Chapter 15

Quadratic Minimization

Large increases in cost with questionable increases in performance can be tolerated only in race horses and women.
— Lord Kalvin

This chapter is mostly the discrete-time counterpart of Chapter 10. In Section 15.1, we first note that the discrete-time form of Chapter 10 is true, and then we present some further discrete-time specific results on minimization and H^2 control problems; this corresponds to Sections 10.1, 10.2 and 10.4. Most results become as simple as for WPLSs with bounded input and output operators (the main exception is that $S \neq D^* JD$ in general).

Section 15.2 contains the discrete-time variant of Section 10.3, i.e., the relations between different classical coercivity assumptions.

In Section 15.3, we present discrete-time variants of Section 10.6, namely necessary and sufficient conditions for the Popov operator $\mathbb{D}^* J\mathbb{D}$ to be uniformly positive ($\geq \varepsilon I$ for some $\varepsilon > 0$). Section 15.4 is the discrete-time counterpart of Section 10.5, containing extended generalized forms of The Strict Bounded (Real) Lemma and The Strictly Positive (Real) Lemma. In Section 15.5, we show that any strongly stabilizing solution of a positive DARE (or of the corresponding Riccati inequality) is the maximal one.

Throughout this chapter, we assume that Standing Hypotheses 14.0.1 and 10.6.6 hold (the latter is satisfied by, e.g., $\widetilde{\mathcal{A}}_+ = \mathrm{tic}$).

15.1 J-critical control and minimization

Everything takes longer, costs more, and is less useful.
— Erwin Tomash

We first state that the discrete-time versions of Chapter 10 hold, and then we give more elegant discrete-time-specific results. For example, there is a unique minimizing exponentially stabilizing control iff the eDARE has an exponentially stabilizing solution with positive signature operator, by Corollary 15.1.3. Under suitable coercivity assumptions, one need not check whether the solution of the eDARE is stabilizing; see, e.g., Corollary 15.1.6; the special case for an LQR is given in Corollary 15.1.7. The minimizing state feedback operator always solves also the H^2 state feedback control problem, as shown in Theorem 15.1.8. For finite-dimensional systems, all this this is essentially known, and a special case of Corollary 15.1.7 was treated in [CZ].

We start by verifying that the results of Chapter 10 hold in their discrete-time forms too:

Theorem 15.1.1 (Chapter 10 applies) *The results of Chapter 10 hold modulo the changes given in (13.63).*

Standing Hypothesis 10.6.6 is satisfied by, e.g., $\widetilde{\mathcal{A}}_+ = \mathrm{tic}$. *Hypothesis 10.6.1(1.)–(6.) are satisfied by all* $\mathbb{D} \in \mathrm{tic}$ *(when we interpret (13.63) so that the requirement "*$(S=)X^*X = D^*JD$*" is removed).*

Recall that CAREs and B_{w}^*-CAREs must be replaced by DAREs; e.g., the LQR-CARE and the LQR-B_{w}^*-CARE become the "LQR-DARE" (15.5). Naturally, this also applies to simplified forms of CAREs (i.e., to CAREs under simplifying assumptions); e.g., (10.3) becomes (15.6). Analogously, equations such as $S = D^*JD$ or $S = R + D^*QD$ must be replaced by $S = D^*JD + B^*\mathcal{P}B$ or by $S = R + D^*QD + B^*\mathcal{P}B$, respectively.

To make things clear, we have rewritten most main results of Chapter 10 into this chapter; due to bounded input and output operators, most of these results are stronger and more elegant than their continuous-time counterparts. The main differences are that a unique minimizing (or J-critical) control is always of state feedback form (as in continuous time in the case of bounded B), and that we need no regularity considerations (nor corresponding assumptions).

This section contains most main results of Sections 10.1 and 10.2, starting from the most general ones. See Section 15.2 for the discrete-time form of Section 10.3, Theorem 15.1.8 for the H^2 problem (Section 10.4), Section 15.3 for Section 10.6, and Section 15.4 for Section 10.5.

Proof of Theorem 15.1.1: The same proofs apply mutatis mutandis. For Hypothesis 10.6.1(1.)–(6.), recall that (13.63) removes any regularity requirements; see the proof of Lemma 10.6.2(c)(8.) for "(6.)" (and recall that (13.63) replaces the B_{w}^*-CARE by the DARE). □

By combining Theorem 14.1.6 and Theorem 9.9.1(a2), we obtain:

Theorem 15.1.2 (Unique minimum) *The following are equivalent:*

(i) there is a unique minimizing control for each $x_0 \in H$;

(ii) $\mathcal{J}(0,\cdot) \geq 0$, and the eDARE has a [unique] $\mathcal{U}_*^*$-stabilizing solution with $S > 0$;

(iii) there is a unique minimizing state feedback operator for Σ.

If $(\mathcal{P}, S, K)$ is as in (ii), then the minimizing state feedback is given by $u_{\min}(t) = Kx(t)$, the minimal cost is $\mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P}x_0 \rangle$, and the cost for closed-loop input $u_{\circlearrowleft} \in \mathrm{c_c}(\mathbf{N}; U)$ is given by

$$\langle x_0, \mathcal{P}x_0 \rangle_H + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle_{\ell^2(\mathbf{N};U)}. \tag{15.1}$$

□

(The condition $\mathcal{J}(0,\cdot) \geq 0$ is redundant at least for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, by Theorem 9.9.1(k). See Theorem 9.9.1 for more on $(\mathcal{P}, S, K)$.)

A solution with $S \geq 0$ would still be minimizing (and $\mathcal{P}$, S and K might all be unique) but the minimizing control would no longer be unique; see Theorem 9.9.1(f2) for details.

Combine Theorem 8.4.2 and Theorem 9.9.1(k) to the above theorem to obtain

Corollary 15.1.3 *There is a unique minimizing control over $\mathcal{U}_{\exp}$ for each $x_0 \in H$ iff the eDARE has an exponentially stabilizing solution with $S > 0$.* □

If $\mathbb{D}$ is positively J-coercive, then the existence of any allowable control implies the existence of a unique minimizing control:

Corollary 15.1.4 (Coercive minimization) *Assume that there is $\varepsilon > 0$ s.t.*

$$\mathcal{J}(0,u) \geq \varepsilon \|u\|_{\mathcal{U}_*^*}^2 \qquad (u \in \mathcal{U}_*^*(0)). \tag{15.2}$$

Assume that Z^{s} is reflexive (e.g., that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\exp}$). Then the following are equivalent:

(i) there is a unique minimizing control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;

(ii) $\mathcal{U}_*^*(x_0) \neq 0$ for each $x_0 \in H$;

(iii) the DARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$.

If (iii) holds, then $S \gg 0$, K is the unique J-critical state feedback operator, and the minimal cost is given by $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$ $(x_0 \in H)$. □

See Lemma 8.4.2 for most common $\|\cdot\|_{\mathcal{U}_*^*}$'s. See Proposition 10.3.1 for equivalent conditions for (15.2) in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.

Proof: By Lemma 8.2.3(c2), D is positively J-coercive, hence Theorems 8.2.5 and 8.1.10 apply. By Corollary 8.1.8, the unique J-critical control is strictly minimizing (on $\mathcal{U}(x)$). □

A special case of this is the standard LQR problem; even in somewhat more general LQR setting, the existence of a solution becomes equivalent to positive J-coercivity:

Corollary 15.1.5 (Standard minimization) *Assume that $J \geq 0$ and $D^*JD \gg 0$. Then the following are equivalent:*

(i) There is a minimizing $u \in \mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The DARE has a $\mathcal{U}_{\mathrm{exp}}$-stabilizing solution $(\mathcal{P}, S, K)$.

(iii) Σ is exponentially stabilizable, and $\mathbb{D}$ is J-coercive (i.e., any (hence all) of (i)–(iii) of Proposition 15.2.2 holds).

