the corresponding changes in proofs too. Some proofs contain references which either can be replaced by the discrete results of this monograph or whose proofs must be verified in the same way; we mention that we have verified for the discrete case [S97b, Lemma 21], all of [S98a] and [S98c] (including the parts of [S98b] that are contained in [S98c] (and more), in particular, Subsections 1–3.4, 3.9(i), 4.1–4.7, and Chapter 5 apply) (with replacements (13.63), both with Remark 6.1.15 and without it.

All this is quite straightforward (the explicit results above contain all the nonstraightforward parts). We sketch below the hardest proofs:

The proof of Lemma 2.2.7 does not need the reference to Lemma D.1.8 in this (discrete) case.

The Corona Theorem for $\mathcal{A} = \mathrm{ti}$ follows directly (use Theorem 13.2.3(b1) for (iv)); case $\mathcal{A} = \ell^1$ follows as shown in the proof of case $\mathcal{A} = \mathrm{MTI_d}$ (take $\mathcal{A} = \heartsuit^{-1}\widetilde{\mathcal{A}}$, prove the theorem, then make the replacements).

Proposition 4.1.7: "(iii)⇔(iv)⇔(iv')⇔(iv'')" follows from Theorem 13.2.3, the rest by transforming the original proof.

Lemma 4.1.8: If $\mathbb{D} \in \mathrm{tic}(U,Y)$ and $\mathbb{D}^*\mathbb{D} \not\geq \varepsilon I$ for any $\varepsilon > 0$, then we can construct $\widehat{u} \in \mathrm{H}^2_1 \setminus \mathrm{H}^2$ as in the proof, but for $\mathbb{F} := T^{-1}\mathbb{D}$, with $|r_1| < \pi/2$ (see

Lemma 13.1.4).

Then $v := Tu$ satisfies $\widehat{v} \in \mathrm{H}(\mathbf{D};U)$, hence $v \in \mathrm{H}^2(r\mathbf{D};U)$ for all $r < 1$, but $\|\widehat{v}\|_2 = \|\widehat{u}\|_{\mathrm{L}^2([-\pi,\pi);U)} = \infty$. However, $\mathbb{D}v = T(\mathbb{F}u) \in \mathrm{L}^2$. Thus, then $\mathbb{D}$ is not quasi-left-invertible. This shows (a); the rest can be shown as in the original proof.

Theorem 6.7.10(d): 1° *(ii)⇒(i):* Assume that $s(I - sA)^{-1}B \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U,H))$ (i.e., $\mathbb{B}\tau \in$ tic) and that Σ is optimizable, hence exponentially stabilizable, by Proposition 13.3.14. Thus, there is $K \in \mathcal{B}(H,U)$ s.t. $\mathbb{A}_\circlearrowleft := \mathbb{A} + \mathbb{B}\tau\mathbb{K}_\circlearrowleft$ is exponentially stable, hence

$$(I - sA)^{-1} = (I - sA_\circlearrowleft)^{-1} - s(I - sA)^{-1}BK(I - sA_\circlearrowleft)^{-1} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(H)). \quad (13.64)$$

Thus, A is exponentially stable, by Lemma 13.3.7(ii'). 2° *(viii)⇒(v):* This follows from Proposition 13.3.14. (The rest of the proof of Theorem 6.7.10 does not require clarification.)

The DT version of part of Chapter 9 is verified in Theorem 14.1.3.

We recommend reading "$[t_1,t_2]$" as "$[t_1,t_2)$" (i.e., $[t_1,t_2-1]$), so that it holds in both discrete-time and continuous-time cases. □

Since B and C are always bounded in discrete-time, several aspects of system theory become as simple as for finite-dimensional systems:

Proposition 13.3.14 (Opt. ⇔ exp. stab.) *A wpls is optimizable iff it is exponentially stabilizable. Thus, a wpls is estimatable iff it is exponentially detectable.*

Proof: Assume that $\Sigma \in \mathrm{wpls}(U,H,Y)$ is optimizable. By Exercise 6.34(i) of [CZ] (with $C = I$), there is $K \in \mathcal{B}(H,U)$ s.t. $A + BK$ is stable (use Lemma 13.3.7(ii) and the fact that $(A+BK)^{\cdot}x_0 \in \ell^2$ for all $x_0 \in H$). The converse is obvious, and the dual claim follows, by duality. □

We have $u, y \in \ell^2 \Rightarrow x \in \ell^2$ for estimatable systems:

Theorem 13.3.15 ($u, y \in \ell^2 \Rightarrow x \in \ell^2$) *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y)$ *be estimatable. Then there is $M < \infty$ s.t. if $u \in \ell^2(\mathbf{R}_+;U)$ and $x_0 \in H$ are s.t. $y := \mathbb{C}x_0 + \mathbb{D}u \in \ell^2$, then $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \ell^2$ and $\|x\|_2 \le M(\|x_0\|_H + \|u\|_2 + \|y\|_2)$.*

Proof: Because Σ is exponentially detectable, we have $\mathbb{A}x_0 + \mathbb{B}\tau u = \mathbb{A}_\sharp x_0 + \mathbb{B}_\sharp \tau u - \mathbb{H}_\sharp \tau y$, where $\Sigma_\sharp$ is the closed-loop system (6.168) corresponding to an exponentially stabilizing output injection pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$, hence $M := \|\mathbb{A}_\sharp\|_{\mathcal{B}(H,\ell^2)} + \|\mathbb{B}_\sharp\tau\|_{\mathrm{tic}} + \|\mathbb{H}_\sharp\tau\|_{\mathrm{tic}} < \infty$, by Lemma 13.3.8. □

It is easy to identify a stabilizing state feedback operator to an exponentially stable system:

Lemma 13.3.16 (*K*) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ *be exponentially stable and* $K \in \mathcal{B}(H,U)$*. Then the following are equivalent:*

(i) K is I/O-stabilizing;

(i') K is output-stabilizing;

(i") K is input-stabilizing;

(ii) K is exponentially r.c.-stabilizing;

(iii) $\sigma(A+BK) \subset \mathbf{D}$*, i.e.,* $\rho(A+BK) < 1$*;*

(iv) $I - Kz(I-zA)^{-1}B \in \mathcal{GB}(U)$ *for* $z \in \overline{\mathbf{D}}$*.*

Proof: Obviously, (i')⇐(ii)⇒(iii)⇒(i). By Lemma 6.7.9, (i') implies (i).

Since $\mathbb{A}$ is exponentially stable, so are $\mathbb{D}$ and $\mathbb{F}$. Therefore, (i) holds iff $(I-\mathbb{F})^{-1} \in$ tic; or equivalently, iff $I - \mathbb{F} \in \mathcal{G}$tic. But $I - \mathbb{F} \in \mathcal{G}$tic implies that the closed-loop system is exponentially stable, i.e., that (iii) holds, by Corollary 6.6.9. On the other hand, condition $(I-\mathbb{F})^{-1} \in$ tic is equivalent to (iv), because the boundedness of $\widehat{\mathbb{X}}^{-1}$ follows from the compactness of $\overline{\mathbf{D}}$ (see also Lemma D.1.2(b2)).

