Chapter 3

Transfer Functions ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$)

Though earth and man were gone,
And suns and universes ceased to be,
And Thou were left alone,
Every existence would exist in Thee.
— Emily Brontë (1818–1848)

In this chapter, we shall study the transfer functions of TI and TIC maps, i.e., the functions $\widehat{\mathbb{E}}$ for which $y = \mathbb{E}u$ corresponds to $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$, when $\mathbb{E} \in \mathrm{TI}(U,Y)$, u is an input signal and y is the corresponding output signal.

For $\mathbb{E} \in \mathrm{TIC}$, this was given in Theorem 2.1.2 (with $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ on $\mathbf{C}^+$, i.e., in $\mathrm{H}^2(\mathbf{C}^+;Y)$); for general $\mathbb{E} \in \mathrm{TI}$, we only know that $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ a.e. $\mathbf{C}^+$, i.e., that $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ in $\mathrm{L}^2(i\mathbf{R};Y)$; this mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ will be established in Theorem 3.1.3.

Thus, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$ and $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$. In Theorem 3.1.6, we shall show that $\widehat{\mathrm{TI}_a \cap \mathrm{TI}_b} = \mathrm{H}^\infty(\{a < \mathrm{Re}\cdot < b\}; \mathcal{B}(U,Y))$. We also provide some further results on these three forms of transfer functions and weaker forms of the them for arbitrary Banach spaces U and Y and L^p in place of L^2 (and "TI^p" in place of TI). These can be considered as extensions of so called *Fourier multiplier theory*.

In Section 3.3, we establish several results on the boundary functions and poles of holomorphic functions.

By H, U and Y we denote arbitrary Hilbert spaces unless something else is indicated.

3.1 Transfer functions of TI ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$)

transfer, *n.:*
A promotion you receive on the condition that you leave town.

To be able to prove that "$\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$", we first recall the definition of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$:

Definition 3.1.1 ($\mathbf{L}^\infty_{\mathbf{strong}}$) *Let U, Y and W be Hilbert spaces, and let Q be a set with a complete positive measure. A function $F : Q \to \mathcal{B}(U,Y)$ is said to be* strongly measurable, *if Fu_0 is Bochner measurable for each $u_0 \in U$.*

We define $\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(U,Y))$ to be the space of (equivalence classes of) strongly measurable functions $Q \to \mathcal{B}(U,Y)$ with norm

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{\|u\|\le 1} \|Fu\|_{\mathrm{L}^\infty} < \infty. \tag{3.1}$$

We define the multiplication in $\mathrm{L}^\infty_{\mathrm{strong}}$ by the formula $[F][G] := [FG]$ for any $F \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(U,Y))$, $G \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(Y,W))$.

Moreover, $[G] \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(Y,U))$ is the adjoint *of $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}$ (i.e. $[F]^* = [G]$) if $\langle Fu,y\rangle = \langle u,Gy\rangle$ a.e. for all $u \in U$ and $y \in Y$.*

All this is well-defined, by Section F.1. Naturally, functions $F,G \in \mathrm{L}^\infty_{\mathrm{strong}}$ are equivalent ($G \in [F]$) iff $\|F-G\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, i.e., iff $Fu = Gu$ a.e. for all $u \in U$. (By $[F]$ we denote the equivalence class of F.)

As above, we write $F \in \mathrm{L}^\infty_{\mathrm{strong}}$ instead of $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}$ when there is no danger of confusion. See Example 3.1.4 for adjoints and Section F.1 for further theory on $\mathrm{L}^\infty_{\mathrm{strong}}$ and strongly (and weakly) measurable operator-valued functions.

Recall that we treat the imaginary axis $i\mathbf{R}$ as $\mathbf{R}$ for measure-theoretic and differentiability aspects etc.; in particular, $m(iE) := m(E)$ for any measurable $E \subset \mathbf{R}$, where m is the one-dimensional Lebesgue measure, and $f \in \mathcal{C}^k(i\mathbf{R};B)$ iff $g \in \mathcal{C}^k(\mathbf{R};B)$, where $g(\cdot) := f(i\cdot)$. The same applies to the applies to any other vertical axis $\omega + i\mathbf{R}$, where $\omega \in \mathbf{R}$. We recall the following from Section F.1:

Lemma 3.1.2 *The space $\mathrm{L}^\infty_{\mathrm{strong}}(\Omega;\mathcal{B}(U,Y))$ is a Banach space and the space $\mathrm{L}^\infty_{\mathrm{strong}}(\Omega;\mathcal{B}(U))$ is a Banach algebra for any measurable $\Omega \subset \mathbf{R}$.* □

(This follows easily from Lemma F.1.3(b) and Theorem F.1.9(s1).)

Now we are able to state that $\mathrm{L}^\infty_{\mathrm{strong}} = \widehat{\mathrm{TI}}$:

Theorem 3.1.3 ($\widehat{\mathbf{TI}} = \mathbf{L}^\infty_{\mathbf{strong}}$) *For any Hilbert spaces U and Y, the following hold:*

(a1) **$\widehat{\mathbf{TI}} = \mathbf{L}^\infty_{\mathbf{strong}}$:** *For each $\mathbb{E} \in \mathrm{TI}(U,Y)$ there is a unique function $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ called the* Fourier transform *(or symbol) of $\mathbb{E}$, s.t. $\widehat{\mathbb{E}}\widehat{f} = \widehat{\mathbb{E}f}$ on $i\mathbf{R}$ for all $f \in \mathrm{L}^2(\mathbf{R};U)$. (We also call the mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ the Fourier transform.)*

The Fourier transform is an isometric isomorphism of $\mathrm{TI}(U,Y)$ onto $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$, and it commutes with adjoints and compositions; in *particular, this mapping is an isometric B^*-algebra isomorphism of* $\mathrm{TI}(U)$ *onto* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$.

(a2) Each $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *has a representative* $F : i\mathbf{R} \to \mathcal{B}(U,Y)$ *s.t.* $\|F(ir)\| \le \|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{u\in U} \|Fu\|_{\mathrm{L}^\infty}$ *for all* $r \in \mathbf{R}$.

(b) If $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, then the boundary function (see Theorem 3.3.1(c1)) of its Laplace transform coincides with its Fourier transform; we identify the two.*

(c) If $\dim U < \infty$*, then* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)) = \mathrm{L}^\infty(i\mathbf{R};\mathcal{B}(U,Y))$.

(d) Let $[F] = \widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$*. Assume that either F is piecewise continuous or U is separable. Then* $\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \operatorname{ess\,sup} \|F\|_{\mathcal{B}(U,Y)}$ *and* $[F^*] = [F]^*$*. Moreover,* $\mathbb{E}^*\mathbb{E} \ge 0$ *iff* $F^*F \ge 0$ *a.e.*

Furthermore, $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ *iff* $F(ir) \in \mathcal{GB}$ *for a.e.* $r \in \mathbf{R}$ *and* $[F^{-1}] \in \mathrm{L}^\infty_{\mathrm{strong}}$*. If F is piecewise continuous, then a third equivalent condition is that* $F(ir) \in \mathcal{GB}$ *for a.e.* $r \in \mathbf{R}$ *and* F^{-1} *is essentially bounded; and a fourth that* $F(ir) \in \mathcal{GB}$ *for all* $r \in \mathbf{R}$ *and* F^{-1} *is bounded.*