If (ii) holds, then $\mathcal{P} \geq 0$, $S \gg 0$, K is the unique J-critical state feedback operator, and the minimal cost is given by $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$ $(x_0 \in H)$.

If we remove the assumptions $J \geq 0$ and $D^*JD \gg 0$, then we must assume that u is unique in (i), and $S := D^*JD + B^*\mathcal{P}B \gg 0$ in (i) and (ii) above; then also (i') and (ii') of Proposition 15.2.2 become merely sufficient in (iii). □

(This follows from Proposition 15.2.2(e)&(f1) and Theorem 14.2.7.)

See Proposition 15.2.2 and the comments following it for additional equivalent conditions. However, even for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, there may be a minimizing control even without J-coercivity, see Example 9.13.3 (which can easily be modified to a discrete-time example).

For the standard LQR cost function $\mathcal{J}(x_0, u) := \|y_1\|_2^2 + \|u\|_2^2$ (i.e., $\mathbb{C} = \left[\begin{smallmatrix} \mathbb{C}_1 \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$, $J = I$) and other similar ones, we have the following:

Corollary 15.1.6 (Coercive minimization: $\mathcal{P}_+$ and $\mathcal{P}_-$) *Assume that $J \gg 0$ and $D^*JD \gg 0$, and that there is $\varepsilon > 0$ s.t.*

$$\mathcal{J}(x_0, u) := \langle \mathbb{C}x_0 + \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}u) \rangle_{\ell^2(\mathbf{N};Y)} \geq \varepsilon \|u\|_2^2 \quad (x_0 \in H,\ u : \mathbf{N} \to U). \tag{15.3}$$

Then the following are equivalent:

(i) the eDARE has a nonnegative solution;

(ii) **(FCC)** *$\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$ (i.e., $\inf_u \mathcal{J}(x_0, u) < \infty$) for all $x_0 \in H$,*

(iii) **($\mathcal{U}_{\mathrm{out}}$-min)** *the eDARE has a smallest nonnegative solution $(\mathcal{P}_-, S_-, K_-)$; $S_- \gg 0$; and K_- is strictly minimizing over all $u : \mathbf{N} \to \infty$ (hence over $\mathcal{U}_{\mathrm{out}}$).*

Assume (iii). Then

(a) **($\mathcal{U}_{\mathrm{exp}}$-min)** *The following are equivalent:*

(i') there is a minimizing control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$;

(ii') the [e]DARE has an exponentially stabilizing solution.

If (i')–(ii') hold, then eDARE has (a) greatest solution $(\mathcal{P}_+, S_+, K_+)$, $S_+ \geq S_- \gg 0$, and K_+ is strictly minimizing over $\mathcal{U}_{\mathrm{exp}}$.

(b) If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable (resp. Σ is exponentially q.r.c.-stabilizable or exponentially detectable; e.g., $C^*C \gg 0$), then $\mathcal{P}_-$ is the unique nonnegative solution of the eDARE, and it is strongly stabilizing (resp. exponentially q.r.c.-stabilizing).

Thus, then this solution is strictly minimizing over $\{u : \mathbf{N} \to U\}$, $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ (resp. and $\mathcal{U}_{\mathrm{exp}}$).

Thus, if $C^*C \gg 0$, then, by (b), any nonnegative solution is the unique nonnegative solution, minimizing over all $u : \mathbf{N} \to U$ and exponentially stabilizing.

By Theorem 14.1.4(e)&(a), $\mathcal{P}_-$ is the unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the eDARE (if (iii) holds), and $\mathcal{P}_+$ is the unique stabilizing solution of the eDARE (if (ii') holds).

See Theorem 9.2.10 (or Section 10.1) for a continuous-time counterpart.

Proof: 1° *(iii)⇒(i)⇒(ii):* Obviously, (iii) implies (i). Since any nonnegative solution of the eDARE is SOS-stabilizing, by Proposition 10.7.3(d), we have "(i)⇒(ii)".

2° *(ii)⇒(iii):* Assume (ii). Since $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$, there is a unique minimizing control over $\mathcal{U}_{\text{out}}(x_0)$ for each $x_0 \in H$, by Theorem 8.4.3, and this control corresponds to the smallest nonnegative solution $(\mathcal{P}_-, S_-, K_-)$ of the eDARE, as shown in (the proof of) Theorem 9.9.1(a2). Since $S_- \geq D^*JD \gg 0$, condition (iii) holds.

(a) If $(\mathcal{P}_+, S_+, K_+)$ is an exponentially stabilizing solution of the eDARE (as in (ii')), then K_+ is minimizing over $\mathcal{U}_{\text{exp}}$, by Theorem 9.8.5 and Theorem 9.9.1(a2), $S_+ \geq S_- \gg 0$ and $\mathcal{P}_+$ is the greatest solution of the eDARI, by Theorem 15.5.2.

To complete to proof of (a), we assume (i') and prove (ii'). The minimizing control for $x_0 = 0$ is obviously unique, namely $u = 0$. Therefore, the minimizing control over $\mathcal{U}_{\text{exp}}(x_0)$ is unique for any $x_0 \in H$, by Lemma 8.3.8. Consequently, there is a $\mathcal{U}_{\text{exp}}$-stabilizing solution of the eDARE, by Theorem 14.1.6.

(b) 1° Assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable. Let $\mathcal{P} \geq 0$ be a solution of the eDARE with closed-loop system $\Sigma_\circlearrowleft$. Then $\mathcal{P}$ is SOS-stabilizing, by Proposition 10.7.3(d), so that $\mathbb{A}_\circlearrowleft = \mathbb{A} + \mathbb{B}\mathbb{K}_\circlearrowleft$ and $\mathbb{B}_\circlearrowleft = \mathbb{B}\mathbb{M}$ are strongly stable, by Theorem 6.7.15(d); thus, $\mathcal{P}$ is strongly stabilizing. Since the strongly stabilizing solution is unique, by Theorem 14.1.4(a), $\mathcal{P}$ must be equal to $\mathcal{P}_-$ (and $\mathcal{P}_+$).

2° Assume that Σ is exponentially q.r.c.-stabilizable or exponentially detectable. Since any nonnegative solution is SOS-stabilizing, it is exponentially q.r.c.-stabilizing, by (b1) or (c1) of Theorem 6.7.15, hence unique, hence equal to $\mathcal{P}_-$ (and $\mathcal{P}_+$).

3° Since $\mathcal{P}_-$ is $\mathcal{U}_{\text{out}}$-stabilizing, it satisfies (PB) over $\mathcal{U}_{\text{out}}$, hence over the smaller classes $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$ too. Since $\mathcal{P}_-$ is strongly (resp. exponentially) stabilizing, we have $\mathbb{K}_\circlearrowleft x_0 \in \mathcal{U}_{\text{str}}(x_0) \subset \mathcal{U}_{\text{sta}}(x_0) \subset \mathcal{U}_{\text{out}}(x_0)$ (resp. and $\mathbb{K}_\circlearrowleft x_0 \in \mathcal{U}_{\text{exp}}(x_0)$) for all $x_0 \in H$ for the corresponding $\mathbb{K}_\circlearrowleft$, so that $\mathcal{P}_-$ is also $\mathcal{U}_{\text{sta}}$-stabilizing and $\mathcal{U}_{\text{str}}$-stabilizing (resp. and $\mathcal{U}_{\text{exp}}$-stabilizing). □

By substitutions $J := \left[\begin{smallmatrix} Q & 0 \\ 0 & R \end{smallmatrix}\right]$, $\mathbb{C} \mapsto \left[\begin{smallmatrix} \mathbb{C} \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} \mapsto \left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right]$, we obtain the cost function

$$\mathcal{J}(x_0, u) := \sum_{k=0}^{\infty} (\langle y_k, Qy_k \rangle_Y + \langle u_k, Ru_k \rangle_U) \tag{15.4}$$

and hence the following corollary (of Corollary 15.1.6):

Corollary 15.1.7 (LQR: $\sum_{j=0}^{\infty}(\|y_j\|_Y^2 + \|u_j\|_U^2)$) *Let $R, Q \gg 0$. Then the following are equivalent:*

(i) there is a $\langle y, Qy\rangle_{\ell^2} + \langle u, Ru\rangle_{\ell^2}$*-minimizing control over all* $u : \mathbf{N} \to U$ *for each* $x_0 \in H$*;*

(ii) for each $x_0 \in H$ *there is* $u \in \ell^2(\mathbf{N}; U)$ *s.t.* $y \in \ell^2$*;*

(iii) the DARE

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C^*QC - K^*SK, \\ S = R + D^*QD + B^*\mathcal{P}B, \\ K = -S^{-1}(D^*QC + B^*\mathcal{P}A), \end{cases} \tag{15.5}$$

has a nonnegative solution $\mathcal{P}$.