If (iii) holds, then $I - \mathbb{F} \in \mathcal{G}\mathrm{tic}_{\mathrm{exp}}$, hence then (ii) holds. If (i") holds, then (iii) holds, by Lemma 6.6.8(c). □

Lemma 13.3.17 (Jointly stabilizing *K* & *H*) *Let* $\Sigma \in \mathrm{wpls}(U,H,Y)$*. Then the following holds:*

(a) Any admissible state feedback and output injection pairs for Σ *are jointly admissible.*

(b) Any output-stabilizing and exponentially detecting pairs for Σ *are exponentially jointly r.c.- and l.c.-stabilizing.*

In particular, the following are equivalent:

(i) Σ *is exponentially jointly r.c.-stabilizable and l.c.-detectable;*

(ii) Σ *is optimizable and estimatable;*

(iii) Σ *is output-stabilizable and estimatable;*

(iv) Σ *is optimizable and input-detectable.*

(c) Let Σ *be estimatable. Then any I/O-stabilizing pair for* Σ *is r.c.-I/O-stabilizing.*

Recall that in classical articles (those with $\dim H < \infty$, i.e., with rational transfer functions) the word "stabilizing" means usually "exponentially stabilizing", hence for them one usually makes the prefix "r.c.-" (etc.) redundant by assuming the system to be "detectable" (then any exponentially stabilizing state feedback pair is exponentially r.c.-stabilizing, by, e.g., Lemma 6.6.26 and (shifted) Theorem 6.6.28).

Proof: (a) Assume that $\begin{pmatrix} K & | & F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ are admissible. Then $\Sigma' := \left(\begin{array}{c|cc} A & H & B \\ \hline C & G & D \\ K & 0 & F \end{array}\right) \in$ wpls, by Lemma 13.3.3, hence $\begin{pmatrix} K & | & F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ are jointly admissible.

(b) By Proposition 13.3.14, we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii); by duality, (iii)$\Leftrightarrow$(iv), so that only (iii)$\Rightarrow$(i) remains to be proved.

Let $\begin{bmatrix} K & | & F \end{bmatrix}$ and $\left[\frac{H}{G}\right]$ be as in (iii). By Lemma 6.7.9, $\begin{bmatrix} K & | & F \end{bmatrix}$ is exponentially stabilizing. Thus, the (closed-loop) state maps $A_\flat := A + B(I - F)^{-1}K$ and $A + H(I-G)^{-1}C$ are exponentially stable, hence so are the closed-loop systems of Σ' corresponding to $L = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ and $L = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, by Lemma 13.3.7. Therefore, (i) holds (the coprimeness follows from the exponentially stable discrete form (cf. Remark 13.3.9 and Theorem 13.3.13) of Theorem 6.6.28.

(c) Let $\begin{pmatrix} K & | & F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ be corresponding pairs; by (a), they are jointly admissible. Thus, if we define $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{X}, \mathbb{Y}$ by (6.172), then $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are exponentially stable and $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ (because $\Sigma_\sharp$ is exponentially stable). $\square$

Lemma 13.3.18 ($u, x \in \ell^2 \Rightarrow y \in \ell^2$) *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \text{wpls}(U,H,Y)$. *If* $u, x \in \pi^+\ell^2$, *then* $y \in \pi^+\ell^2$ *and* $\|y\|_2 \le M(\|u\|_2 + \|x\|_2)$, *where* $x_0 \in H$ *is arbitrary,* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$, *and* $M := \max\{\|C\|, \|D\|\}$. $\square$

(This follows from equation $y = Cx + Du$, (equation (13.40)).)

The discrete-time version of Lemma 6.3.20 is obvious, but we shall record it for future use:

Lemma 13.3.19 *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \text{wpls}(U,H,Y)$. *Assume that* $r > 0$, $u \in \ell^2_r(\mathbf{N}; U)$, *and* $x := \mathbb{B}\tau u \in \ell^2_r(\mathbf{N}; H)$. *Then* $(z^{-1} - A)\widehat{x}(z) = B\widehat{u}(z) \in H$ *for a.e.* $z \in r^{-1}\partial\mathbf{D}$.

Assume, in addition, that $y := \mathbb{D}u \in \ell^2_r$. *Then* $\widehat{y} = C\widehat{x} + D\widehat{u} \in Y$ *a.e. on* $r^{-1}\partial\mathbf{D}$. *In particular, for* $r = 1$ *and* $J \in \mathcal{B}(Y)$ *we have*

$$\langle \mathbb{D}u, J\mathbb{D}u \rangle_{\ell^2(\mathbf{Z};Y)} = (2\pi)^{-1} \left\langle \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix}, \kappa \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix} \right\rangle_{\mathrm{L}^2(\partial\mathbf{D};Y)}, \tag{13.65}$$

where $\kappa := \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix}$. $\square$

(The proof is a much simpler version of the proof of Lemma 6.3.20, and hence omitted.)

Notes

In the form (13.40), the wpls's in have been studied for several decades; two of the cornerstones being [KFA] and [Fuhrmann81]; whose Sections III.1–III.5 contain a further study on their realizations theory.

Olof Staffans [S99] has formulated stable wpls's essentially as in Definition 13.3.1. Jarmo Malinen [Mal00] has defined wpls's with different domain and range spaces and presented a theory on them; his results include part of Lemma 13.3.3 and Lemma 13.3.12.

The monographs [SF], [RR], [Nikolsky] and [FF] contain rather general operator theory and harmonic analysis, but many of their results are applicable for wpls's. The article [S01] and Chapter 11 of [Sbook] contain applications of [SF] to both continuous-time system (especially for ones with contractive semigroups) and discrete-time systems. In Chapter 11 of [Sbook] Staffans shows how to use the Cayley transform to convert a complete WPLS to a wpls or vice versa, whereas we have only treated the Cayley transform of the I/O map (Theorem 13.2.3); that chapter was written two years after this one.

Observe that Theorem 13.3.13 (and the theorems mentioned right below it) contains the discrete-time variants of most continuous-time results of this monograph. Also for most other continuous-time results the discrete-time variants are true and rather easily verified (usually the same proofs apply, mutatis mutandis). Much of our theory is well known in the finite-dimensional case (see, e.g., [LR] or [IOW]).

13.4 Time discretization (Δ^{S} : WPLS $\to$ wpls)

Our problem is within ourselves. We have found the means to blow the world physically apart. Spiritually, we have yet to find the means to put the world's pieces back together again.

— Thomas E. Dewey (1902–1971) US lawyer, politician

In this section, we shall present *discretization*, a method to convert a WPLS to a wpls (Theorem 13.4.4). In other chapters of this book, we often use discretization to deduce properties of WPLSs from those of wpls's, because the latter ones have bounded generators and can hence be more easily explored. Note that discretization differs from the methods of Theorem 13.2.3 (the Cayley transform) and of Lemma 13.1.4.

The principle is well-known, and it has been used (implicitly) to deduce that any semigroup control system (as defined in [Sal89]) is a WPLS (i.e., that a locally L^2-bounded system is actually bounded w.r.t. L^2_ω for some $\omega \in \mathbf{R}$).

Theorem 13.4.4 describes the preservation of properties of systems and Theorem 13.4.5 of those of I/O maps.

As mentioned above, U, W, H, Y and Z denote Hilbert spaces of arbitrary dimensions.

Definition 13.4.1 *For* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ *we define its* discretization $\Delta^{\ell^2}u : \mathbf{Z} \to \mathrm{L}^2([0,1);U)$ *by* $(\Delta^{\ell^2}u)_n := \pi_{[0,1)}\tau(n)u$ $(n \in \mathbf{Z})$.

Note that this discretization is completely different from the Cayley transform $\heartsuit$: TIC $\leftrightarrow$ tic of Theorem 13.2.3.

The map Δ^{ℓ^2} is obviously a linear map of $\mathrm{L}^2_{\mathrm{loc}}$ one-to-one and onto "$\ell^2_{\mathrm{loc}}(\mathbf{Z};\mathrm{L}^2([0,1);U))$", the space of all sequences $\mathbf{Z} \to \mathrm{L}^2([0,1);U)$. We identify Δ^{ℓ^2} with its restrictions (to, e.g., $\mathrm{L}^2_\omega(\mathbf{R};U) \to \ell^2_{e^\omega}(\mathbf{Z};U)$ or to $\mathrm{L}^2_\omega(\mathbf{R}_+;U) \to \ell^2_{e^\omega}(\mathbf{N};U)$ for some $\omega \in \mathbf{R}$).