(e1) Let $\mathbb{E} \in \mathrm{TI}(U)$*. Then* $\mathbb{E} \ge 0$ *iff* $\langle \widehat{\mathbb{E}}u, u\rangle \ge 0$ *a.e. for all* $u \in U$ *(iff* $\widehat{\mathbb{E}} \ge 0$ *a.e., provided that either* $\widehat{\mathbb{E}}$ *is piecewise continuous or U is separable).*

(e2) Let $\mathbb{E}_k \in \mathrm{TI}(U,*)$ *($k = 1,2,3,4$). Then* $\mathbb{E}_1^*\mathbb{E}_2 \ge \mathbb{E}_3^*\mathbb{E}_4$ *iff* $\langle \widehat{\mathbb{E}}_1 u, \widehat{\mathbb{E}}_2 u\rangle \ge \langle \widehat{\mathbb{E}}_3 u, \widehat{\mathbb{E}}_4 u\rangle$ *a.e. for all* $u \in U$ *(iff* $\widehat{\mathbb{E}}_1^*\widehat{\mathbb{E}}_2 \ge \widehat{\mathbb{E}}_3^*\widehat{\mathbb{E}}_4$ *a.e., provided that either* $\widehat{\mathbb{E}}_k$ *is piecewise continuous for* $k = 1,2,3,4$*, or U is separable).*

Of course, we have $\widehat{\mathrm{TI}_\omega}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(\omega + i\mathbf{R};\mathcal{B}(U,Y))$ in a similar way (because $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2$ and hence $\widehat{\mathrm{L}^2_\omega} = \tau(-\omega)(\widehat{\mathrm{L}^2})$); see also Remark 2.1.6.

All above results also hold for $\partial\mathbf{D}$ mutatis mutandis, by Theorem 13.2.3. See Theorem 3.2.4 for a result weaker than (a1) in the case of separable Banach spaces U and Y. Note also that $\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(U,Y))$ is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

Given $\mathbb{E} \in \mathrm{TI}(U,Y)$ and $u_0 \in U$, we have $\widehat{\mathbb{E}}u_0 = \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0$ a.e. on $i\mathbf{R}$, where, e.g., $\phi(t) = \mathrm{e}^{-t^2/2}$ (see Lemma D.1.25), because the Fourier transform is one-to-one on L^2. The following proof of (a1) is based on this.

Proof of Theorem 3.1.3: (a1) Let $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. By Theorem F.1.7(b) and The Plancherel Theorem, $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)) \subset \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$, isometrically, through $f \mapsto \mathbb{E}f := \mathcal{L}^{-1}\widehat{\mathbb{E}}\widehat{f}$. Moreover, $\mathbb{E}\tau^t f = \mathcal{L}^{-1}\widehat{\mathbb{E}}\mathrm{e}^{t\cdot}\widehat{f}\mathcal{L}^{-1}\mathrm{e}^{t\cdot}\widehat{\mathbb{E}}\widehat{f} = \tau^t\mathbb{E}f$ for all $f \in \mathrm{L}^2$. Thus, $\mathcal{L}^{-1}\mathrm{L}^\infty_{\mathrm{strong}} \subset \mathrm{TI}(U,Y)$, isometrically.

Therefore, we only have to show the converse. Thus, below we assume that $\mathbb{E} \in \mathrm{TI}(U,Y)$, find a candidate $\widehat{\mathbb{E}}$ and then show that $\mathcal{L}\mathbb{E}f = \widehat{\mathbb{E}}\widehat{f}$ for all $f \in \mathrm{L}^2$.

(We note that the proof does require that U and Y are Hilbert spaces, because we want all $\mathrm{L}^2(\mathbf{R};Y)$ functions to have a Fourier(–Plancherel) transform and we want to be able to extend $\mathcal{B}(U_0,Y)$ operators to $\mathcal{B}(U,Y)$ operators whenever U_0 is a subspace of Y.)

1° Define $\phi(t) := \mathrm{e}^{-t^2/2}$. Then $\widehat{\phi}(ir) = \sqrt{2\pi}\mathrm{e}^{-r^2/2} > 0$ for $r \in \mathbf{R}$, by Lemma D.1.25; in particular $\widehat{\phi}(ir)^{-1}$ is everywhere defined and continuous.

2° By L we shall denote the operator defined in Lemma B.5.3.

3° Set $A_t := \{u \mid it \in \mathrm{Leb}(L\mathcal{L}\mathbb{E}\phi u)\} \subset U$. Then, for a fixed $u \in U$, we have $u \in A_t$ for a.e. t. By (B.56), A_t is a subspace of U.

If $u \in U$ and $\Lambda \in Y^*$, then $g \mapsto \Lambda\mathbb{E}gu$ is in $\mathrm{TI}(\mathbf{C})$ and of norm $\leq \|\mathbb{E}\|\,\|\Lambda\|\,\|u\|$, hence, by the corresponding scalar result [BL, Theorem 6.1.2, p. 132], there is $T_{\Lambda,u} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. $\|T_{\Lambda,u}\| \leq \|\mathbb{E}\|\,\|\Lambda\|\,\|u\|$ and

$$T_{\Lambda,u}\widehat{g} = \mathcal{L}\Lambda\mathbb{E}gu = \Lambda\mathcal{L}\mathbb{E}gu \text{ for } g \in \mathrm{L}^2(\mathbf{R}). \tag{3.2}$$

If follows that for all $\Lambda \in Y^*$ s.t. $\|\Lambda\| \leq 1$ we have

$$|\Lambda\widehat{\phi}(ir)^{-1}L(\mathcal{L}\mathbb{E}\phi u)(ir)| \leq |\widehat{\phi}(ir)^{-1}LT_{\Lambda,u}(ir)\widehat{\phi}(ir)| \leq \|\mathbb{E}\|\|u\|, \tag{3.3}$$

hence

$$\|\widehat{\phi}(ir)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(ir)\|_Y \leq \|\mathbb{E}\|\|u\|. \tag{3.4}$$

Let $F(it) \in \mathcal{B}(U,Y)$ be a linear extension of $A_t \ni u \mapsto \widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it) \in Y$ (that mapping is linear, by (B.56)) satisfying $\|F(it)\| \leq \|\mathbb{E}\|$ (by (3.4), this is possible, e.g., extend to the closure of A_t by continuity, and extend by zero on $A_t^\perp$).

Because for $u \in U$ we have $F(it)u = \widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it)$ for $t \in A_t$, hence for a.e. $t \in \mathbf{R}$, the function F is in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$, in particular, $F \in \mathcal{B}(\mathrm{L}^2(i\mathbf{R};U),\mathrm{L}^2(i\mathbf{R};Y))$, by Theorem F.1.7(b).

For $\widehat{f} \in \mathrm{L}^2(i\mathbf{R})$ and $u \in U$ we have

$$\Lambda F(it)\widehat{f}(it)u = \widehat{f}(it)\widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it) = \widehat{f}(it)T_{\Lambda,u}(it) = \Lambda(\mathcal{L}\mathbb{E}fu)(it) \text{ a.e.} \tag{3.5}$$

for all $\Lambda \in Y^*$ (the first inequality follows from the definition of $F(it)$ and from the fact that $u \in A_t$ for a.e. t, the second and third inequality follow from (3.2)). Because Ff and $\mathcal{L}\mathbb{E}fu$ are $\mathrm{L}^2(i\mathbf{R};Y)$ functions, hence measurable, it follows from (3.5) that $(Ff)(it)u = (\mathcal{L}\mathbb{E}fu)(it)$ for (a.e.) $t \in \mathbf{R}$, by Lemma B.2.6.

We can extend $Ffu = Tfu$ to $Fg = Tg$ for arbitrary simple functions g by linearity, then for arbitrary $g \in \mathrm{L}^2(i\mathbf{R};U)$, by density (Theorem B.3.11) and continuity.