If (iii) holds, then the smallest nonnegative solution is minimizing over all $u : \mathbf{N} \to U$.

There is a minimizing control over $\mathcal{U}_{\exp}$ *iff the DARE has an exponentially stabilizing solution* $\mathcal{P}_+$*; such a solution is strictly minimizing over* $\mathcal{U}_{\exp}$ *and the greatest nonnegative solution of the DARE.*

If Σ *is exponentially detectable (e.g.,* $C^*C \gg 0$*), then the DARE has at most one nonnegative solution, and such a solution is necessarily strictly minimizing over* $\mathcal{U}_{\exp}$. □

(This follows from Corollary 15.1.6. Above, as elsewhere, $x_{n+1} := Ax_n + Bu_n$, $y_n := Cx_n + Du_n$, and $\mathcal{U}_{\exp}(x_0) = \{u \in \ell^2(\mathbf{N}; U) \,|\, x \in \ell^2\}$. Note that the cost is finite for $u \in \mathcal{U}_{\text{out}}(x_0)$ only.)

Thus, in the LQR problem, we only have to find a maximal or minimal solution, depending whether we wish to require the state to be stable or not. Then we shall check whether this condition satisfies (PB) for $\mathcal{U}_{\text{out}}$ or whether it is exponentially stabilizing; see Theorem 10.1.4(b1)&(b2).

If $D = 0$, then (15.5) reduces to

$$\mathcal{P} = A^*\mathcal{P}A + C^*QC - A^*\mathcal{P}B(R + B^*\mathcal{P}B)^{-1}B^*\mathcal{P}A. \tag{15.6}$$

and the minimizing state feedback reduces to $u = -(R + B^*\mathcal{P}B)^{-1}B^*\mathcal{P}Ax$,

The minimizing state feedback operator always solves also the H^2 state feedback and full-information control problems:

Theorem 15.1.8 ($\mathbf{H}^2$ problem) *Assume that there is a minimizing state feedback operator* K *over* $\mathcal{U}_*^*$*. Let* $B_2 \in \mathcal{B}(W, H)$*,*

Then K *solves the* H^2 *problem (strictly if* K *is strictly minimizing), i.e., it leads to the minimization of the cost* $\|\mathbb{D}u + \mathbb{C}B_2 w_0\|_{\ell^2(\mathbf{N};Y)}$ $(= (2\pi)^{-1/2}\|\widehat{\mathbb{D}}\widehat{u} + C(I - zA)^{-1}B_2\|_{\mathrm{H}^2(\mathbf{D};Y)})$ *(where* $u_k := Kx_k$ *for all* $k \in \mathbf{N}$*), for each* $w_0 \in W$*; see Figure 10.2.* □

(This is Theorem 10.4.2 in its discrete-time form.) Note from Theorem 15.1.1 that also the rest of Section 10.4 holds in its discrete-time form.

Notes

Finite-dimensional minimization problems have been studied extensively, see e.g., [LR] for rather up-to-date results and historical remarks. See also the notes to the sections of Chapter 10.

A classical article on the stable infinite-dimensional discrete-time minimization problems is [Helton76b], where William Helton uses a spectral factorization approach with the factors allowed to be noninvertible, so that also some singular cost functions (with $D^*JC = 0$) are covered, at the cost of less direct results. Under certain assumptions on reachability and cost, Helton shows that a certain positive "eDARE" has a solution iff the Popov operator $\mathbb{D}^*J\mathbb{D}$ can be written as $\mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in$ tic.

A standard LQR result can be found from Exercise 6.34 of [CZ] (close to the LQR case of Corollary 15.1.4).

The finite-dimensional discrete-time H^2-problem is explained and treated on pp. 271–274 of [IOW].

15.2 Standard assumptions in discrete time

The explanation requiring the fewest assumptions is the most likely to be correct.
— William of Occam, (c. 1285 – c. 1349)

This is the discrete-time equivalent of Section 10.3, with essentially the same results.

Positive J-coercivity is the standard coercivity assumption in minimization problems (e.g., LQR and H^2), and it is also posed in H^∞ problems on a part of the system. In this section, we present several equivalent (or sufficient or necessary) conditions for positive J-coercivity over $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{exp}}$. The former means that the output is coercive w.r.t. to the input ($\|y\|_{\ell^2} \geq \varepsilon\|u\|_{\ell^2}$ for some $\varepsilon > 0$ whenever $u, y \in \ell^2$ (and $x_0 = 0$); see Proposition 10.3.1), and the latter means that the output is coercive w.r.t. to both the input and the state ($\|y\|_{\ell^2} \geq \varepsilon\left(\|u\|_{\ell^2} + \|x\|_{\ell^2}\right)$ for some $\varepsilon > 0$ whenever $u, x, y \in \ell^2$ (and $x_0 = 0$); see Proposition 15.2.2).

We start with the simpler one, namely $\mathcal{U}_{\text{out}}$:

Proposition 15.2.1 ($\boldsymbol{\mathcal{U}_{\text{out}}}$: $\boldsymbol{y \in \ell^2 \Rightarrow u \in \ell^2}$) *Proposition 10.3.1 holds also in its discrete-time form. However, Proposition 10.3.2 becomes Proposition 15.2.2, since now S takes the role of D^*JD.*

Proof: (We only need to prove Proposition 10.3.1, since Proposition 15.2.2 is proved below.)

(a)&(c)&(d) The original proofs apply (mutatis mutandis).

(b) Assume that $\dim U \times H \times Y < \infty$ and that $\Sigma = \left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \text{wpls}(U, H, Y)$. Since now $\widehat{\mathbb{D}}$ is rational, so is $\widehat{\mathbb{F}} := \heartsuit^{-1}\widehat{\mathbb{D}}$ (by Proposition 13.2.5(b3); use Lemma 13.2.6 to guarantee that $\|\widehat{\mathbb{F}}(\infty)\| < \infty$ if -1 is among the poles of $\widehat{\mathbb{D}}$). Consequently, $\widehat{\mathbb{F}}$ has a finite-dimensional realization (see, e.g., Section 6.4 of [LR] or Theorem 1.13.2 of [IOW]), and we can apply Proposition 10.3.1(b) to this realization to obtain the same claims (in their discrete-time forms) for Σ. $\square$

The case for $\mathcal{U}_{\text{exp}}$ is more tricky, though yet essentially simpler than that in the continuous time:

Proposition 15.2.2 ($\boldsymbol{\mathcal{U}_{\text{exp}}}$: $\boldsymbol{y \in \ell^2 \Rightarrow u, x \in \ell^2}$) *Let $\Sigma := \left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \text{wpls}(U, H, Y)$. Let $J = J^* \in \mathcal{B}(Y)$, and set*

$$\kappa(x_0, u_0) := \langle (Cx_0 + Du_0), J(Cx_0 + Du_0) \rangle, \qquad \mathcal{J}(0, u) := \langle \mathbb{D}u, J\mathbb{D}u \rangle. \tag{15.7}$$

We have the following implications between the conditions (i)–(iii') given below:

(a) Each of conditions (i)–(iii') is invariant under admissible state feedback (in the sense that if $\Sigma_\flat$ is the corresponding closed-loop system, then $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies (i) (resp. (i'), ...) iff $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ satisfies (i) (resp. (i'), ...)).