It will be shown in Theorem 13.4.5 that for $\omega \in \mathbf{R}$, $r := e^\omega$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ we have $u \in \mathrm{L}^2_\omega \Leftrightarrow \Delta^{\ell^2}u \in \ell^2_r$, and that Δ^{ℓ^2} is an isomorphism of L^2_ω onto ℓ^2_r, i.e., $\Delta^{\ell^2} \in \mathcal{GB}(\mathrm{L}^2_\omega,\ell^2_r)$. Before going into further technical details we define the discretization of systems:

Definition 13.4.2 *For* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *we define its* discretization

$$\Delta^{\mathrm{S}}\Sigma := \left[\begin{array}{c|c} \Delta^{\mathrm{S}}\mathbb{A} & \Delta^{\mathrm{S}}\mathbb{B} \\ \hline \Delta^{\mathrm{S}}\mathbb{C} & \Delta^{\mathrm{S}}\mathbb{D} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}(1)^{\cdot} & \mathbb{B}(\Delta^{\ell^2})^{-1} \\ \hline \Delta^{\ell^2}\mathbb{C} & \Delta^{\ell^2}\mathbb{D}(\Delta^{\ell^2})^{-1} \end{array}\right] \in \mathrm{wpls}(U_\Delta,H,Y_\Delta), \tag{13.66}$$

where $U_\Delta := \mathrm{L}^2([0,1);U)$, $Y_\Delta := \mathrm{L}^2([0,1);Y)$.

If $\Sigma \in$ WPLS, then $\Delta^{\mathrm{S}}\Sigma \in$ wpls; the converse is not true without additional assumptions:

Proposition 13.4.3 *Let* $\omega \in \mathbf{R}$, $r := e^{\omega}$. *If* $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$, *then* $\Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}_r(U_\Delta,H,Y_\Delta)$.

Conversely, assume that $\left(\frac{A\,|\,B}{C\,|\,D}\right) \in \mathrm{wpls}_r(U_\Delta,H,Y_\Delta)$. *Then* $\Sigma := \left[\frac{\mathbb{A}\,|\,\mathbb{B}}{\mathbb{C}\,|\,\mathbb{D}}\right] := \Delta^{\mathrm{S}^{-1}}\left(\frac{A\,|\,B}{C\,|\,D}\right) \in \mathrm{WPLS}(U,H,Y)$ *iff* $A = \mathbb{A}(1)$ *for some* C_0*-semigroup* $\mathbb{A}$, *and*

$$\mathbb{A}^t\mathbb{B} = \mathbb{B}\tau^t, \qquad \mathbb{C}\mathbb{A}^t = \pi_+\tau^t\mathbb{C}, \qquad \tau^t\mathbb{D} = \mathbb{D}\tau^t \qquad (t \in (0,1)). \tag{13.67}$$

If this is the case, then $\Sigma \in \mathrm{WPLS}_\omega$.

However, Δ^{S} does not map WPLS onto wpls; in fact, none of the four Tauberian conditions above is redundant:

We have $\left(\frac{A\,|\,B}{C\,|\,D}\right) \in$ wpls but $\Delta^{\mathrm{S}}\left(\frac{A\,|\,B}{C\,|\,D}\right) \notin$ WPLS when, e.g., 1) A is s.t. it does not have a square root "$\mathbb{A}(1/2)$" or 2) D is s.t. it is not "causal" on $\pi_{[0,1)}\mathrm{L}^2$ (see also Theorem 13.4.5(f)), or 3) $H = \mathrm{L}^2(\mathbf{R}_+)$, $\mathbb{A} = \pi_+\tau$, $Cx_0 := x_0(1-\cdot) \in \mathcal{B}(H,\mathrm{L}^2([0,1)))$ (use the dual of "3)" for the $\mathbb{B}$-condition; we can take $B = 0 = D = C$ in 1), $A = 0 = B = C$ in 2) or $B = 0 = D$ in 3) to guarantee that only one condition is violated).

Proof of Proposition 13.4.3: The other claims are obvious, so we only sketch the proof of the converse claim.

By Theorem 13.4.5, we have $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega,H)$, $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$, $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$, $\pi_-\mathbb{D}\pi_+ = 0$, $\tau^t\mathbb{D} = \mathbb{D}\tau^t$ and $\mathbb{C}\mathbb{A}^t x_0 = \pi_+\tau^t\mathbb{C}x_0$ for $t \in \mathbf{Z}$. Combine this with the assumptions to the get the axioms of Definition 6.1.1 satisfied (by density and continuity, the axioms hold for ω iff they hold for some $\omega' \in \mathbf{R}$). $\square$

Above we used the fact that $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y) \Leftrightarrow \Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}_{e^\omega}(U_\Delta,H,Y_\Delta)$ (when Σ is known to be a WPLS):

Theorem 13.4.4 *Let* $\Sigma = \left[\frac{\mathbb{A}\,|\,\mathbb{B}}{\mathbb{C}\,|\,\mathbb{D}}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega \in \mathbf{R}$, $r := e^{\omega}$. *Then the following holds:*

(a1) The equations $\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \\ \mathbb{C}x_0 + \mathbb{D}u \end{bmatrix}$ *become equivalent to* $\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} A^j & \mathbb{B}\tau^j \\ \mathbb{C} & \mathbb{D} \end{bmatrix}\begin{bmatrix} x_0 \\ u \end{bmatrix}$ *(i.e., to* $\begin{cases} x_{j+1} = Ax_j + Bu_j \\ y_j = Cx_j + Du_j \end{cases}$*), and* $\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{B}\tau(t)u \\ \mathbb{D}u \end{bmatrix}$ *become equivalent to* $\begin{bmatrix} x_n \\ (y_\cdot) \end{bmatrix} = \begin{bmatrix} \Delta^{\mathrm{S}}\mathbb{B}\tau^n(u_\cdot) \\ \Delta^{\mathrm{S}}\mathbb{D}(u_\cdot) \end{bmatrix}$, *where the discrete and continuous time input, state and output correspond to each other through*

$$u_n = \pi_{[0,1)}\tau(n)u, \;\; x_n = x(n), \;\; y_n = \pi_{[0,1)}\tau(n)y, \tag{13.68}$$

(for all n), i.e., $(u_\cdot) = \Delta^{\ell^2}u$, $x_\cdot := x(\cdot)$, $(y_\cdot) = \Delta^{\ell^2}y$.

(In both settings, we must have $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$*; in the initial value setting we assume that* $\pi_-u = 0$ *and* $x_0 \in H$*, in the time-invariant setting we must have* $\pi_-u \in \mathrm{L}^2_\alpha$ *(equivalently,* $\pi_-\Delta^{\mathrm{S}}u \in \ell^2_{e^r}$*), where* α *is s.t.* $\mathbb{B}$ *and* $\mathbb{D}$ *are* α*-stable. Also (a2) and (a3) use the same assumptions.)*

(a2) In both settings described in (a1), we have

$$u \in \mathrm{L}^2_{\omega'} \Leftrightarrow \Delta^{\rm S} u \in \ell^2_{r'}, \qquad y \in \mathrm{L}^2_{\omega'} \Leftrightarrow \Delta^{\rm S} y \in \ell^2_{r'}, \qquad x,u \in \mathrm{L}^2_{\omega'} \Leftrightarrow x, \Delta^{\rm S} u \in \ell^2_{r'} \tag{13.69}$$

for any $\omega' \in \mathbf{R}$, $r' := e^{\omega'}$. *(Here "$x \in \ell^2_{r'}$" means that the restriction* $(x(n))_{n\in\mathbf{Z}}$ *of* x *to* $\mathbf{Z}$ *(that is, "the discretized state") belongs to* $\ell^2_{r'}$*.)*