The second paragraph of (a1) is easy to prove, e.g., by an application to functions of the form $\chi_E u$ and $\chi_E y$, one easily notes that $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^*$ for an arbitrary $\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}$ (in particular, $\mathrm{L}^\infty_{\mathrm{strong}}$ is a B^*-algebra).

(a2) The representative F constructed in the proof of (a1) obviously satisfies the conditions of (a2).

(b) This will be proved in Theorem 3.3.1(c1).

(c) See Lemma F.1.3(d).

(d) The claim on $\mathbb{E}^*\mathbb{E} \geq 0$ follows from (e1) and the claim $[F^*] = [F]^*$, because $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} = [F^*][F] = [F^*F]$.

The rest follows from Lemma F.1.3(f1)&(f2). In the proof of the $\mathcal{G}$TI-equivalence, we also need the fact $\mathbb{E} \in \mathcal{G}\mathrm{TI} \Leftrightarrow [F] \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$, from (a1) (the extra condition imposed on "G" in (f2) follows from (a2) above).

For the last remark we note that if F is piecewise continuous and F^{-1} exists a.e. and F^{-1} is bounded, then F^{-1} exists everywhere, by Lemma A.3.3(A3), and $[F^{-1}] \in \mathrm{L}^\infty_{\mathrm{strong}}$.

(Instead of piecewise continuity, it suffices to assume that $i\mathbf{R}$ is divided into an at most countable number of intervals of positive measure, and F is continuous on each of them.)

(e) For clarity, we only prove (e1) (which is a special case of (e2)); the same proof of applies for (e2) with slight changes.

1° If $\langle \widehat{\mathbb{E}}u, u\rangle_U \geq 0$ a.e. for all $u \in U$, then $\langle \widehat{\mathbb{E}}f, f\rangle_{\mathrm{L}^2(i\mathbf{R};U)} \geq 0$ for all simple functions $f \in \mathrm{L}^2$, hence for all $f \in \mathrm{L}^2$.

Conversely, if $u \in U$ and $\langle \widehat{\mathbb{E}}u\chi_E, u\chi_E\rangle_{\mathrm{L}^2(i\mathbf{R};U)} \geq 0$ for all finite-measurable sets E, then $\langle \widehat{\mathbb{E}}u, u\rangle \geq 0$ a.e.

2° If $\widehat{\mathbb{E}}$ is piecewise continuous, then the $\widehat{\mathbb{E}} \geq 0$ claim is obvious. If U is separable and $\langle \widehat{\mathbb{E}}u, u\rangle \geq 0$ a.e. for all $u \in U$, then there is a null set N s.t. $\langle \widehat{\mathbb{E}}u, u\rangle \geq 0$ on N^c for all u in a countable, dense subset of U, hence for all $u \in U$. The converse is trivial. □

In the case of unseparable Y (and discontinuous $\mathrm{L}^\infty_{\mathrm{strong}}$ functions), there are some peculiarities, e.g., even the element $0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ may have a representative $F : \mathbf{R} \to \mathcal{B}(U,Y)$ with $\operatorname{ess\,sup} \|F\|_{\mathcal{B}(U,Y)} = \infty$ and F^* nonmeasurable:

Example 3.1.4 **[$\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$ & $\|F\|_{\mathrm{L}^\infty} = \infty$]** Let $\{e_r\}_{r\in\mathbf{R}}$ be the natural base of $U := \ell^2(\mathbf{R})$, and let $d_0 \in Y$ be s.t. $\|d_0\| = 1$; here Y can be any unseparable Hilbert space.

For all $f : \mathbf{R} \to \mathbf{R}$, we define $F_f : i\mathbf{R} \to \mathcal{B}(U)$ by $F_f(ir)u := f(r)u_r d_0$ (where $u = (u_r)_{r\in\mathbf{R}}$), so that $\|F_f(ir)\|_{\mathcal{B}(U,Y)} = |f(r)|$ for all $r \in \mathbf{R}$. Consequently, $F_f u = 0$ a.e. for all $u \in U$, because $u_r = 0$ for a.e. r; in particular, $[F_f] = [0] \in \mathrm{L}^\infty_{\mathrm{strong}}$, i.e., $\|[F_f]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, even though $\|F\|_{\mathrm{L}^\infty} := \|\|F\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^\infty(\mathbf{R})} = \|f\|_\infty$ may be infinite (and F may be nonmeasurable). Moreover,

$$\langle F_f(ir)u, y\rangle = \langle f(r)u_r d_0, y\rangle = \langle f(r)u_r, y_0\rangle = \langle u, f(r)y_0 e_r\rangle \text{ for all } u \in U,\ y \in Y, \tag{3.6}$$

where $y_0 := \langle d_0, y\rangle =: \Lambda y$, hence, $F_f^* = f(r)e_r\Lambda$, in particular, $F_f(ir)^* d_0 = f(r)e_r$ and $\|F_f(ir)^*\|_{\mathcal{B}(U,Y)} = |f(r)| = \|F_f(ir)\|$ ($r \in \mathbf{R}$). The following holds:

(a) If $f(r) = r$, then $F_f(ir)^* d_0 = re_r$, hence $\|F_f^* d_0\|_\infty = \infty$. Consequently,

$$\text{“}\|[F_f^*]\|\text{''}_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{y_0\in Y} \operatorname*{ess\,sup}_{r\in\mathbf{R}} \|F_f(ir)^* y_0\|_U = \infty. \tag{3.7}$$

(b) If $g(r) \equiv 1$, then “$\|[F_g^*]\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$”$= 1$. However, even in this case, we have $[F_f] \notin \mathrm{L}^\infty_{\mathrm{strong}}$, because F_g^* is not even strongly measurable: obviously, the function $r \mapsto F_g(ir)^* d_0 = e_r \in U$ is not almost separably-valued, hence neither measurable (nor is the function F_f of (a) or any other F_f except those with $f = 0$ a.e.).

◁

Thus, in general, the adjoint of a representative of some $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ need not be in the class of $\widehat{\mathbb{E}}^*$, not even in the class of any $\mathrm{L}^\infty_{\mathrm{strong}}$ function, even if this representative were bounded (see (b) above).

However, there is a unique $\widehat{\mathbb{F}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ s.t. $\langle u, E^* y\rangle = \langle Eu, y\rangle = \langle u, Fy\rangle$ a.e. for all u, y whenever E and F are representatives of $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{F}}$, respectively, by Theorem 3.1.3(a1). Note that, if Fu is strongly measurable, then F^* is weakly measurable in the sense that $\langle F^* y, u\rangle$ is measurable for all $u \in U$ and $y \in Y$.

Sometimes TI operators (and WPLSs) are studied over Banach spaces and general L^p (see, e.g., [Sbook] and several articles of G. Weiss). Many of our results generalize to that setting with ease but some do not at all. The emphasis of this book is in L^2 signals over Hilbert spaces, because this allows one to formulate the standard control problems. However, we give here certain results in a wider setting for future reference.