(b) If Σ is estimatable, then $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm exp}$; hence then (i) becomes equivalent to (ai) of Proposition 10.3.1.

(c) (i')⇒(i)⇔(i")⇐(ii)⇐(iii)⇒(iii'); and (ii')⇒(ii) (without further assumptions).

(d) **(dim$<\infty$)** *Assume that $\dim U \times H \times Y < \infty$. Then (iii)⇔(iii')⇐(vii).*

Assume, in addition, that Σ is exponentially stabilizable. Then (i), (i"), (ii), (iii), (iii'), (v) and (vi) are equivalent to each other (and to (ai) and (bii)–(biv) of Proposition 10.3.1 if Σ is exponentially detectable). Moreover, in (ii), (ii'), (iii) and (iv), we may replace "$z \in \partial\mathbf{D}$" by "$z \in E$", where $E \subset \partial\mathbf{D}$ is dense.

(e) Assume Σ is exponentially stabilizable, $D^*JD \gg 0$ and $J \geq 0$.

Then (i)–(iii), (v) and (vi) are equivalent, and the word "unique" is redundant in (v).

(f1) If Σ is exponentially stabilizable, then (i')⇔(ii'), and (i)⇔(i")⇔(ii)⇔(iii)⇔(vi)⇒(iii')&(v).

(f2) If Σ is exponentially stabilizable and $\dim U < \infty$, then (i)⇔(v) (see also (f1)).

(i) $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\rm exp}(0)$; i.e., $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\rm exp}$.

(i') $\mathcal{J}(0,u) \geq \varepsilon\|\mathbb{B}\tau u\|_2^2$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\rm exp}(0)$, and $D^*JD \gg 0$.

(i") $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2 + \|\mathbb{D}u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\rm exp}(0)$.

(ii) There is $\varepsilon > 0$ s.t.

$$(z-A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2) \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{15.8}$$

(ii') $D^*JD \gg 0$ and there is $\varepsilon > 0$ s.t.

$$(z-A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) \geq \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{15.9}$$

(iii) There is $\varepsilon > 0$ s.t. $T_z^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_z \geq \varepsilon I$ $(z \in \partial\mathbf{D})$ on $H \times U$, where $T_z := \left[\begin{smallmatrix} A-z & B \\ C & D \end{smallmatrix}\right]$.

(iii') $T_z^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_z > 0$ $(z \in \partial\mathbf{D})$. Equivalently,

$$z \in \partial\mathbf{D} \ \&\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in H \times U \ \&\ (z-A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) > 0. \tag{15.10}$$

(iv) There is $\varepsilon > 0$ s.t. $\langle u_0, \widehat{\mathbb{D}}(z)^* J \widehat{\mathbb{D}}(z) u_0 \rangle \geq \varepsilon\left(\|u_0\|_U^2 + \|(z^{-1}-A)^{-1}Bu_0\|_H^2\right)$ for a.e. $z \in \partial\mathbf{D}$.

(v) There is a unique minimizing $u \in \mathcal{U}_{\rm exp}(x_0)$ for each $x_0 \in H$.

(vi) The DARE has an exponentially stabilizing solution with $S \gg 0$.

(vii) (C,A) has no unobservable nodes on $\partial\mathbf{D}$, $J = I$, $D^*D > 0$ and $D^*C = 0$.

By Example 15.2.3, $D^*JD \gg 0$ (hence neither (i') and (ii')) is not implied by any of (i)–(vi) except (trivially) by (i') and (ii'). (Intuitively, this follows from the fact that $(z^{-1}-A)^{-1}B = z$ is bounded away from zero on $\partial\mathbf{D}$ in the example; the same cannot happen for $(\cdot - A)^{-1}B$ on $i\mathbf{R}$ (for WPLSs).)

Example 15.2.3 Let $\Sigma := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\Sigma \in \mathrm{wpls}(\mathbf{C},\mathbf{C},\mathbf{C})$ is exponentially stable and $\mathbb{D} = \tau^{-1}$ (note that $y_n = x_n = u_{n-1}$ for all n).

Moreover, Σ and $J := 1$ do not satisfy (i') nor (ii') (since $D^*JD = 0$) but do satisfy the other assumptions of Proposition 15.2.2. $\triangleleft$

Thus, the "correct" assumption in (i') and (ii') would be "$S \gg 0$", not "$D^*JD \gg 0$", but the former cannot be formulated without $\mathcal{P}$, hence we prefer having sufficient conditions instead of necessary and sufficient conditions.

This important difference makes several DARE results either "weaker" or more complicated than their CARE counterparts. To make things simpler, we shall often assume that $D^*JD \gg 0$ & $J \geq 0$ even when this condition is not necessary.

If $J \gg 0$ (equivalently, $J = I$), we get the following equivalent forms of above conditions:

(i) $\|\mathbb{D}u\|_2 \geq \varepsilon(\|\mathbb{B}\tau u\|_2 + \|u\|_2)$;

(ii) $(z-A)x_0 = Bu_0 \Rightarrow \|Cx_0 + Du_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U)$;

(ii') $D^*D \gg 0$, and $(z-A)x_0 = Bu_0 \Rightarrow \|Cx_0 + Du_0\|_Y \geq \varepsilon\|x_0\|_H$;

(iii') $T_z := \left[\begin{smallmatrix} A-z & B \\ C & D \end{smallmatrix}\right]$ has a full column rank (i.e., $T_z^*T_z \geq \varepsilon I$) on $H \times U$ for all $z \in \partial\mathbf{D}$;

(iv) $\|\widehat{\mathbb{D}}(z)u_0\| \geq \varepsilon(\|u_0\| + \|(z^{-1}-A)^{-1}Bu_0\|$ for a.e. $z \in \partial\mathbf{D}$ and all $u_0 \in U$

(for some $\varepsilon > 0$).

Proof of Proposition 15.2.2: The original proofs apply mutatis mutandis (this requires slightly more than (13.63)).

In fact, the proofs become much easier in most cases, since we do not have to worry about regularity; use Lemma 13.3.19 in place of Lemma 6.3.20.

(Note that we would have to replace z by z^{-1} outside (iv) and (iv') to have all z's correspond to each other.)

We have combined the modified versions of (g1) and (g2) with (c).

The first condition in (vii) means that $\mathrm{Ker}(\left[\begin{smallmatrix} z-A \\ C \end{smallmatrix}\right]) = \{0\}$ for all $z \in \partial\mathbf{D}$.

In 2.1° of the proof of (f), use the Cayley transform of the function $\widehat{f}$ provided by Lemma D.1.24.

The only "new" claim is the redundancy of uniqueness in (e). However, when $J \geq 0$ and $D^*JD \gg 0$, then

$$\langle y, Jy\rangle_{\mathrm{L}^2} = \langle Cx + Du, J(Cx+Du)\rangle_{\mathrm{L}^2} \geq \langle u, D^*JDu\rangle \geq \varepsilon\|u\|_2^2 \quad (u \in \ell^2(\mathbf{N};U)) \tag{15.11}$$

for some $\varepsilon > 0$, where $x := \mathbb{B}\tau u$. Consequently, then $\mathcal{J}(0,\cdot)$ has a unique minimum at $u = 0$. It follows Lemma 8.3.8 that the word "unique" is redundant in (v). $\square$

(See the notes on p. 583.)