(a3) Let $\omega' \in \mathbf{R}$, $r' := e^{\omega'}$. *Then there is* $M = M_{\omega'} \in (0,\infty)$ *s.t. in both settings described in (a1), we have*

$$\|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)} \le M(\|x\|_{\ell^2_{r'}(N;H)} + \|\Delta^{\rm S} u\|_{\ell^2_{r'}(N;U)}) \tag{13.70}$$

$$\|x\|_{\ell^2_{r'}(N+1;H)} + \|\Delta^{\rm S} u\|_{\ell^2_{r'}(N;U)} \le M(\|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)}). \tag{13.71}$$

Here we must take $J = \mathbf{R}_+$, $N = \mathbf{N}$ *in the initial value setting and* $J = \mathbf{R}$, $N = \mathbf{Z}$ *in the time-invariant setting (there is no bound for* $\|x_0\|_H$ *in the initial value setting, hence the "$N+1$").*

(b1) The generators of $\Delta^{\rm S}\Sigma$ *are given by*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{c|c} \mathbb{A}(1) & \mathbb{B}\tau(1)\pi_{[0,1)} \\ \hline \pi_{[0,1)}\mathbb{C} & \pi_{[0,1)}\mathbb{D}\pi_{[0,1)} \end{array}\right) \in \mathcal{B}(H \times U_\Delta, H \times Y_\Delta). \tag{13.72}$$

(b2) We have $D \in \mathcal{GB} \Leftrightarrow \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$.

(c) The discretization $\Delta^{\rm S}$ *commutes with valid compositions and inversions of operators.*

(See also Theorem 13.4.5. Note that $\Delta^{\ell^2 -1}\mathbb{B}_{\rm d}$ *is not valid for* $\mathbb{B}_{\rm d} \in \mathcal{B}(\ell^2_r, H)$*, hence neither is* $\Delta^{\mathrm{S}^{-1}}(\mathbb{B}_{\rm d}\tau)$*; on the other hand,* $(\Delta^{\rm S}\mathbb{B})\tau = (u \mapsto \mathbb{B}\tau\Delta^{\ell^2 -1} u) = \Delta^{\rm S}(\mathbb{B}\tau) \in \mathrm{tic}_\infty$. *Note also that the discretization of* $\left[\begin{smallmatrix} A & B \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{A} & \mathbb{B}\tau \end{smallmatrix}\right]$ *is not* $\left(\begin{smallmatrix} A_{\rm d} & B_{\rm d} \\ I & 0 \end{smallmatrix}\right)$ *in general, where* $\left(\begin{array}{c|c} A_{\rm d} & B_{\rm d} \end{array}\right) = \Delta^{\rm S}\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$*).)*

(d1) We have $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y) \Leftrightarrow \Delta^{\rm S}\Sigma \in \mathrm{wpls}_r(U_\Delta, H, Y_\Delta)$.

(d2) Σ *and* $\Delta^{\rm S}\Sigma$ *have the same stability properties (see Definitions 6.1.3 and 13.3.1).*

To be exact, a component of Σ *is [exponentially/strongly/weakly]* ω*-stable iff the corresponding component of* $\Delta^{\rm S}\Sigma$ *is [exponentially/strongly/weakly]* r*-stable*

(e1) An output feedback operator $L \in \mathcal{B}(Y,U)$ *is admissible [stabilizing] for* Σ *iff* L *is admissible [stabilizing] for* $\Delta^{\rm S}\Sigma$. *If* L *is admissible, then* $(\Delta^{\rm S}\Sigma)_L = \Delta^{\rm S}\Sigma_L$.

An analogous result holds for other forms of feedback for Σ *(but not for those for* $\Delta^{\rm S}\Sigma$*, because the discretization* $\Delta^{\rm S}$: WPLS $\to$ wpls *is not onto):*

Any dynamic feedback (resp. state feedback, output injection) for Σ *is admissible [stabilizing] for* Σ *iff its discretization is admissible [stabilizing] for* $\Delta^{\rm S}\Sigma$.

The prefices "I/O-", "SOS-", "weakly", "strongly", "exponentially", "q.r.c.-" and "q.l.c.-" apply (whereas "r.c.-", "l.c.-", "d.c.-" and "jointly" *possibly do not).*

(e2) $\Delta^{\mathrm{S}}\Sigma$ *has all the stabilizability properties of* Σ.

(e3) $\Delta^{\mathrm{S}}\Sigma$ *is optimizable (resp. estimatable) iff* Σ *is optimizable (resp. estimatable).*

(f1) **(J-critical control over $\mathcal{U}$)** *Let* $x_0 \in H$ *and* $J = J^* \in \mathcal{B}(Y)$. *Then* $\mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) = \Delta^{\ell^2}\mathcal{U}_{\mathrm{out}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) = \Delta^{\ell^2}\mathcal{U}_{\mathrm{exp}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{sta}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) \supset \Delta^{\ell^2}\mathcal{U}_{\mathrm{sta}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{str}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) \supset \Delta^{\ell^2}\mathcal{U}_{\mathrm{str}}^{\Sigma}(x_0)$, *where the superindex corresponds to the underlying system.*

The same equalities and inclusions also hold for the subsets of the corresponding J-critical controls. Thus if such controls exist for Σ and for each $x_0 \in H$, then corresponding J-critical cost operators are equal for Σ and $\Delta^{\mathrm{S}}\Sigma$.

In particular, if $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is J-critical for Σ and J over $\mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{exp}}$), then* $\Delta^{\mathrm{S}}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is J-critical for Σ and J over $\mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{exp}}$).*

(f2) **(J-critical control over $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$)** *Let* $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$ *be as in Definition 8.3.2 and let* $J = J^* \in \mathcal{B}(Y)$. *Then*

$$\Delta^{\ell^2}\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}(x_0) = \mathcal{U}^{\mathrm{e}^{\vartheta},\Delta^{\mathrm{S}}\Sigma}_{[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}]}(x_0) \quad (x_0 \in H); \tag{13.73}$$

the same holds for corresponding subsets of J-critical controls. Moreover, this maps $\mathcal{U}_{\mathrm{out}}^{\Sigma} \mapsto \mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}$ *and* $\mathcal{U}_{\mathrm{exp}}^{\Sigma} \mapsto \mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}$.

(g) *The map $\mathbb{D}$ is [positively] J-coercive over* $\mathcal{U}_{\mathrm{exp}}^{\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{out}}^{\Sigma}$, $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$*) iff* $\Delta^{\mathrm{S}}\mathbb{D}$ *is [positively] J-coercive over* $\mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}$, $\mathcal{U}^{\mathrm{e}^{\vartheta},\Delta^{\mathrm{S}}\Sigma}_{[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}]}$*) (cf. (f1)–(f2)).*

If $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{str}}^{\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}^{\Sigma}$*), then* $\Delta^{\mathrm{S}}\mathbb{D}$ *is [positively] J-coercive over* $\mathcal{U}_{\mathrm{str}}^{\Delta^{\mathrm{S}}\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}^{\Delta^{\mathrm{S}}\Sigma}$*).*

Because Δ^{S} maps WPLS into wpls (but not onto), we can use the theorem to obtain continuous-time analogies of uniqueness results (including equality of formulae and stability of operators) only, not of existence results (including words “jointly”, “r.c.”, ...). E.g., the interaction operator and coprime multipliers provided by Lemma 13.3.17 need not be images of any continuous-time operators (their preimages need not be time-invariant). Cf. the proof of (e1).