Theorem 3.1.5 ($\mathbf{TI}^p_\omega$) *Let X and Y be Banach spaces, $p, q \in [1, \infty]$ and $\omega \in \mathbf{R}$. Define*

$$\mathrm{TI}^{p,q}_\omega(X,Y) := \{\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_\omega(\mathbf{R}; X), \mathrm{L}^q_\omega(\mathbf{R}; Y)) \mid \mathbb{E}\tau(t) = \tau(t)\mathbb{E} \text{ for all } t \in \mathbf{R}\}, \tag{3.8}$$

$\mathrm{TI}^p_\omega := \mathrm{TI}^{p,p}_\omega$ and $\mathrm{TI} := \mathrm{TI}^{2,2}_\omega$. Then $\mathrm{TI}^{p,q}_\omega$ is a closed subspace of $\mathcal{B}(\mathrm{L}^p_\omega, \mathrm{L}^q_\omega)$. Moreover, $\|\mathbb{E}f\|_{\mathrm{W}^{n,q}_\omega} \le \|\mathbb{E}\|_{\mathrm{TI}^{p,q}} \|f\|_{\mathrm{W}^{n,p}_\omega}$ and $\mathbb{E}\partial^n f = \partial^n \mathbb{E} f$ for all $f \in \mathrm{W}^{n,p}_\omega(\mathbf{R}; U)$, $\mathbb{E} \in \mathrm{TI}^{p,q}_\omega$.

Finally, $\mathbb{E}[\mathcal{S}(\mathbf{R}; X)] \subset \mathrm{W}^{\infty,q}_\omega \subset \mathcal{C}^\infty(\mathbf{R}; Y)$ for all $\mathbb{E} \in \mathrm{TI}^{p,q}(X,Y)$, and $\mathbb{E}[\mathcal{C}_0(\mathbf{R}; X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R}; Y)$ (even $\mathbb{E}[\mathcal{C}_0(\mathbf{R}; X)] \subset \mathcal{C}_0(\mathbf{R}; Y)$ if Y is a Hilbert space) for all $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$.

(We have set $\mathrm{W}^{\infty,p}_\omega := \cap_{k\in\mathbf{N}} \mathrm{W}^{k,p}_\omega$. Note also that the definition of TI^∞ in [Sbook] differs from that of ours: it requires the L^∞ functions to vanish at infinity.)

The TI^p operators correspond to Fourier multipliers (see Section 3.2; note that the multiplier does not determine the operator uniquely in case $p = \infty$). Analogously, $\mathrm{TI}^{p,\infty}(U,Y) = \mathcal{B}(\mathrm{L}^p(\mathbf{R}; B), B_2)*$, in particular, $\mathrm{TI}^{p,\infty}(\mathbf{C}) = \mathrm{L}^{p'}(\mathbf{R})*$, where $p^{-1} + p'^{-1} = 1$, for $p < \infty$. but since the case $q \ne p$ is only rarely treated, we shall not consider it further (and we omit the proofs).

Proof: 1° Obviously, $\mathrm{TI}^{p,q}_\omega$ is a closed subspace of $\mathcal{B}(\mathrm{L}^p_\omega, \mathrm{L}^q_\omega)$. By Lemma B.7.8, a function $f \in \mathrm{L}^p_\omega(\mathbf{R}; X)$ is in $\mathrm{W}^{1,p}_\omega$ iff $(\tau(h)f - f)/h$ converges in L^p_ω as $h \to 0$. Thus, for $f \in \mathrm{W}^{1,p}_\omega$ we have (here $\overset{X}{\lim}$ means a limit in the space X)

$$\begin{aligned}\mathbb{E}\partial f &= \mathbb{E} \overset{\mathrm{L}^p_\omega}{\lim_{h\to 0}} (\tau(h)f - f)/h = \overset{\mathrm{L}^q_\omega}{\lim_{h\to 0}} \mathbb{E}(\tau(h)f - f)/h \\ &= \overset{\mathrm{L}^q_\omega}{\lim_{h\to 0}} (\tau(h)\mathbb{E}f - \mathbb{E}f)/h = \partial(\mathbb{E}f),\end{aligned}$$

hence $\partial(\mathbb{E}f)$ exists and is equal to $\mathbb{E}\partial f$. Thus, $\mathbb{E}[\mathrm{W}^{1,p}_\omega] \subset \mathrm{W}^{1,q}_\omega$.

By induction, $\mathbb{E}\partial^n \subset \partial^n \mathbb{E}$ and $\mathbb{E}\mathrm{W}^{n,p}_\omega \subset \mathrm{W}^{n,q}_\omega$ for any $n \in \mathbf{N}$. Consequently, $\|\mathbb{E}f\|_{\mathrm{W}^{n,q}_\omega} \le \|\mathbb{E}\|_{\mathrm{TI}^{p,q}} \|f\|_{\mathrm{W}^{n,p}_\omega}$

2° *$\mathbb{E}[\mathcal{S}(\mathbf{R}; X)] \subset \mathcal{C}^\infty(\mathbf{R}; Y)$:* Let $\mathbb{E} \in \mathrm{TI}^p(X,Y)$. Obviously, $\mathcal{S} \subset \mathrm{W}^{\infty,p}$. By 1°, $\mathbb{E}[\mathcal{S}(\mathbf{R}; X)] \subset \mathrm{W}^{\infty,q}$. By Corollary B.7.7, we have $\mathrm{W}^{\infty,q} \subset \mathcal{C}^\infty$.

3° *$\mathbb{E}[\mathcal{C}_0(\mathbf{R}; X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R}; Y)$:* Let $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$. Then $\mathbb{E}[\mathcal{S}(\mathbf{R}; X)] \subset \mathcal{C}^\infty \cap \mathrm{L}^\infty \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R}; Y)$, hence $\mathbb{E}[\mathcal{C}_0(\mathbf{R}; X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R}; Y)$, by continuity (by Theorem B.3.11(c), $\mathcal{C}_0$ is the closure of $\mathcal{S}$ in L^∞).

4° $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_0(\mathbf{R};Y)$*:* Assume that Y is a Hilbert space. Set $T\phi := (\mathbb{E}\text{Я}\phi)(0)$ for all $\phi \in \mathcal{C}_0(\mathbf{R};X)$. One easily verifies that $T \in M := \mathcal{B}(\mathcal{C}_0(\mathbf{R};X),Y)$ and $\mathbb{E} = T*$, where $(T*\phi)(t) := T\tau^{-t}\text{Я}\phi$ ($t \in \mathbf{R}$, $\phi \in \mathcal{C}_0(\mathbf{R};X)$). By Lemma D.1.14(d), we have $T*\phi \in \mathcal{C}_0$ for all $\phi \in \mathcal{C}_0$, hence $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_0(\mathbf{R};Y)$.

Remark — This is not true for general Y: Let X be any Banach space (e.g., $X = \mathbf{C}$). Let $Y := \ell^\infty(\mathbf{N};\mathrm{L}^\infty)$, where $\mathrm{L}^\infty := \mathrm{L}^\infty(\mathbf{R};X)$. One easily verifies that $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$, where $(\mathbb{E}f)_n := \tau^{-n}f$ ($f \in \mathrm{L}^\infty$). Obviously, $f \neq 0 \Longrightarrow \mathbb{E}f \notin \mathcal{C}_0(\mathbf{R};\mathrm{L}^\infty)$. (Note also that $\pi_-\mathbb{E}\pi_+ = 0$.) □

If $\mathbb{E} \in \mathrm{TI}_a \cap \mathrm{TI}_b(U,Y)$, $a < b$, then $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(r + i\mathbf{R};\mathcal{B}(U,Y))$ for all $r \in (a,b)$. Actually, $\widehat{\mathbb{E}}$ can be redefined so that it becomes holomorphic on $\mathbf{C}_{a,b} := (a,b) + i\mathbf{R} = \{s \in \mathbf{C} \mid a < \mathrm{Re}\, s < b\}$:

Theorem 3.1.6 ($\widehat{\mathbf{TI}_a \cap \mathbf{TI}_b} = \mathbf{H}^\infty(\{a < \mathbf{Re}\cdot < b\},\mathcal{B})$) *Let U and Y be Hilbert spaces and $-\infty < a \leq b < \infty$.*

(a) Let $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$. Then there is a unique $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, s.t.

$$\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u} \quad \text{on } \mathbf{C}_{a,b}, \tag{3.9}$$

for all $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} = \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$, $\widehat{\mathbb{E}}$ has "strong" nontangential boundary functions (again denoted by $\widehat{\mathbb{E}}$) on $a + i\mathbf{R}$ and $b + i\mathbf{R}$, and $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. in $r + i\mathbf{R}$ for each $r \in [a,b]$ and $u \in \mathrm{L}^2_r(\mathbf{R};U)$.