15.3 Positive DAREs

I have yet to see any problem, however complicated, which, when you looked at it in the right way, did not become still more complicated.
— Poul Anderson

Now we present discrete-time equivalents of the main results of Section 10.6, namely necessary and sufficient conditions, in terms of Riccati equations and inequalities, for the Popov operator $\mathbb{D}^* J \mathbb{D}$ to be (stable and) uniformly positive ($\geq \varepsilon I$ for some $\varepsilon > 0$). The reader may find the equivalent condition "$\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} \geq \varepsilon I$ on $\partial \mathbf{D}$" (in $\mathrm{L}^\infty_{\text{strong}}(\partial \mathbf{D}; \mathcal{B}(U))$) more familiar. The proofs are analogous to their continuous-time counterparts and hence omitted. See Section 10.6 for additional remarks too.

We first note that uniform positivity is equivalent to the existence of a positive spectral factorization, as well as to the existence of an I/O-stabilizing solution of the DARE:

Lemma 15.3.1 ($\mathbb{D}^* J \mathbb{D} \gg 0 \Leftrightarrow$ SpF $\Leftrightarrow$ DARE) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U, H, Y)$ be strongly stable and $J = J^* \in \mathcal{B}(Y)$. Then the following are equivalent:*

(i) $\mathbb{D}^* J \mathbb{D} \gg 0$.

(ii) $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{tic}(U)$.

(iii) The DARE has an I/O-stabilizing solution $\mathcal{P} = \mathcal{P}^*$, *with* $S \gg 0$.

(iv) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *s.t.* $S \gg 0$, *and* $\widehat{\mathbb{M}}(z) = I + K(z^{-1} - (A + BK))^{-1} B$ *is in* $\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$.

Moreover, if a solution $\mathcal{P}$ of (iii) or (iv) exists, then a stable and strongly stabilizing solution $\mathcal{P}$ of the DARE exists with $S \gg 0$.

If Σ is exponentially stable, then we have one more equivalent condition:

(v) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *with* $S \gg 0$ *and* $\sigma(A + BK) \subset \mathbf{D}$.

Condition (vi) is sufficient for $\mathbb{D}^* J \mathbb{D} \geq 0$:

(vi) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *with* $S \geq 0$. □

In many applications, we have $C^* J C \leq 0$ and $\Sigma \in \mathrm{SOS}$, and this allows us to drop any stability assumptions and avoid checking any stabilization conditions, as long as (non-uniform) nonnegativity ($\mathbb{D}^* J \mathbb{D} \geq 0$) is sufficient for us. Indeed, then the nonnegativity of the Popov operator is "practically equivalent" (cf. Proposition 15.4.2) to the existence of a nonpositive solution to the Riccati inequality:

Proposition 15.3.2 ($\mathbb{D}^* J \mathbb{D} \geq 0 \Leftrightarrow$ DARI) *Assume that $C^* J C \leq 0$. Then we have (ii)⇒(iii)⇒(i)⇐(iv) for the following conditions:*

(i) $\mathbb{D}^{t\,*} J \mathbb{D}^t \geq 0$ *for all* $t \geq 0$.

(ii) There is $\mathcal{P} \leq 0$ *s.t.* $S := D^* J D + B^* \mathcal{P} B \gg 0$ *and*

$$\begin{bmatrix} A^* \mathcal{P} A - \mathcal{P} + C^* J C & A^* \mathcal{P} B + C^* J D)^* \\ B^* \mathcal{P} A + D^* J C & S \end{bmatrix} \geq 0. \tag{15.12}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.*

$$S := D^*JD + B^*\mathcal{P}B \gg 0, \quad \text{and} \tag{15.13}$$

$$(B^*\mathcal{P}A + D^*JC)^*S^{-1}(B^*\mathcal{P}A + D^*JC) \le A^*\mathcal{P}A - \mathcal{P} + C^*JC. \tag{15.14}$$

(iv) $\mathbb{D} \in \mathrm{tic}$ *and* $\mathbb{D}^*J\mathbb{D} \ge 0$.

Moreover, the following hold:

(a) If $\mathbb{D} \in \mathrm{tic}$*, then we have (i)⇔(iv).*

(b) If $\Sigma \in \mathrm{sos}$ *and* $\mathbb{D}^*J\mathbb{D} \gg 0$*, then (i)–(iv) hold (in fact, we can have equality in (10.81)).*

(c) If $\mathbb{D} \in \mathrm{tic}$ *and* $\mathbb{B}$ *is strongly stable, then we can replace "*$\mathcal{P} \le 0$*" by* $\mathcal{P} = \mathcal{P}^*$ *everywhere in this proposition.* □

(Use Lemma 15.5.1 in the proof of "(iii)⇒(i)".)

Similarly, uniform positivity ($\mathbb{D}^*J\mathbb{D} \ge \varepsilon I$) with exponential stability is equivalent to the existence of a nonpositive solution to the uniform Riccati inequality (still assuming that $C^*JC \le 0$). This time neither the system nor the solution need be stabilizing (a priori):

Theorem 15.3.3 ($\mathbb{D}^*J\mathbb{D} \gg 0$ ⇔ DARI) *Assume that* $C^*JC \le 0$*. Then the following are equivalent:*

(i) Σ *is exponentially stable and* $\mathbb{D}^*J\mathbb{D} \gg 0$.

(ii) There is $\mathcal{P} \le 0$ *s.t.*

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} + C^*JC & A^*\mathcal{P}B + C^*JD \\ B^*\mathcal{P}A + D^*JC & D^*JD + B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.15}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.* $S := D^*JD + B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A + D^*JC)^*S^{-1}(B^*\mathcal{P}A + D^*JC) \ll A^*\mathcal{P}A - \mathcal{P} + C^*JC. \tag{15.16}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ *(and there is an exponentially stabilizing solution if (i) holds).* □

(See the notes on p. 595.)

15.4 Real lemmas

Progress means replacing a theory that is wrong with one more subtly wrong.

In this section, we present the Bounded Real Lemma (in two forms) and the Strict Positive Real Lemma; see Section 10.5 for an introduction and corresponding continuous-time results. The proofs are analogous to their continuous-time counterparts and hence omitted.

We start with necessary and sufficient conditions for the norm of the I/O map to be less than a given constant:

Theorem 15.4.1 (Generalized Strict Bounded Real Lemma) *Assume that* $\gamma > 0$*. Then the following are equivalent:*

(i) Σ *is exponentially stable and* $\|\mathbb{D}\| < \gamma$*.*

(ii) There is $\mathcal{P} \leq 0$ *s.t.*

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} - C^*C & A^*\mathcal{P}B - C^*D \\ B^*\mathcal{P}A - D^*C & \gamma^2 I - D^*D + B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.17}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.* $S := \gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A - D^*C)^*S^{-1}(B^*\mathcal{P}A - D^*C) \ll A^*\mathcal{P}A - \mathcal{P} - C^*C. \tag{15.18}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ *(and there is an exponentially stabilizing solution if (i) holds).* □

As before, $\|\mathbb{D}\| := \|\mathbb{D}\|_{\mathrm{ti}} := \|\mathbb{D}\|_{\mathrm{tic}} := \|\mathbb{D}\|_{\mathcal{B}(\ell^2(\mathbf{Z};U),\ell^2(\mathbf{Z};Y))}$.