By (13.83), we could have made Δ^{ℓ^2} (and hence $\Delta^{\mathrm{S}} \in \mathcal{B}(\mathrm{TIC}_{\omega}, \mathrm{tic}_r)$ too) isometric $\mathrm{L}^2_{\omega} \to \ell^2_r$ by using $\mathrm{L}^2_{\omega}([0,1);U)$-valued sequences instead of $\mathrm{L}^2([0,1);U)$-valued ones. However, we have chosen the latter ones in order to make the discretization independent of ω (cf. Lemma 13.3.2).

Proof of Theorem 13.4.4: (a1) These claims are quite obvious.

(a2) The claims on u and y follow from Theorem 13.4.5(a1); the last one follows from (a3).

(a3) (As obvious from the proof, in fact any $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ will do. On the other hand, to see that we cannot get a bound for $\|x_0\|$ in the initial value setting, let $H = \ell^2(\mathbf{N})$ and $\mathbb{A}^t e_n = \mathrm{e}^{-nt} e_n$ ($n \in \mathbf{N}$), so that $\|\mathbb{A}e_n + \mathbb{B}0\|_{\mathrm{L}^2}^2 = 1/2n$ even though $\|e_n\|_{\ell^2} = 1$.)

By isomorphism claim of Theorem 13.4.5(a1), we have $\|u\|_{\mathrm{L}^2_{\omega'}(J;U)} \le M''\|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(N;U)}$ and $\|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(N;U)} \le M''\|u\|_{\mathrm{L}^2_{\omega'}(J;U)}$ for some $M'' < \infty$ and all $u \in \mathrm{L}^2_{\omega'}$. Therefore, we only need upper bounds for the norms of x.

Set $M' := \max_{0\le t\le 1}\{\|\mathbb{A}(t)\|_{\mathcal{B}(H)}, \|\mathbb{B}^t\|_{\mathcal{B}(\mathrm{L}^2,H)}\}$. Let x_0, u and x be as in (either setting of) (a1) (in the initial value setting we extend them by zero on $\mathbf{R}_-$ and $\mathbf{Z}_-$).

1° Assume that $x, \Delta^{\mathrm{S}} u \in \ell^2_{r'}$, so that $u \in \mathrm{L}^2_{\omega'}$. Then, by (6.9),

$$\|x(n+t)\| \le \|\mathbb{A}^t x(n)\| + \|\mathbb{B}^t \tau^n u\| \le M'\big(\|x(n)\| + \|\pi_{[n,n+1)} u\|_2\big), \tag{13.74}$$

hence $\|\pi_{[n,n+1)} x\|_2 \le M'\big(\|x(n)\| + \|\pi_{[n,n+1)} u\|\big)$, for any $n \in N$. Consequently,

$$\|\Delta^{\mathrm{S}} x\|_{\ell^2_{r'}} \le M'\big(\|x\|_{\ell^2_{r'}} + \|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}}\big). \tag{13.75}$$

Thus, $M := \sqrt{2}M'M'' + M''$ will do for the first inequality (note that $2(a^2 + b^2) \ge (a+b)^2$; the addition $+M''$ is for u).

2° Let $x, u \in \mathrm{L}^2_{\omega'}$, so that $\Delta^{\mathrm{S}} u \in \ell^2_{r'}$. Then

$$\|x(n+1)\| \le \min_{0\le t\le 1} \|\mathbb{A}^t x(n+1-t)\| + \max_{0\le t\le 1} \|\mathbb{B}^t \tau^{n+1-t} u\|\big) \tag{13.76}$$

$$\le M'\big(\|\pi_{[n,n+1)} x\|_2 + \|\pi_{[n,n+1)} u\|_2\big), \tag{13.77}$$

(because $\min_{0\le t\le 1}\|x(n+1-t)\| \le \|\pi_{[n,n+1)} x\|_2$). Therefore, $\|x\|_{\ell^2_{r'}(N+1;H)} \le M'\|\Delta^{\mathrm{S}} x\|_{\ell^2_{r'}(N;H)} + \|\Delta^{\mathrm{S}} x\|_{\ell^2_{r'}(N;U)}$. Thus, M will do for the second inequality too.

(b1) This follows from (a) (alternatively, from (13.39)).

(b2) This follows from Lemma 13.1.7 (which says that $D \in \mathcal{GB} \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty$) and (c) (which says that $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty$).

(c) This is obvious from the definition (e.g., $(\Delta^{\mathrm{S}}\mathbb{C})(\Delta^{\mathrm{S}}\mathbb{B}) = \Delta^{\mathrm{S}}(\mathbb{C}\mathbb{B})$); note that to $\mathbb{A}(t)$ this applies for $t \in \mathbf{Z}$ only.

(d1) This follows from (d2) (recall that we have assumed that $\Sigma \in$ WPLS).

(d2) 1° $\mathbb{C}$ *and* $\mathbb{D}$*:* Obviously, $\Delta^{\mathrm{S}}\mathbb{C}[H] \subset \ell^2_r \Leftrightarrow \mathbb{C}[H] \subset \mathrm{L}^2_\omega$, and $\Delta^{\mathrm{S}}\mathbb{D}[\ell^2_r] \subset \ell^2_r \Leftrightarrow \mathbb{D}[\mathrm{L}^2_\omega] \subset \mathrm{L}^2_\omega$, so that the stability of $\mathbb{C}$ (resp. $\mathbb{D}$) equals that of $\Delta^{\mathrm{S}}\mathbb{C}$ (resp. $\Delta^{\mathrm{S}}\mathbb{D}$) (see Lemma 6.1.12).

2° $\Delta^{\mathrm{S}}\mathbb{A}$ *and* $\Delta^{\mathrm{S}}\mathbb{B}$ *are at least as stable as* $\mathbb{A}$ *and* $\mathbb{B}$*:* If $\mathrm{e}^{-\omega t}\mathbb{A}^t x_0$ (resp. $\mathrm{e}^{-\omega t}\mathbb{B}\tau^t u$) is bounded or converges to zero strongly or weakly, as $t \to +\infty$, then so does $r^{-n}\mathbb{A}^n x_0$ (resp. $r^{-n}\mathbb{B}\tau^n u$), as $\mathbf{N} \ni n \to +\infty$. Thus, we only have to show that $\Delta^{\mathrm{S}}\mathbb{A}$ (resp. $\Delta^{\mathrm{S}}\mathbb{B}$) cannot be more stable than $\mathbb{A}$ (resp. $\mathbb{B}$).

3° $\Delta^{\mathrm{S}}\mathbb{A}$ *is as stable as* $\mathbb{A}$*:* If $\|r^{-n}\mathbb{A}^n\| \le M$ for all n, then

$$\|\mathrm{e}^{-\omega(n+h)}\mathbb{A}^{n+h}\| = \|r^{-n}\mathbb{A}^n \mathrm{e}^{-\omega h}\mathbb{A}^h\| \le M \max_{h\in[0,1]} \|\mathrm{e}^{-\omega h}\mathbb{A}^h\|. \tag{13.78}$$

Thus, the r-stability of A implies the ω-stability of $\mathbb{A}$. From (13.78) one also observes that if $r^{-n}A^n x_0 \to 0$ strongly, then $\mathrm{e}^{-\omega t}\mathbb{A}(t)x_0 \to 0$ strongly.

Assume then that A is weakly r-stable, i.e., that $\|r^{-n}A^n\| \le M$ for all n and $\langle z_0, r^{-n}A^n x_0\rangle \to 0$ for $x_0, z_0 \in H$.

Let $x_0, z_0 \in H$ be arbitrary. Set $K := \{e^{-\omega h}\mathbb{A}(h)x_0 \,|\, h \in [0,1]\}$, and $T_n x := \langle z_0, r^{-n}A^n x\rangle$ (thus, $T_n \in H^*$). It follows that $T_n x \to 0$ uniformly on K, by Lemma A.3.4(H2), hence $\langle z_0, e^{-\omega t}\mathbb{A}(t)x_0\rangle \to 0$, as $t \to +\infty$. Thus, $\mathbb{A}$ is weakly ω-stable.