(b) Conversely, if $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, then there is a unique $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$ s.t. $\widehat{\mathbb{E}}_{|r+i\mathbf{R}}$ is its transform for some (hence all) $r \in (a,b)$. If this is the case, then $\mathbb{E}$ and $\widehat{\mathbb{E}}$ are as in (a).

The above boundary functions exist in the sense that $\widehat{\mathbb{E}}u_0 \in \mathrm{H}^\infty$ has the boundary function $\widehat{\mathbb{E}}u_0$ on $a + i\mathbf{R}$ in the sense of (1.) and (2.) of Theorem 3.3.1(a), for each $u_0 \in U$. It follows that $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}_{a,b};Y)$ has the boundary function $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{L}^2$ on $a + i\mathbf{R}$ in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a), for each $u \in \mathrm{L}^2_a \cap \mathrm{L}^2_b(\mathbf{R};U)$, by Proposition D.1.21(a). The "mirror images" of these two claims hold at $b + i\mathbf{R}$.

Note also that both sides of (3.9) are holomorphic on $\mathbf{C}_{a,b}$, by Proposition D.1.21(a). By the last claim in (a), $\widehat{\mathbb{E}}_{|r+i\mathbf{R}}$ is the Fourier transform of $\mathbb{E} \in \mathrm{TI}_r$ for all $r \in [a,b]$; this justifies our notation (and that of Theorem 2.1.2).

Proof: (The proofs of Theorems 3.1.6 and 3.1.7 use implicitly Theorem 3.3.1(a); naturally, the converse is not true. Part of (a) could also be obtained from Theorem 3.1.7 as a corollary, but we prefer to the simpler proof below.)

(a) 1° *We have $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$:* Let ϕ be as in Lemma D.1.25. We define the function $\widehat{\mathbb{E}}u_0 := \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for each $u_0 \in U$ (since $\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$, by Proposition D.1.21(a), we have $\widehat{\mathbb{E}}(s)u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$).

For any fixed $s \in \mathbf{C}_{a,b}$, the operator $\widehat{\mathbb{E}}(s) : U \to Y$ is obviously linear; by the norm inequality in Lemma D.1.10(a), it is also bounded (for this fixed s). By Lemma D.1.1(b), it follows that $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b}; \mathcal{B}(U,Y))$.

2° $\|\widehat{\mathbb{E}}\|_\infty \le M$: Let $F(r+i\cdot)$ be a representative of the Fourier transform (see Theorem 3.1.3(a)) of $\mathbb{E}$, for each $r \in [a,b]$, so that

$$\max_{r\in[a,b]} \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}(r+i\mathbf{R};\mathcal{B}(U,Y))} = \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\} =: M \tag{3.10}$$

by (2.11). Then, given $r \in (a,b)$, we have $F\widehat{\phi} u_0 = \widehat{\mathbb{E}}\widehat{\phi} u_0$ a.e. on $r+i\mathbf{R}$, hence $Fu_0 = \widehat{\mathbb{E}} u_0$ a.e. on $r+i\mathbf{R}$, i.e., $F = \widehat{\mathbb{E}}$ as an element of $\mathrm{L}^\infty_{\mathrm{strong}}(r+i\mathbf{R};\mathcal{B}(U,Y))$; in particular, $\sup \|\widehat{\mathbb{E}}(r+i\mathbf{R})\|_{\mathcal{B}(U,Y)} = \|\mathbb{E}\|_{\mathrm{TI}_r} \le M$ (since $\widehat{\mathbb{E}}$ is continuous in $\mathbf{C}_{a,b}$, by 1°). Because r was arbitrary, we have $\|\widehat{\mathbb{E}}\|_{\mathcal{B}(U,Y)} \le M$ on $\mathbf{C}_{a,b}$.

3° *Identity (3.9) holds:* Set $\widehat{\mathbb{E}} := F$ on $a+i\mathbf{R}$ and on $b+i\mathbf{R}$, so that $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. in $r+i\mathbf{R}$ for each $r \in [a,b]$ and $u \in \mathrm{L}^2_r(\mathbf{R};U)$ (recall from 2° that $\widehat{\mathbb{E}} = F$ on $r+i\mathbf{R}$ as elements of $\mathrm{L}^\infty_{\mathrm{strong}}$). If $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$, then both sides of (3.9) are holomorphic, by Proposition D.1.21(a), hence then the equality holds on the whole $\mathbf{C}_{a,b}$.

4° *Boundary functions:* By Proposition D.1.21(c), the function $\widehat{\mathbb{E}}\widehat{u}$ is the nontangential boundary function of itself at $a+i\mathbf{R}$ and at $b+i\mathbf{R}$, in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a1). Set $u = \phi u_0$ for an arbitrary $u_0 \in U$ and divide by $\widehat{\phi}^{-1}$ to obtain that $\widehat{\mathbb{E}} u_0$ is the nontangential boundary function of itself at $a+i\mathbf{R}$ and at $b+i\mathbf{R}$, in the sense of (2.) (and (1.)) of Theorem 3.3.1(a1).

(b) The function $\widehat{\mathbb{E}}|_{r+i\mathbf{R}}$ defines a unique $\mathbb{E}_r \in \mathrm{TI}_r(U,Y)$, and

$$\|\mathbb{E}_r\|_{\mathrm{TI}_r} \le \|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} =: M, \tag{3.11}$$

for each $r \in (a,b)$, by Theorem 3.1.3.

Let $u \in \mathcal{F} := \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$ and $r \in (a,b)$ be arbitrary. Then $\widehat{\mathbb{E}}_r \widehat{u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $r+i\mathbf{R}$. But $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}_{a,b};Y)$, hence $\widehat{\mathbb{E}}\widehat{u} = \widehat{f}$ for some unique $f \in \mathrm{L}^2_a \cap \mathrm{L}^2_b$, by Proposition D.1.21(c). By uniqueness, $\mathbb{E}_r u = f$ a.e., hence (3.9) holds with $\mathbb{E}_r$ in place of $\mathbb{E}$. Because $r \in (a,b)$ was arbitrary, we have $\mathbb{E}_r u = \mathbb{E}_{r'} u$ a.e. for any $r, r' \in (a,b)$. Because $u \in \mathcal{F}$ was arbitrary, it follows from Lemma 2.1.10(c) that $\mathbb{E}_r = \mathbb{E}_{r'} \in \mathrm{TI}_r \cap \mathrm{TI}_{r'}$, for any $r, r' \in (a,b)$.