For $\Sigma \in$ sos, we have an "almost equivalence" that can be used to find an estimate of the norm of $\mathbb{D}$:

Proposition 15.4.2 (Nonexp. $\|\mathbb{D}\|_{\mathrm{tic}} < \gamma$) *Assume* $\gamma > 0$*.*

If (ii) or (iii) holds, then $\mathbb{D} \in$ tic *and* $\|\mathbb{D}\| \leq \gamma$*.*

Conversely, if $\Sigma \in$ sos *and* $\|\mathbb{D}\| < \gamma$*, then (ii) and (iii) hold (also with "$=$" in place of "$\geq$").*

Here we have referred to the following conditions:

(ii) There is $\mathcal{P} \leq 0$ *s.t.* $\gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ *and*

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} - C^*C & A^*\mathcal{P}B - C^*D \\ B^*\mathcal{P}A - D^*C & \gamma^2 I - D^*D + B^*\mathcal{P}B \end{bmatrix} \geq 0. \tag{15.19}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.* $S := \gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A - D^*C)^*S^{-1}(B^*\mathcal{P}A - D^*C) \leq A^*\mathcal{P}A - \mathcal{P} - C^*C. \tag{15.20}$$

Moreover, (ii) and (iii) are equivalent. If $\mathbb{B}$ *is strongly stable, then we can replace "$\mathcal{P} \leq 0$" by $\mathcal{P} = \mathcal{P}^*$ everywhere in this proposition.* □

Theorem 15.4.3 (Generalized Strictly Positive (Real) Lemma) *The following are equivalent:*

(i) Σ *is exponentially stable and* $\mathbb{D}$ *is strictly positive (i.e.,* $\mathbb{D}+\mathbb{D}^* \gg 0$*);*

(ii) There is $\mathcal{P} \leq 0$ *s.t.*

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} & A^*\mathcal{P}B + C^* \\ B^*\mathcal{P}A + C & D + D^* + B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.21}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.* $S := D + D^* + B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A + C)^* S^{-1} (B^*\mathcal{P}A + C) \ll A^*\mathcal{P}A - \mathcal{P}. \tag{15.22}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ *(and there is an exponentially stabilizing solution if (i) holds).* □

Notes for Sections 15.3 and 15.4

The restriction of Theorem 15.4.1 to finite-dimensional systems is essentially contained in Section 7.7 of [IOW] (multiply the inequalities by -1 and replace $\mathcal{P}$ by $-\mathcal{P}$); that result is the most general discrete-time "real lemma" that we have found in the literature. In particular, also the others are contained in our results. See also the notes on p. 595.

15.5 Riccati inequalities and the maximal solution

Like a great poet, Nature knows how to produce the greatest effects with the most limited means.
— Heinrich Heine

In this section, we extend to infinite-dimensional systems the standard Riccati inequality result (cf., e.g., [LR], Theorem 13.1.1 for discrete time and Theorem 9.1.1 for continuous time): if a Riccati inequality of a strongly stabilizable system has any solution with a uniformly positive signature operator, then it has a greatest solution. Moreover, the solutions of standard minimization problems correspond to this solution (assuming that we require strong or exponential stabilization).

Traditionally, this greatest solution has been called modestly "maximal solution", but we use the more specific term "greatest solution". See Theorem 9.8.13 and Corollary 9.2.11 for continuous-time analogies.

Lemma 15.5.1 (DARI) *We call* $(\mathcal{P},S,K)$ *a solution of the* extended Discrete-time Algebraic Riccati inequality (eDARI) *if* $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $S\in\mathcal{B}(U)$, $K\in\mathcal{B}(H,U)$ *and*

$$\begin{cases} \mathcal{P}\le A^*\mathcal{P}A+C^*JC-K^*SK, \\ S=D^*JD+B^*\mathcal{P}B, \\ SK=-(D^*JC+B^*\mathcal{P}A). \end{cases} \tag{15.23}$$

Such a solution necessarily satisfies

$$\mathbb{K}^{t\,*}S\mathbb{K}^t\le\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t-\mathcal{P}+\mathbb{C}^{t\,*}J\mathbb{C}^t \quad \text{and} \tag{15.24}$$

$$\mathbb{X}^{t\,*}S\mathbb{X}^t\le\mathbb{D}^{t\,*}J\mathbb{D}^t+\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t. \tag{15.25}$$

for all $t\in\mathbf{N}$. *Inequality (15.24) holds even if we drop the assumptions that* $\mathcal{P}=\mathcal{P}^*$, $S=D^*JD+B^*\mathcal{P}B$ *and* $SK=-(D^*JC+B^*\mathcal{P}A)$. □

(The proof of Lemma 14.2.1 applies with certain "=" symbols replaced by "≤" or "≥" symbols. Note that the third equation of the IARE is lost due to its unsymmetry.)

Theorem 15.5.2 (Greatest solution $\mathcal{P}_+$ of DARE and DARI) *Assume that* $\Sigma\in$ wpls(U,H,Y) *is strongly (*$\left[\begin{smallmatrix}\mathbb{A}&\mathbb{B}\\ \mathbb{C}&\mathbb{D}\end{smallmatrix}\right]$*-)stabilizable, and that the eDARI has a solution* $(\mathcal{P},S,K)$ *s.t.* $S\gg 0$.

Then the DARE has a solution $(\mathcal{P}_+,S_+,K_+)$ *s.t.* $S_+\gg 0$ *and* $\mathcal{P}_+\ge\mathcal{P}$ *for all solutions* $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$ *of the eDARI having* $S\ge 0$.

Moreover, if $\mathcal{P}$ *is a strongly* $\left[\begin{smallmatrix}\mathbb{A}&\mathbb{B}\\ \mathbb{C}&\mathbb{D}\end{smallmatrix}\right]$*-stabilizing solution of the eDARE, then* $\mathcal{P}=\mathcal{P}_+$.

Corollary 15.5.3 (Greatest solution $\mathcal{P}_+$ of the DARE) *If the DARE has a strongly stabilizing (or strongly* $\left[\begin{smallmatrix}\mathbb{A}&\mathbb{B}\\ \mathbb{C}&\mathbb{D}\end{smallmatrix}\right]$*-stabilizing) solution s.t.* $S\gg 0$, *then this solution is the greatest solution of the eDARE having* $S\ge 0$.

In particular, if $D^*JD\gg 0$ *and the DARE has a strongly* $\left[\begin{smallmatrix}\mathbb{A}&\mathbb{B}\\ \mathbb{C}&\mathbb{D}\end{smallmatrix}\right]$*-stabilizing solution* $\mathcal{P}\ge 0$, *then* $\mathcal{P}\ge\mathcal{P}'$ *for any nonnegative solution* $\mathcal{P}'$ *of the eDARE (or the eDARI).* □

(The corollary follows from Theorem 15.5.2.)

The example $A = 1$, $B = 0 = C$, $D = 1 = J$, $\mathcal{P} \in \mathbf{R}$ shows that "strongly" is not redundant in the above corollary; modify (or discretize) Example 9.13.12(b) to observe that "strongly" cannot be replaced by "weakly" (in the infinite-dimensional case).

N.B. The number of all self-adjoint solutions of a Riccati equation can be infinite (the simplest example is the two-dimensional CARE $\mathcal{P}^2 = I$ with self-adjoint solutions $\mathcal{P} = \left[\begin{smallmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{smallmatrix}\right]$ $(\theta \in [0, 2\pi))$).

Proof of Theorem 15.5.2:

0° *Notes on the theorem:* We have required above a solution of the eDARI to be self-adjoint. We have also assumed the existence of a solution of the eDARI that has $S \gg 0$. Still, the operator $\mathcal{P}_+$ need not be positive (take, e.g., $J = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $A = -I$, so that $S = I$ and $\mathcal{P}_+ = -\int_0^\infty e^{-2t} I\,dt \ll 0$ is the unique solution of the DARE, by Lemma 9.12.2(c)).

Note that we have allowed instead of strong stabilizability the weaker condition of strong $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$-stabilizability: there must be a state feedback operator K_0 for Σ s.t. $\left(\begin{array}{c|c} A+BK_0 & B \\ \hline C+DK_0 & D \end{array}\right)$ is strongly stable; the third row (generated by $\left(\begin{array}{c|c} K_0 & 0 \end{array}\right)$) of the corresponding closed-loop system Σ_0 need not be stable (the same applies to the "moreover" claim).

1° *About the proof:* This is Theorem 13.1.1 of [LR] except that 1. it assumes that $\dim U, \dim H, \dim Y < \infty$, 2. it requires Σ to be exponentially stabilizable, 3. it assumes that $D^*JD \in \mathcal{GB}(U)$, 4. its statements incorrectly do not (the proof does) require $S \gg 0$ for solutions $\mathcal{P}$ that are to be shown to satisfy $\mathcal{P} \leq \mathcal{P}_+$ (note that we do assume $S \geq 0$), 5. it also states that $\mathcal{P}_+$ is "almost stabilizing" (our 10° is formally weaker, but if $\dim H < \infty$, then this implies that $\mathcal{P}_+$ is "almost stabilizing", i.e., that $\rho(A_+) \leq 1$, as noted in 8°).