4° $\Delta^S\mathbb{B}$ *is as stable as* $\mathbb{B}$*:* If $\mathbb{B}(\Delta^{\ell^2})^{-1} \in \mathcal{B}(\ell^2_r; H)$, then $\mathbb{B} \in \mathcal{B}(L^2_\omega; H)$, since $\Delta^{\ell^2} \in \mathcal{GB}(L^2_\omega, \ell^2_r)$. Thus, $\Delta^S\mathbb{B}$ is r-stable iff $\mathbb{B}$ is ω-stable.

Assume that $\mathbb{B}(\Delta^{\ell^2})^{-1}$ is strongly r-stable. Let $u \in L^2_\omega(\mathbf{R}; U)$. Since $K := \{e^{-\omega h}\tau^h u \,|\, h \in [0,1]\} \subset L^2_\omega$ is compact and $e^{-\omega n}\mathbb{B}\tau^n v \to 0$, as $\mathbf{N} \ni n \to \infty$, for all $v \in K$ (in fact, for all $v \in L^2_\omega$), this convergence is uniform on K, hence $\|e^{-\omega(n+h)}\mathbb{B}\tau^(n+h)u\| < \varepsilon$ for all $n > N_\varepsilon$ and $h \in [0,1]$, so that $\mathbb{B}$ is strongly ω-stable.

The map $\mathbb{B}$ is weakly ω-stable whenever $\mathbb{B}(\Delta^{\ell^2})^{-1}$ is weakly r-stable, as one observes by adding an arbitrary $\Lambda \in H^*$ before $\mathbb{B}$ in the above proof.

(e1) This follows from (c), (d2) and Theorem 13.4.5(g) (it is enough to verify this for static output feedback, because other forms of feedback and injection can be reduced to static output feedback, as in Summary 6.7.1.

(Note that prefices "r.c.", "l.c.", "d.c." and "jointly" would require existence of certain kinds of TIC or tic operators. Because $\Delta^S\mathbb{X} \in \text{tic}_\infty \not\Rightarrow \mathbb{X} \in \text{TIC}_\infty$ (cf. Theorem 13.4.5), existence results for $\Delta^S\Sigma$ do not necessarily tell anything about Σ. If $\begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ admissible for Σ, then so are $\Delta^S \begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\Delta^S \left[\frac{\mathbb{H}}{\mathbb{G}}\right]$, they are even jointly admissible, by Lemma 13.3.17(a), but we do not know whether $\begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are then jointly admissible. Even if they were jointly admissible with some $\mathbb{E} \in \text{TIC}_\infty$, and $\Delta^S \begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\Delta^S \left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ were jointly stabilizing, we do not know whether $\begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ would then be jointly stabilizing (unless $\Delta^S \begin{bmatrix} \mathbb{K} \,|\, \mathbb{F} \end{bmatrix}$ and $\Delta^S \left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are known to be jointly stabilizing with some $\Delta^S\mathbb{E}'$, where $\mathbb{E}' \in \text{TIC}_\infty$).)

(e2) This follows easily from (e1) and (d2). (Note that the converse holds (at least) for static feedback.)

(e3) For optimizability this follows from (f). For estimatability, one could very carefully verify this directly, but the easiest way is to note that the final state estimation problems (FSEPs) for Σ and $\Delta^S\Sigma$ are obviously equivalent. (By Theorem 5.3 of [WR00], the FSEP for Σ has a solution iff Σ is estimatable; it is even easier to verify this in discrete time.)

(f1)&(f2) The equalities and inclusions follow from (a2); also the rest of (f) follows easily from (a)–(d1) and Theorem 13.4.5(a1)&(a2). (Note that $\Delta^{\ell^2}x$ may be bounded even if x is unbounded, hence we can state mere inclusions for $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$ in both (f1) and (f2).)

(g) This follows from the facts that the norms $\|\cdot\|_{\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}}$ and $\|\Delta^{\ell^2}\cdot\|_{\mathcal{U}^{e^{\vartheta},\Delta^S\Sigma}_{[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}]}}$ are obviously equivalent (use Theorem 13.4.5(a1)) (they are equal if $\vartheta = 0$), and that, similarly, $\|\cdot\|'_{\mathcal{U}_{\text{exp}}} = \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\text{exp}}}$, $\|\cdot\|'_{\mathcal{U}_{\text{out}}} = \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\text{out}}}$, $\|\cdot\|'_{\mathcal{U}_{\text{sta}}} \geq \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\text{sta}}}$ and $\|\cdot\|'_{\mathcal{U}_{\text{str}}} \geq \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\text{str}}}$ (see Lemma 8.4.2 for the norms) □

We end this section by listing the basic properties of the discretization of I/O maps:

Theorem 13.4.5 ($\mathbf{L^2_\omega(R;}U\mathbf{)} \cong \boldsymbol{\ell}^2_{\mathbf{e^\omega}}\mathbf{(Z;L^2([0,1);}U\mathbf{))}$) *Let $\omega \in \mathbf{R}$, and set $r := e^{\omega}$. Then the following holds:*

(a1) Let $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$. Then $u \in \mathrm{L}^2_\omega \Leftrightarrow \Delta^{\ell^2} u \in \ell^2_r$, and $u \in \mathrm{L}^2_{\mathrm{c}} \Leftrightarrow \Delta^{\ell^2} u \in \mathrm{c_c}$.

Moreover, Δ^{ℓ^2} is an isomorphism of L^2_ω onto ℓ^2_r (i.e., $\Delta^{\ell^2} \in \mathcal{GB}(\mathrm{L}^2_\omega, \ell^2_r)$). For $\omega = 0$ this isomorphism is an isometry.

(a2) We have $\langle \Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle = \langle f, g\rangle$ for $f \in \mathrm{L}^2_\omega(\mathbf{R};U)$, $g \in \mathrm{L}^2_{-\omega}(\mathbf{R};U)$.

(b1) The map $\Delta^{\mathrm{S}}_{U\to Y} : \mathbb{E} \mapsto \Delta^{\ell^2}_Y \mathbb{E}(\Delta^{\ell^2}_U)^{-1}$ is an isomorphism of $\mathcal{B}(\mathrm{L}^2_\omega, \mathrm{L}^2_\omega)$ onto $\mathcal{B}(\ell^2_r, \ell^2_r)$. For $\omega = 0$ this isomorphism is an isometry.

(b2) Moreover, Δ^{S} commutes with (anticausal) adjoints and valid compositions of operators. Thus,

$$\Delta^{\mathrm{S}}\mathbb{E}^* = (\Delta^{\mathrm{S}}\mathbb{E})^*, \ \ \Delta^{\mathrm{S}}(\mathbb{E}\mathbb{F}) = (\Delta^{\mathrm{S}}\mathbb{E})(\Delta^{\mathrm{S}}\mathbb{F}), \quad \Delta^{\mathrm{S}}\mathbb{E}^{-1} = (\Delta^{\mathrm{S}}\mathbb{E})^{-1} \tag{13.79}$$

for $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathrm{L}^2_\omega(\mathbf{R};Y))$ and $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};Y), \mathrm{L}^2_\omega(\mathbf{R};H))$. (But $\Delta^{\mathrm{S}}\mathbb{E}^{\mathrm{d}} \neq (\Delta^{\mathrm{S}}\mathbb{E})^{\mathrm{d}}$ in general.)

(b3) The map Δ^{S} is an isomorphism of TI *into* ti *and of* TIC *into* tic.

(c) $\Delta^{\ell^2}\pi_\pm = \pi^\pm\Delta^{\ell^2}$, $\ \Delta^{\mathrm{S}}\pi_\pm = \pi^\pm$, $\ \Delta^{\ell^2}\tau(n) = \tau^n\Delta^{\ell^2}$, $\ \Delta^{\mathrm{S}}\tau(n) = \tau^n$.