Fix some $r \in (a,b)$ and set $\mathbb{E} := \mathbb{E}_r$. By the above, (3.9) holds for all $u \in \mathcal{F}$, and $\mathbb{E} \in \mathrm{TI}_{r'}(U,Y)$ and $\|\mathbb{E}\|_{\mathrm{TI}_{r'}} \le M$ for all $r' \in (a,b)$. By Lemma 2.1.10(g), it follows that $\mathbb{E} \in \mathrm{TI}_a \cap \mathrm{TI}_b$. Thus, $\mathbb{E}$ and $\widehat{\mathbb{E}}$ are as in (a), so that we obtain the rest of (b) from (a) (if $\widehat{\mathbb{E}}$ is the Fourier transform of $\widetilde{\mathbb{E}} \in \mathrm{TI}_r$ for some $r \in (a,b)$, then $\widetilde{\mathbb{E}} = \mathbb{E}$, by uniqueness). □

We observe that we have achieved an alternative proof of Theorem 2.1.2: it is a corollary of Theorem 3.1.6 and Lemma 2.1.11. A similar weaker claim (the transform being a contractive linear isometry into; also this claim is given in [W91a]) is true also when U and Y are general Banach spaces and L^2 is replaced by L^p, $1 \le p < \infty$, as one obtains from the following (weaker) generalization of the above theorem:

Theorem 3.1.7 ($\widehat{\mathbf{TI}_a \cap \mathbf{TI}_b} \subset \mathbf{H}^\infty(\{a < \mathrm{Re}\cdot < b\}, \mathcal{B})$) *Assume, for this theorem, that X and Y are Banach spaces, $1 \le p < \infty$, and $-\infty < a \le b < \infty$. Let $\mathbb{E} \in \mathrm{TI}^p_a(X,Y) \cap \mathrm{TI}^p_b(X,Y)$. Then there is a unique $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{(a,b)}; \mathcal{B}(X,Y))$ s.t.*

$$\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u} \quad \text{on } \mathbf{C}_{a,b} \tag{3.12}$$

for all simple $u \in \mathrm{L}^2(\mathbf{R};X)$. Moreover, (3.12) holds for all $u \in \mathrm{L}^p_a(\mathbf{R};X) \cap \mathrm{L}^p_b(\mathbf{R};X)$. Finally, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} \le \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$.

The converse is not true; indeed, for $Y := \ell^\infty(\mathbf{N})$, there is $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(Y))$ (continuous to the boundary) s.t. $\widehat{\mathbb{E}}\widehat{u} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R};Y)]$ for some $u \in \mathrm{L}^2(\mathbf{R};Y)$, by Example 3.3.4. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))}$ can be arbitrarily small compared to $\max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$, by the third remark of Example 3.3.4.

Recall from Proposition D.1.21(a) that $\widehat{u}, \widehat{\mathbb{E}u} \in \mathrm{H}(\mathbf{C}_{a,b};*)$ for all $u \in \mathrm{L}^p_a(\mathbf{R};X) \cap \mathrm{L}^p_b(\mathbf{R};X)$. Note also that if $u \in \mathrm{L}^2(\mathbf{R};X)$ is simple, then $u \in \mathrm{L}^q_r(\mathbf{R};X)$ for all $q \in [1,\infty]$, $r \in \mathbf{R}$.

We observe from Definition E.1.3 and Proposition E.1.8 that Remark 2.1.9 also applies in this general case, in particular, $\mathrm{TI}^p_a \cap \mathrm{TI}^p_b$ is well defined.

If Y is separable, then one can obtain an analogous theorem in case $p = \infty$ for $u \in \mathcal{C}_{0,a}(\mathbf{R};X) \cap \mathcal{C}_{0,b}(\mathbf{R};X)$ by slightly modifying the proof below and that of Theorem 3.2.4. However, such functions are not dense in $\mathrm{L}^\infty_a \cap \mathrm{L}^\infty_b$ and the counter-example of Section 3 of [W91a] shows that even Theorem 2.1.2 (which is a corollary Theorem 3.1.7, as explained before the theorem) is false for $p = \infty$ (also when $Y = \mathbf{C} = U$), hence so is Theorem 3.1.7.

Proof of Theorem 3.1.7: *Part I — Preparations:*

I.1° Defining $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$: This goes as in the proof of Theorem 3.1.6: Let ϕ be as in Lemma D.1.25. We define the function $\widehat{\mathbb{E}}u_0 := \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for each $u_0 \in X$ (since $\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$, by Proposition D.1.21(a), we have $\widehat{\mathbb{E}}(s)u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$).

For any fixed $s \in \mathbf{C}_{a,b}$, the operator $\widehat{\mathbb{E}}(s) : X \to Y$ is obviously linear; by the norm inequality in Lemma D.1.10(a), it is also bounded (for this fixed s). By Lemma D.1.1(b), it follows that $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$.

I.2° Defining M: By Proposition E.1.8, we have

$$M := \sup_{r \in [a,b]} \|\mathbb{E}\|_{\mathrm{TI}^p_r} = \max\{\|\mathbb{E}\|_{\mathrm{TI}^p_a}, \|\mathbb{E}\|_{\mathrm{TI}^p_b}\}. \tag{3.13}$$

Part II — Separable X and Y:

II.1° $\|\widehat{\mathbb{E}}\|_\infty \le M$, and (3.14) holds with $\widehat{\mathbb{E}}$ in place of F: Choose S as in Theorem 3.2.4. By Theorem 3.2.4, for each $r \in [a,b]$ we can choose $F(r+\cdot) : i\mathbf{R} \to \mathcal{B}(X,S^*)$ s.t. $\|F(r+it)\|_{\mathcal{B}(X,S^*)} \le M$ for all $t \in \mathbf{R}$ and

$$(F\widehat{u})\Lambda = \mathcal{L}\Lambda\mathbb{E}u \text{ a.e. on } r + i\mathbf{R} \text{ for all } \Lambda \in S \text{ and all finite-dimensional } u \in \mathrm{L}^2_r(\mathbf{R};X) \cap \mathrm{L}^p_r(\mathbf{R};X). \tag{3.14}$$

This defines a function $F : \overline{\mathbf{C}_{a,b}} \to \mathcal{B}(X,S^*)$.

By $I.1°$, we have $\widehat{\mathbb{E}}\widehat{\phi}u_0 = \mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for all $u_0 \in X$. By Proposition D.1.21(a), we have $\mathbb{E}\phi u_0 \in \mathrm{L}^1_r(\mathbf{R};Y)$ for all $r \in (a,b)$, hence $\mathcal{L}\Lambda\mathbb{E}\phi u_0 =$

$\Lambda\mathcal{L}\mathbb{E}\phi u_0$, hence $\mathcal{L}\Lambda\mathbb{E}\phi u_0 = \Lambda\widehat{\mathbb{E}}\widehat{\phi}u_0$, for each $u_0 \in X$. Combine this with (3.14) to observe that (for an arbitrary fixed $r \in (a,b)$)

$$(\widehat{\mathbb{E}}\widehat{\phi}u_0)\Lambda = (\mathcal{L}\mathbb{E}\phi u_0)\Lambda = (F\widehat{\phi}u_0)\Lambda \quad \text{a.e. on } r + i\mathbf{R}, \tag{3.15}$$

hence $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ a.e. on $r+i\mathbf{R}$, for all $\Lambda \in S$ and $u_0 \in X$, hence $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ a.e. on $r+i\mathbf{R}$. Choose a null set N_k for Λ_k for each $k \in \mathbf{N}$ to observe that $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ on $(r+i\mathbf{R}) \setminus N$, where $N := \cup_{k\in\mathbf{N}} N_k$, for all $\Lambda \in \{\Lambda_k\}$, hence for all $\Lambda \in S$, by linearity.