We shall follow the proof from [LR], mutatis mutandis.

It is easy to bypass 3. (see below); to bypass 1., we have to use additional tricks for the stability of A_n ($n \in \mathbf{N}$) (see 4° below); by developing these tricks further we can bypass 2. too.

The symbols of [LR] map to those of ours as follows: $X_k \mapsto \mathcal{P}_k$, $\widetilde{X} \mapsto \mathcal{P}$, $L_k \mapsto -K_k$, $R \mapsto D^*JD$, $Q \mapsto C^*JC$, $C \mapsto D^*JC$, $E \mapsto -SK$.

2° $\mathcal{P}_0 \geq \mathcal{P}$: For each $n \in \mathbf{N}$, we shall denote by Σ_n the strongly stable closed-loop system of Σ corresponding to the state feedback operator K_n (specified below).

Let K_0 be strongly stabilizing for Σ (or at least s.t. $\left(\begin{array}{c|c} A+BK_0 & B \\ \hline C+DK_0 & D \end{array}\right)$ becomes stable). By Lemma 9.12.2(c), the operator $\mathcal{P}_0 := \mathbb{C}_0^* J \mathbb{C}_0$ is the unique solution of the DARE $\mathcal{P}_0 = A_0^* \mathcal{P}_0 A_0 - C_0^* J C_0$. It is straightforward to verify that

$$(\mathcal{P}_0 - \mathcal{P}) - A_0^*(\mathcal{P}_0 - \mathcal{P})A_0 = \mathcal{R}(\mathcal{P}) + (K_0 - K)^* S (K_0 - K) \geq 0 \tag{15.26}$$

for any solution $\mathcal{P}$ of the eDARI s.t. $S \geq 0$, where $\mathcal{R}(\mathcal{P}) := A^*\mathcal{P}A - \mathcal{P} + C^*JC - K^*SK \geq 0$. Multiply (15.26) by $(A_0^*)^k$ to the left and by A_0^k to the right, and add the results for $k = 0, 1, \ldots, n$ to obtain that $\mathcal{P}_0 - \mathcal{P} \geq (A_0^*)^{n+1}(\mathcal{P}_0 - \mathcal{P})A_0^{n+1} \to 0$, as $n \to +\infty$, so that $\mathcal{P}_0 \geq \mathcal{P}$.

3° *Induction — $\mathcal{P}_n$:* Let $n \in \mathbf{N} + 1$. Assume (inductively) that $\mathcal{P}_0 \geq \mathcal{P}_1 \geq$

$\cdots \geq \mathcal{P}_{n-1}$ s.t. $\mathcal{P}_k \geq \mathcal{P}$ ($k \leq n-1$) for all solutions $\mathcal{P}$ of the eDARI having $S \geq 0$ (in particular, $S_k := D^*JD + B^*\mathcal{P}_k B \geq S \gg 0$, where S is as in the assumptions) and $\left[\begin{array}{c|c} \mathbb{A}_k & \mathbb{B}_k \\ \hline \mathbb{C}_k & \mathbb{D}_k \end{array}\right] := \left(\begin{array}{c|c} A+BK_k & B \\ \hline C+DK_k & D \end{array}\right)$ is strongly stable; these are the top two rows of the closed-loop system Σ_K corresponding to the state feedback operator

$$K_k := -S_{k-1}^{-1}(D^*JC + B^*\mathcal{P}_{k-1}A) \quad (k \geq 1) \tag{15.27}$$

for $k \leq n-1$. Define K_n and Σ_n as above.

4° *Induction — $\left(\begin{array}{c|c} A_n & B \\ \hline C_n & D \end{array}\right)$ is strongly stable:* We obtain for any solution $\mathcal{P}$ of the DARI, after a straightforward computation, that

$$\begin{aligned}(\mathcal{P}_{n-1} - \mathcal{P}) &- A_n^*(\mathcal{P}_{n-1} - \mathcal{P})A_n \\ &= (K_n - K_{n-1})^* S_{n-1}(K_n - K_{n-1}) + \mathcal{R}(\mathcal{P}) + (K_n - K)^* S(K_n - K).\end{aligned} \tag{15.28}$$

Multiply this by $(A_n^*)^k$ to the left and by A_n^k to the right, and add the results for $k = 0, 1, \ldots, m$ to obtain that

$$(\mathcal{P}_{n-1} - \mathcal{P}) \geq (A_n^*)^{m+1}(\mathcal{P}_{n-1} - \mathcal{P})A_n^{m+1} + \sum_{k=0}^{m}(A_n^*)^k(K_n - K_{n-1})^* S_{n-1}(K_n - K_{n-1})A_n^k \tag{15.29}$$

for all $m \in \mathbf{N}$ (recall $\mathcal{P}_{n-1} - \mathcal{P} \geq 0$). Since $S_{n-1} \gg 0$, it follows that $\|TA_n^{\cdot}\|_{\mathcal{B}(H,\mathrm{L}^2)} < \infty$, where $T := K_n - K_{n-1}$.

Now add $(A_n^*)^k \cdot (15.28) \cdot A_n^k |z|^{2k+2}$ for $k = 0, 1, \ldots, m$ to obtain that

$$\sum_{k=0}^{m}(A_n^*)^k T^* S_{n-1} T A_n^k |z|^{2k+2} \tag{15.30}$$

$$\leq \sum_{k=0}^{m}(A_n^*)^k(\mathcal{P}_{n-1} - \mathcal{P})A_n^k|z|^{2k+2} - \sum_{k=0}^{m}(A_n^*)^{k+1}(\mathcal{P}_{n-1} - \mathcal{P})A_n^{k+1}|z|^{2k+2} \tag{15.31}$$

$$= (\mathcal{P}_{n-1} - \mathcal{P})|z|^2 - \sum_{k=1}^{m}(A_n^*)^k(\mathcal{P}_{n-1} - \mathcal{P})A_n^k|z|^{2k}(1 - |z|^2) \tag{15.32}$$

$$- (A_n^*)^{m+1}(\mathcal{P}_{n-1} - \mathcal{P})A_n^{m+1}|z|^{2m+2} \tag{15.33}$$

$$\leq (\mathcal{P}_{n-1} - \mathcal{P})|z|^2 \leq (\mathcal{P}_{n-1} - \mathcal{P}) \qquad (|z| \leq 1). \tag{15.34}$$

Let Σ_n be the closed-loop system for Σ_{n-1} corresponding to the state feedback operator $T := K_n - K_{n-1}$, i.e.,

$$\Sigma_n' := \left[\begin{array}{c|c} \mathbb{A}_n & \mathbb{B}_n \\ \hline \mathbb{C}_n & \mathbb{D}_n \\ \mathbb{T}_n & \mathbb{E}_n \end{array}\right] := \left(\begin{array}{c|c} A_n & B \\ \hline C_n & D \\ T & 0 \end{array}\right) \tag{15.35}$$

(indeed, from the generators we see that the two top rows of Σ_n' are equal to those of Σ_n; observe also that $A_n = A + BK_n = A_{n-1} + BT$, $C_n = C + DT$). Below (15.29) we noted that $\mathbb{T}_n = TA_n^{\cdot}$ is stable. By Lemma 6.6.8, it follows that $\mathbb{A}_n = \mathbb{A}_{n-1} + \mathbb{B}_{n-1}\tau\mathbb{T}_n$ is strongly stable, From $B^* \cdot (15.30) \cdot B \leq B^*(\mathcal{P}_{n-1} - \mathcal{P})B$ we obtain (since $S_{n-1} \gg 0$) that

$$\| \sum_{k=0}^{m} TA_n^k z^{k+1} B \|_{\mathcal{B}(U)} \leq M_n < \infty \quad (m \in \mathbf{N}). \tag{15.36}$$

By (13.52), this implies that $\widehat{\mathbb{E}}_n \in \mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$, i.e., that $\mathbb{E}_n \in \mathrm{tic}(U)$. Consequently, $\mathbb{B}_n = \mathbb{B}_{n-1}(\mathbb{E}_n + I)$ and $\mathbb{D}_n = \mathbb{D}_{n-1}(\mathbb{E}_n + I)$ are stable. Therefore, also $\mathbb{C}_n = \mathbb{C}_{n-1} + \mathbb{D}_{n-1}\mathbb{T}_n$ is stable. Now we have shown that two top rows of Σ_n are strongly stable.