(d) Let $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U))$. Then $\pi_\pm\mathbb{E}\pi_\pm$ is invertible on $\pi_\pm\mathrm{L}^2_\omega$ iff $\pi^\pm(\Delta^{\mathrm{S}}\mathbb{E})\pi^\pm$ is invertible on $\pi^\pm\ell^2_r$.

(e) Let $\mathbb{E}, P \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$. Then $\Delta^{\mathrm{S}}\mathbb{E} \geq 0$ [$\gg 0$] on $(\Delta^{\mathrm{S}}P)\ell^2$ iff $\mathbb{E} \geq 0$ [$\gg 0$] on $P\mathrm{L}^2$.

(f) Δ^{S} is an isomorphism of $\mathrm{TI}_\omega(U,Y)$ into $\mathrm{ti}_r(\mathrm{L}^2([0,1);U), \mathrm{L}^2([0,1);Y))$. Moreover, if $\mathbb{E} \in \mathrm{TI}_{\omega'}(U,Y)$ for some $\omega' \in \mathbf{R}$, then

$$\mathbb{E} \in \mathrm{TI}_\omega(U,Y) \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathrm{ti}_r, \ \ \mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{G}\mathrm{ti}_r, \tag{13.80}$$

$$\mathbb{E} \in \mathrm{TIC}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathrm{tic}_r, \ \ \mathbb{E} \in \mathcal{G}\mathrm{TIC}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{G}\mathrm{tic}_r, \tag{13.81}$$

$$\mathbb{E} \in \mathcal{B}(U,Y) \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{B}(U_\Delta, Y_\Delta). \tag{13.82}$$

However, the time-invariance (resp. staticity) of $\Delta^{\mathrm{S}}\widetilde{\mathbb{E}}$ does not imply that of $\widetilde{\mathbb{E}}$ for general $\widetilde{\mathbb{E}} \in \mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$.

(g) Operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ and $\mathbb{M} \in \mathrm{TIC}(U)$ are q.r.c. iff $\Delta^{\mathrm{S}}\mathbb{N}$ and $\Delta^{\mathrm{S}}\mathbb{M}$ are q.r.c.

(h1) Let $\mathbb{D} \in \mathrm{TI}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$. Then $\mathbb{D}$ is minimax J-coercive iff $\Delta^{\mathrm{S}}\mathbb{D}$ minimax J-coercive.

(h2) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$. Then $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$ iff $\Delta^{\mathrm{S}}\mathbb{D}$ [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$.

(m) **(MTIC$_\mathbf{d} \to \boldsymbol{\ell}^1$)** *Let $T > 0$. Let $\mathbb{E} := \sum_j A_j\delta_{jT}* \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U), \mathrm{L}^2(\mathbf{R};Y))$ and $\mathbb{F} := \sum_j A_j e_j * \in \mathcal{B}(\ell^2, \ell^2)$, where $A_j \in \mathcal{B}(U,Y)$, $\sum_j \|A_j\| < \infty$, and*

$e_j = \chi_{\{j\}} \in \ell^1(\mathbf{Z})$ (i.e., $(e_j * g)_k = g(k-j)$ for all $g \in \ell^2(\mathbf{Z};U_\Delta)$). Redefine $\Delta_U^{\ell^2} u := (\tau(nT)\pi_{[nT,nT+1)}u)$.

Then $\Delta^{\mathrm{S}}\mathbb{E} = \mathbb{F}$, and $\widehat{\mathbb{E}}(-it/T) = \widehat{\mathbb{F}}(\mathrm{e}^{it})$ for $t \in \mathbf{R}$; in particular, $\widehat{\mathbb{E}}(ir + i2\pi/T) = \widehat{\mathbb{F}}(\mathrm{e}^{-iTr})$ for all $r \in \mathbf{R}$. If $A_j = 0$ for $j < 0$, then we also have $\widehat{\mathbb{E}}(s) = \widehat{\mathbb{E}}(s + i2\pi/T) = \widehat{\mathbb{F}}(\mathrm{e}^{-Ts})$ for all $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$.

The results hold also when we use $[0,T)$ for arbitrary $T > 0$ instead of $T = 1$ (this is illustrated in (m)); except for (e), (a2) and the adjoint formula of (b2), we can even let U and Y be arbitrary Banach spaces.

However, $\mathbf{Я}_{-1}\Delta^{\mathrm{S}} \neq \Delta^{\mathrm{S}}\mathbf{Я} \neq \mathbf{Я}\Delta^{\mathrm{S}}$ and $\Delta^{\mathrm{S}}\mathbb{E}^{\mathrm{d}} \neq (\Delta^{\mathrm{S}}\mathbb{E})^{\mathrm{d}}$ in general.

Because of the (algebraic and topologic) isomorphism, things such as exponential stability and coprimeness are preserved under the discretization (in both ways).

The formula $s \mapsto \mathrm{e}^{-sT}$ in (m) maps any strip of $\overline{\mathbf{C}^+} \cup \{\infty\}$ of height $2\pi/T$ one-to-one and onto $\overline{\mathbf{D}}$. However, for general $\mathbb{D} \in \mathrm{TIC}$, the connection between $\widehat{\mathbb{D}}$ and $\widehat{\Delta^{\mathrm{S}}\mathbb{D}}$ seems to be rather complicated.

Note also the differences to Theorem 13.2.3: the transform Δ^{ℓ^2} (or Δ^{S}) treats also the unstable case and commutes with time-shifts, but it does not map TI onto ti, it does not commute with time reflection, and U and Y are different from U_Δ and Y_Δ.

Thus, the Cayley transform is usually better for transferring stable I/O results, whereas the discretization can be used to transform uniqueness results (including equality results) from discrete time to continuous time (and existence results in the other direction), including the unstable results and those concerning more than just the I/O maps of systems.

Also systems (not merely I/O maps) can be mapped to each other by using the Cayley transform (see, e.g., p. 212–213 and 331–332 of [CZ]). However, the transform of $\left(\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \\ \hline C_{\mathrm{d}} & D_{\mathrm{d}} \end{array}\right)$ requires that $I + A_{\mathrm{d}} \in \mathcal{GB}$, and we only know the preservation of I/O-stability and exponential stability, not, e.g., internal [P-]stability. Moreover, this does not apply to continuous-time systems with unbounded generators.

Proof of Theorem 13.4.5: (a1) Clearly $u \mapsto (\Delta_U^{\ell^2} u)_k \in \mathrm{L}^2([0,1);U)$ is linear, continuous and onto. Obviously, $u \in \mathrm{L}^2_{\mathrm{c}} \Leftrightarrow \Delta^{\ell^2} u \in \mathrm{c}_{\mathrm{c}}$. For $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ we have

$$\|\Delta^{\ell^2} u\|^2_{\ell^2_r} = \sum_{n\in\mathbf{Z}} \|r^n \pi_{[0,1)}\tau(n)u\|_2^2 = \sum_{n\in\mathbf{Z}} \int_n^{n+1} \|\mathrm{e}^{-\omega n} u(t)\|^2 \, dt, \tag{13.83}$$

$\|u\|^2_{\mathrm{L}^2_\omega} = \sum_{n\in\mathbf{Z}} \int_n^{n+1} \|\mathrm{e}^{-\omega t} u(t)\|^2 \, dt$, and the quotient $\mathrm{e}^{-\omega t}/\mathrm{e}^{-\omega n} = \mathrm{e}^{-\omega(t-n)}$ is between 1 and $\mathrm{e}^{-\omega} = r^{-1}$. Therefore, (a1) holds. (Note that there are no norm equivalence constants that would suit for every $\omega \in \mathbf{R}$.)