Thus, $F(r+it)u_0 = \widehat{\mathbb{E}}(r+it)u_0$ as elements of S, i.e., $F(r+it)u_0 = \widehat{\mathbb{E}}(r+it)u_0 \in Y$, for all t s.t. $r+it \notin N$, hence $Fu_0 = \widehat{\mathbb{E}}u_0$ a.e. on $r+i\mathbf{R}$. Since $r \in (a,b)$ was arbitrary, we conclude that (3.14) holds with $\widehat{\mathbb{E}}$ in place of F, for all $r \in (a,b)$. We also conclude that

$$\|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,Y)} = \|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,S^*)} = \|F(r+it)\|_{\mathcal{B}(X,S^*)} \leq M \tag{3.16}$$

for a.e. $t \in \mathbf{R}$, hence $\|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,Y)} \leq M$ for all $t \in \mathbf{R}$, for each $r \in (a,b)$. Thus, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))} \leq M$.

II.2° (3.12) and the "moreover" claim hold: By Theorem 3.2.4(b2), condition (3.14) actually holds for all finite-dimensional $u \in \mathrm{L}^1_r \cap \mathrm{L}^p_r$ (since then $u \in \mathrm{L}^2_r$ or $p \leq 2$), when $r \in (a,b)$.

But $\mathrm{L}^p_a \cap \mathrm{L}^p_b \subset \mathrm{L}^1_r \cap \mathrm{L}^p_r$, by Proposition D.1.21(a1), hence (3.14) holds for all finite-dimensional $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b$. But $\mathcal{L}\Lambda\mathbb{E}u = \Lambda\mathcal{L}\mathbb{E}u$, and S separates points (since it is norming), hence $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $r+i\mathbf{R}$, for such u and all $r \in (a,b)$. By continuity, we have $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ everywhere on $\mathbf{C}_{a,b}$ for such u

Given a general $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X)$, there are finite-dimensional $\{u_n\} \subset \mathcal{C}^\infty_c(\mathbf{R};X)$ s.t. $u_n \to u$ in L^p_a and in L^p_b, by Theorem B.3.11(b2). Then $\mathbb{E}u_n \to \mathbb{E}u$ in L^p_a and in L^p_b. By Proposition D.1.21(a2), it follows that $\mathcal{L}\Lambda\mathbb{E}u_n \to \mathcal{L}\Lambda\mathbb{E}u$ and $\widehat{\mathbb{E}}\widehat{u}_n \to \widehat{\mathbb{E}}\widehat{u}$ pointwise on $\mathbf{C}_{a,b}$. Therefore, $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$.

Part III — the general case:

III.1° $\|\widehat{\mathbb{E}}\|_\infty \leq M$: Let X_0 be any closed separable subspace of X. Choose Y_0 as in Lemma 3.2.6, so that $\mathbb{E}_{X_0,Y_0} := \mathbb{E}_{|\mathrm{L}^p(\mathbf{R};X_0)} \in \mathrm{TI}^p(X_0,Y_0)$.

By Part II, $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for all $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X_0)$. Take $\widehat{u} = \phi x_0$ for arbitrary $x_0 \in X_0$ to observe that $\widehat{\mathbb{E}}(s_0)x_0 \in Y_0$. Thus, $\widehat{\mathbb{E}}_{X_0} := \widehat{\mathbb{E}}_{|X_0} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X_0,Y_0))$.

We deduce from this and Part II that $\|\widehat{\mathbb{E}}x_0\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};Y)} \leq M$ for all $x_0 \in X_0$. Since X_0 was arbitrary, we have $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$ and $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))} \leq M$.

III.2° (3.12) and the "moreover" claim hold: Given $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X)$, choose a closed separable subspace X_0 of X s.t. $u(t) \in X_0$ for a.e. $t \in \mathbf{R}$, so that $u \in \mathrm{L}^p(\mathbf{R};X_0)$ (after redefinition on a null set). Choose Y_0 as in *III.1°*, so that $\widehat{\mathbb{E}}_{X_0,Y_0} \in \mathrm{TI}^p(X_0,Y_0)$. Now we observe from Part II that (3.12) and the "moreover" claim hold.

III.3° Uniqueness: Assume that also some $\widehat{\mathbb{F}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$ is as in the theorem (in place of $\widehat{\mathbb{E}}$). Then $(\widehat{\mathbb{E}} - \widehat{\mathbb{F}})\widehat{f}u_0 = 0$ on $\mathbf{C}_{a,b}$ for all simple

$f \in \mathrm{L}^2(\mathbf{R})$ and all $u_0 \in U$. Given $s_0 \in \mathbf{C}_{a,b}$, take $f := \chi_{[0,r]}$ (so that $f \in \mathrm{H}(\mathbf{C})$, $\widehat{f}(s) = (1 - \mathrm{e}^{-rs})/s)$, where $r > 0$ is s.t. $r \operatorname{Re} s_0 \notin 2\pi\mathbf{N}$, so that $\widehat{f}(s_0) \neq 0$ to observe that $(\widehat{\mathbb{E}} - \widehat{\mathbb{F}})(s_0) = 0$. □

Notes

The $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$ result of Theorem 3.1.3(a1) is well known in the case of separable Hilbert spaces, see, e.g., Theorem 1 of [FS], which also provides an analogous result on unbounded closed operators. Lemma 13.1.5 provides the discrete-time counterpart of our result and Appendix F provides further results on $\mathrm{L}^\infty_{\mathrm{strong}}$.

By Example 3.3.4, not all $\mathrm{L}^\infty_{\mathrm{strong}}$ functions (not even all $\widehat{\mathrm{CTIC}}$ functions) correspond to TI operators when U and Y are allowed to be Banach spaces.

We conjecture that all $\mathrm{TI}^p(U,Y)$ maps have $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ transfer functions also when U and Y are general Banach spaces and $1 \leq p < \infty$. However, we cannot prove this; see Section 3.2 for weaker analogies.

The claims on $\mathrm{W}^{n,p}_\omega$ in Theorem 3.1.5 are from [Sbook]. The special case $\mathbb{E} \in \mathrm{TIC}$ of Theorems 3.1.6 and 3.1.7 is essentially contained in Theorem 2.3 of [W91a].

See Appendix F and the notes on p. 1023 for $\mathrm{L}^\infty_{\mathrm{strong}}$.

3.2 $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$ for Banach spaces (Fourier Multipliers)

I had a feeling once about mathematics – that I saw it all. Depth beyond depth was revealed to me – the Byss and the Abyss. I saw – as one might see the transit of Venus or even the Lord Mayor's Show – a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why tergiversation was inevitable – but it was after dinner and I let it go.

— Winston Churchill, (1874–1965)

In this section, we study Banach space Fourier multiplier theory; we shall mainly give Banach and TI^p equivalents of Theorem 3.1.3, $1 \le p < \infty$, with some partial results and guidelines for the case $p = \infty$, to which a complete extension would be false. Some of these results were used in the proof of Theorem 3.1.7.

We start by recalling the scalar case from [BL]:

Lemma 3.2.1 ($\mathbf{TI}^p(\mathbf{C}) \subset \mathbf{TI}^2(\mathbf{C})$) *Let $\mathbb{E} \in \mathrm{TI}^p_\omega(\mathbf{C})$, where $p \in [1,\infty)$ and $\omega \in \mathbf{R}$. Then $\mathbb{E}|_{C_c^\infty}$ has a unique extension to $\mathrm{TI}^2_\omega(\mathbf{C})$, and this extension coincides with $\mathbb{E}$ on $\mathrm{L}^p_\omega \cap \mathrm{L}^2_\omega$.*

In particular, there is a unique $\widehat{\mathbb{E}} \in \mathrm{L}^\infty(\omega + i\mathbf{R})$ s.t. $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}) \cap \mathrm{L}^p_\omega(\mathbf{R})$. In fact, we have $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R})$; if $p \in [1,2]$, then $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^p_\omega(\mathbf{R})$ too. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty(\omega+i\mathbf{R})} \le \|\mathbb{E}\|_{\mathrm{TI}^p_\omega}$.