5° *$\mathcal{P}_n \geq \mathcal{P}$:* By Lemma 9.12.2(c), the operator $\mathcal{P}_n := \mathbb{C}_n^* J \mathbb{C}_n$ is the unique solution of the DARE $\mathcal{P}_n = A_n^* \mathcal{P}_n A_n + C_n^* J C_n$. If $\mathcal{P}$ is a solution of the eDARI having $S \geq 0$, then (as in 2°)

$$(\mathcal{P}_n - \mathcal{P}) - A_n^*(\mathcal{P}_n - \mathcal{P})A_n = \mathcal{R}(\mathcal{P}) + (K_n - K)S(K_n - K) \geq 0, \tag{15.37}$$

hence $\mathcal{P}_n - \mathcal{P} \geq (A_n^*)^{m+1}(\mathcal{P}_n - \mathcal{P})A_n^{m+1} \to 0$ (as in 2°), as $m \to +\infty$, so that $\mathcal{P}_n \geq \mathcal{P}$.

6° *$\mathcal{P}_{n-1} \geq \mathcal{P}_n$:* One can analogously to p. 310 of [LR] compute that

$$(\mathcal{P}_{n-1} - \mathcal{P}_n) - A_n^*(\mathcal{P}_{n-1} - \mathcal{P}_n)A_n = (K_n - K_{n-1})^* S_{n-1}(K_n - K_{n-1}) \geq 0. \tag{15.38}$$

As above, we obtain that $\mathcal{P}_{n-1} - \mathcal{P}_n \geq (A_n^*)^{m+1}(\mathcal{P}_{n-1} - \mathcal{P}_n)A_n^{m+1} \to 0$, hence $\mathcal{P}_{n-1} \geq \mathcal{P}_n$.

7° *$\mathcal{P}_+$:* Let $\mathcal{P}$ be a solution of the eDARI. Then $\mathcal{P}_n \geq \mathcal{P}_{n+1} \geq \mathcal{P}$ $(n \in \mathbf{N})$, so that there is a unique $\mathcal{P}_+ \in \mathcal{B}(H)$ s.t. $\mathcal{P}_n x_0 \to \mathcal{P}_+ x_0$ for all $x_0 \in H$, by Lemma A.3.1(b5). Since $\mathcal{P}_n = \mathcal{P}_n^* \geq \mathcal{P}$ for all n, we have $\mathcal{P}_+ = \mathcal{P}_+^* \geq \mathcal{P}$.

8° *$S_n \to S_+$, $S_n^{-1} \to S_+^{-1}, K_n \to K_+, A_n \to A_+$, strongly:* Set $S_+ := D^*JD + B^*\mathcal{P}_+ B \geq S \gg 0$ (where $S \gg 0$ corresponds to the solution in the assumptions), $K_+ := S_+^{-1}(D^*JC + B^*\mathcal{P}_+ A)$. For all $n \in \mathbf{N}$, we have $S_n \geq S_+ \geq \varepsilon I$ for some $\varepsilon > 0$, hence $0 \ll S_n^{-1} \leq \varepsilon^{-1} I$, by Lemma A.3.1(b1), hence $\|S_n^{-1}\| \leq \varepsilon^{-1/2} < \infty$. Therefore, $S_n^{-1} \to S_+^{-1}$ strongly, as $n \to +\infty$, by Lemma A.3.1(b5). Therefore, $K_n \to K_+$ strongly (see Lemma A.3.1(j3)), hence $A_n \to A_+$, $C_n \to C_+$ strongly, as $n \to +\infty$, where Σ_+ is the closed-loop system corresponding to K_+ (i.e., $A_+ := A + BK_+$, $C_+ := C + DK_+$).

(Note that if $\dim H < \infty$, then "strong equals uniform", hence then $\rho(A_+) \leq 1$ (since $\rho(A_n) \leq 1$ for all $n \in \mathbf{N}$), but even then we may have $A_+ = I$ as in Example 13.2.1 of [LR], where $C = 0$, $A = B = R = I$, and hence $\mathcal{P}_n = (2^{n+1} - 1)^{-1} I$ $(n \in \mathbf{N})$.)

9° *$\mathcal{P}_+$ solves the DARE:* By 8° and 5°, we have $\mathcal{P}_+ = A_+^* \mathcal{P}_+ A_+ + C_+^* J C_+$; combine this to the definitions of K_+ and S_+ to observe that $\mathcal{P}_+$ is a solution of the DARE, by Lemma 9.10.1(b4)(iv).

10° *$\mathcal{P}_+ = \mathcal{P}$:* Let $\mathcal{P}$ be a strongly stabilizing (at least for the top two rows) solution of the eDARE. Then we can choose $\mathcal{P}_0 := \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = \mathcal{P}$ in 2°. It follows that $\mathcal{P} = \mathcal{P}_0 \geq \mathcal{P}_+ \geq \mathcal{P}$, hence $\mathcal{P} = \mathcal{P}_+$.

11° *Remark — exponentially stable A_n:* One observes from 4° (and Lemma 13.3.7(ii)) that if A_0 is exponentially stable, then so is A_n for all $n \in \mathbf{N}$. □

Notes

The classical result (e.g., Theorem 13.1.1 of [LR]) assumes that $\mathcal{P}$ is exponentially stabilizing, hence our result seems to be new (a generalization) also for finite-dimensional systems.

In continuous-time, the classical result (e.g., Theorem 9.1.1 of [LR]) was extended to WPLSs with bounded B and C by Ruth Curtain and Leiba Rodman in [CR], which also contains comparison theorems for the solutions of two different Riccati inequalities (for two different WPLSs).

Laurence Dumortier has enhanced some of the results of [CR] (for WPLSs having bounded B and C and finite-dimensional U and Y). In [Dumortier], she has shown that the maximal solution $\mathcal{P}_+$ of the standard LQR CARE stabilizes all strictly intact poles of A as well as all observable ones, but it leaves the critical (meaning $\sigma(A) \cap i\mathbf{R}$) nonobservable poles unstable. Thus, at least in this special case, $\mathcal{P}_+$ is exponentially stabilizing iff $\mathrm{Ker}(\mathbb{C}) \cap \sigma(A) \cap i\mathbf{R} = \emptyset$.

Unbounded input and output operators make the classical proof virtually impossible. This is surely the reason why there does not seem to be any earlier results for WPLSs having an unbounded input or output operator. We have obtained such results, Corollary 9.2.11 and Theorem 9.8.13, by reducing them to the results of this section, using discretization. This way we obtain analogous results for the "IARI" in the general case (replace the first equation of the IARE by an inequality). However, this way we have obtained results for the CARE (or for the IARE or for the "IARI") only, not for the corresponding inequality ("CARI"), since the "CARI" (9.217) does not lead to the IARI, as explained in Proposition 9.11.9. Therefore, a "CARI" result would require a continuous-time proof.

It seems that such a proof would be possible for bounded B, as well as for the parabolic systems of Hypothesis 9.5.1, since in both of these cases we would have $\mathrm{Dom}(A_n) = \mathrm{Dom}(A)$ for each n in the proof (generalize the proof of Theorem 9.1.1 of [LR]; note that each K_n would become bounded). These might be worth of further investigations.