(a2) Now $\langle \Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle_{\ell^2_r, \ell^2_{-r}} := \sum_n \int_n^{n+1} \langle f, g\rangle_U \, dt = \langle f, g\rangle_{\mathrm{L}^2_\omega, \mathrm{L}^2_{-\omega}}$ (cf. (13.83)).

(b1) This follows from (a1) and (a2): Δ^{S} has the inverse $\mathcal{B}(\ell^2(\mathbf{Z};Y_\Delta), \ell^2(\mathbf{Z};U_\Delta)) \ni \mathbb{E} \mapsto (\Delta_U^{\ell^2})^{-1}\mathbb{E}\Delta_Y^{\ell^2} \in \mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$ and it is isometric by the equation $\|\Delta^{\mathrm{S}}\mathbb{F}\Delta_U^{\ell^2} f\| = \|\Delta_U^{\ell^2}\mathbb{F}f\| = \|\mathbb{F}f\|$, valid for $f \in \mathrm{L}^2(\mathbf{R};U)$, $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)$.

(b2) This is obvious from the definition except for $\Delta^{\mathrm{S}}\mathbb{E}^* = (\Delta^{\mathrm{S}}\mathbb{E})^*$, which holds because the equation $\langle \mathbb{E}f, g\rangle = \langle f, \mathbb{E}^* g\rangle$ is equivalent to $\langle (\Delta^{\mathrm{S}}\mathbb{E})\Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle = \langle \Delta^{\ell^2} f, (\Delta^{\mathrm{S}}\mathbb{E}^*)\Delta^{\ell^2} g\rangle$, by (a2).

(b3) This follows from (b1) and (g).

(c) The $\Delta_U^{\ell^2}$ formulae are obvious, the Δ^{S} formulae follow. It is obvious that Δ^{ℓ^2} does not commute with time reflection.

(d) By (b), $\mathbb{G}\pi_+\mathbb{E}\pi_+ = \pi_+ = \pi_+\mathbb{E}\pi_+\mathbb{G}$ for some $\mathbb{G} \in \mathcal{B}(\pi_\pm \mathrm{L}^2_\omega)$ (we may identify $\mathbb{G}$ with $\pi_+\mathbb{G}\pi_+ \in \mathcal{B}(\mathrm{L}^2_\omega)$) iff $(\Delta^{\mathrm{S}}\mathbb{G})\pi^+(\Delta^{\mathrm{S}}\mathbb{E})\pi^+ = \pi^+ = \pi^+(\Delta^{\mathrm{S}}\mathbb{E})\pi^+(\Delta^{\mathrm{S}}\mathbb{G})$.

(e) Now $\langle \Delta^{\mathrm{S}}\mathbb{E}\Delta^{\mathrm{S}} P\Delta^{\ell^2} f, \Delta^{\mathrm{S}} P\Delta^{\ell^2} f\rangle \geq 0$ for all $\Delta^{\ell^2} f \in \ell^2$ iff $\langle \mathbb{E}Pf, Pf\rangle \geq 0$ for all $f \in \mathrm{L}^2$. By replacing $\mathbb{E}$ by $\mathbb{E} - \varepsilon I$ we get the "$\gg 0$" claim.

(f) By (c), Δ^{S} preserves time-invariance, hence (see (b) too) Δ^{S} is an isomorphism (into).

Because of (b), the "$\Rightarrow$" parts of (13.80) and the first "$\Leftarrow$" are trivial. The second "$\Leftarrow$" follows from the first, because $\mathbb{E}^{-1} \in \mathcal{GB}(\mathrm{L}^2)$ inherits the time-invariance of $\mathbb{E}$ (since $\tau \in \mathcal{GB}$).

Because $\Delta^{\mathrm{S}}\pi_-\mathbb{E}\pi_+ = \pi^-(\Delta^{\mathrm{S}}\mathbb{E})\pi^+$ (by (b)), the next two equivalences follow.

By the above results, $\mathbb{E} \in \mathrm{TIC} \cap \mathrm{TIC}^* \Leftrightarrow \mathbb{E} \in \mathrm{tic} \cap \mathrm{tic}^*$. By Lemmas 2.1.7 and 13.1.2, $\mathrm{TIC} \cap \mathrm{TIC}^* = \mathcal{B}$ and $\mathrm{tic} \cap \mathrm{tic}^* = \mathcal{B}$.

The counter-example is obtained by choosing a static $\Delta^{\mathrm{S}}\mathbb{E}$ so that it is not "time-invariant on $\pi_{[0,1)}\mathrm{L}^2$", e.g., take $E = \pi_{[0,1)}\tau^{1/2}\pi_{[0,1)} \in \mathcal{B}(U_\Delta)$, $(\mathbb{E}u)(t) := Eu(t)$ $(t \in \mathbf{Z})$.

(g) This follows from (a1).

(h1) This follows from (a1), (e), and Definition 11.4.1.

(h2) This follows from (a1), (e) and Lemma 8.4.11(a1)&(a2).

(m) We have

$$\begin{aligned}
[\Delta^{\mathrm{S}}(A_j\delta_{jT}*)]\Delta_U^{\ell^2} f &= \Delta_U^{\ell^2} A_j\delta_{jT} * f = \Delta_U^{\ell^2} A_j\tau(-jT)f \\
&= (\tau((k-j)T)\pi_{[(k-j)T,(k-j+1)T)}A_j f)_{k\in\mathbf{Z}} \\
&= A_j((\Delta_U^{\ell^2} f)_{k-j})_{k\in\mathbf{Z}} = A_j e_j * \Delta_U^{\ell^2} f,
\end{aligned}$$

for each $f \in \mathrm{L}^2(\mathbf{R};U)$, i.e., for each $\Delta_U^{\ell^2} f \in \ell^2(\mathbf{Z};U_\Delta)$. Therefore $\Delta^{\mathrm{S}}\mathbb{E} = \mathbb{F}$.

We have $(\mathcal{L}\delta_{jT})(s) = \mathrm{e}^{-jTs} = (\mathrm{e}^{-Ts})^j = (\mathcal{Z}e_j)(\mathrm{e}^{-Ts})$ for each $j \in \mathbf{Z}$ and $s \in i\mathbf{R}$ (or for each $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$ if $A_j = 0$ for $j < 0$), hence $(\mathcal{L}\mathbb{E})(s) = (\mathcal{Z}\mathbb{F})(\mathrm{e}^{-Ts})$ for such s; in particular $(\mathcal{L}\mathbb{E})(-it/T) = (\mathcal{Z}\mathbb{F})(\mathrm{e}^{-it})$ for all $t \in \mathbf{R}$ (set $s := -it/T$). $\square$

Just to simplify the notation, we have used $T = 1$ above, although the discretization could be written for a general $[0,T)$:

Remark 13.4.6 (Discretization over $[0,T)$) *As obvious from the proofs, all results of this section could be formulated for discretization over $[0,T)$ ($T > 0$) instead of $[0,1)$ (e.g., $(\Delta^{\ell^2} u)_n := \pi_{[0,T)}\tau(nT)u$ ($n \in \mathbf{Z}$), hence Δ^{ℓ^2} maps $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U) \to \ell^2_{\mathrm{loc}}(\mathbf{Z};\mathrm{L}^2([0,T);U))$).* □

(An alternative proof would be to compress time; the operator $\mathcal{T}_T \in (\mathrm{L}^2_\omega, \mathrm{L}^2_{\omega/T})$ defined by $(\mathcal{T}_T u)(t) := u(Tt)$ is an isomorphism with inverse $\mathcal{T}_{1/T}$; see [Sbook] for details.)

Notes

Time discretization has more or less implicitly been used in [Sal89] and [W94a]. Some basic facts are given in Section 2.4 of [Sbook], but most of this section seems to be new.