The above also holds with $\mathrm{TI}^{C_0}_\omega$ in place of TI^p_ω and $C_{0,\omega}$ in place of L^p_ω (for $p = \infty$).

We conclude that $\mathrm{TI}^p(\mathbf{C}^n, \mathbf{C}^m) \subset \mathrm{TI}^2(\mathbf{C}^n, \mathbf{C}^m)$ for all $n, m \in \mathbf{N}$ (apply the lemma to each component of $\mathbb{E}$). Naturally, the space $\mathrm{TI}^{C_0}_\omega$ mentioned above refers to

$$\mathrm{TI}^{C_0}_\omega(U,Y) := \{\mathbb{E} \in \mathcal{B}(C_{0,\omega}(\mathbf{R};U), C_{0,\omega}(\mathbf{R};Y)) \mid \mathbb{E}\tau(t) = \tau(t)\mathbb{E} \text{ for all } t \in \mathbf{R}\}, \tag{3.17}$$

where $C_{0,\omega}(\mathbf{R},U) := \mathrm{e}^{\omega\cdot} C_0(\mathbf{R};U)$ (with norm $\|f\|_{C_{0,\omega}} := \|\mathrm{e}^{-\omega\cdot} f\|_{C_0} := \|\mathrm{e}^{-\omega\cdot} f\|_\infty$ for all $f \in C_{0,\omega}$). Since C_c^∞ is dense in $C_{0,\omega}$ (by Theorem B.3.11(c), since $\mathrm{e}^{-\omega\cdot} C_c^\infty = C_c^\infty$), this is in accordance with standard Fourier multiplier theory (and this allows a density argument unlike L^∞_ω would do). We observe from Theorem 3.1.5 that $\mathbb{E}|_{C_{0,\omega}} \in \mathrm{TI}^{C_0}_\omega(U,Y)$ for all $\mathbb{E} \in \mathrm{TI}^\infty_\omega(U,Y)$.

As shown in Example 3.2.3, we may have $\widehat{\mathbb{E}} = 0$ (equivalently, $\mathbb{E}|_{C_c^\infty} = 0$, hence $\mathbb{E}|_{C_0} = 0$) for $\mathbb{E} \in \mathrm{TI}^\infty \setminus \{0\}$.

Proof of Lemma 3.2.1: (W.l.o.g. we assume that $\omega = 0$.)

1° This follows easily from Theorem 6.1.2 of [BL] (and Theorem B.3.11 and Lemma A.3.10) if we require that $u \in \mathcal{S}(\mathbf{R})$. In particular, if $u \in \mathrm{L}^p \cap \mathrm{L}^2$, then there are $\{u_n\} \subset C_c^\infty(\mathbf{R})$ s.t. $u_n \to u$ in L^p_ω and in L^2_ω, thus, then $\mathbb{E}_2 u \leftarrow \mathbb{E}_2 u_n = \mathbb{E}u_n \to \mathbb{E}u$, as $n \to \infty$, where $\mathbb{E}_2$ is the extension of $\mathbb{E}|_{C_c^\infty}$ to $\mathrm{TI}^2(\mathbf{C})$. Therefore $\mathbb{E} = \mathbb{E}_2$ on $\mathrm{L}^p \cap \mathrm{L}^2$.

2° Assume that $p \in [1,2]$ and that $u \in \mathrm{L}^p(\mathbf{R})$. Choose $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbf{R})$ s.t. $u_n \to u$ in $\mathrm{L}^p(\mathbf{R})$. Since $\mathbb{E}u_n \to \mathbb{E}u$ in $\mathrm{L}^p(\mathbf{R})$, we have $\widehat{\mathbb{E}u_n} \to \widehat{\mathbb{E}u}$ in $\mathrm{L}^q(i\mathbf{R})$, where p and q are as in Theorem E.1.7. But $\widehat{\mathbb{E}u_n} = \widehat{\mathbb{E}}\widehat{u}_n \to \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$, hence $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$.

3° For general p, we have $\mathbb{E} \in \mathrm{TI}^2$, hence $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$ for all $u \in \mathrm{L}^2(\mathbf{R})$, by 2°.

4° Parts 1° and 3° apply to $\mathrm{TI}_0^{\mathcal{C}_0}$ too, since for any $f \in \mathcal{C}_0 \cap \mathrm{L}^2$, there is $\{f_n\} \subset \mathcal{C}_c^\infty$ s.t. $f_n \to f$ in L^2 and in $\mathcal{C}_0$ (multiply the convolution from Lemma 2.18 of [Adams] with suitable ϕ from Lemma B.3.10). □

Corollary 3.2.2 *Let $\mathbb{E} \in \mathrm{TI}^p_\omega(X,Y)$, where X and Y are arbitrary Banach spaces and $1 \le p < \infty$. Then $\Lambda\mathbb{E}f \in \mathrm{L}^2_\omega(\mathbf{R}) \cap \mathrm{L}^p_\omega(\mathbf{R})$ for each $\Lambda \in Y^*$ and each finite-dimensional $f \in \mathrm{L}^2_\omega(\mathbf{R};X) \cap \mathrm{L}^p_\omega(\mathbf{R};X)$ (for $f \in \mathrm{L}^2_\omega(\mathbf{R};X) \cap \mathcal{C}_{0,\omega}(\mathbf{R};X)$ we can allow for $p = \infty$).*

Proof: (We assume that $p < \infty$; the case $p = \infty$ is analogous.) Let $f \in \mathrm{L}^2_\omega(\mathbf{R};X_0) \cap \mathrm{L}^p_\omega(\mathbf{R};X_0)$, where X_0 is a finite-dimensional subspace of X.

If $X_0 = \mathrm{span}\{x_0\}$ for some $x_0 \in X$, then $\Lambda\mathbb{E}P^*_{x_0} \in \mathrm{TI}^p_\omega(\mathbf{C})$, where $P_{x_0}\alpha := \alpha x_0$ ($\alpha \in \mathbf{C}$), hence then the claim follows from Lemma 3.2.1.

For a general n-dimensional X_0, we can apply the above to the n one-dimensional elements of $\Lambda\mathbb{E}P^*_{X_0}$, where P_{X_0} is an isomorphism $X_0 \to \mathbf{C}^n$. □

In case $p = \infty$, the operator $\widehat{\mathbb{E}}$ does no longer define $\mathbb{E}$ uniquely:

Example 3.2.3 [$\mathbf{0 \ne \mathbb{E} \in TIC^\infty(C)}$ but $\widehat{\mathbb{E}} = \mathbf{0}$] Let $\Lambda \in \mathrm{L}^\infty(\mathbf{R})^*$ be a "Banach limit at $-\infty$". Define $\mathbb{E} \in \mathrm{TIC}^\infty(\mathbf{C})$ by $(\mathbb{E}f)(t) := \Lambda f$ ($t \in \mathbf{R}$), so that $\|\mathbb{E}\|_{\mathrm{TIC}^\infty} = 1$. (Note that $\mathbb{E}f$ is a constant function for each $f \in \mathrm{L}^\infty(\mathbf{R})$.)

Then $\mathbb{E}f = 0$ for all $f \in \mathcal{C}_0$ and for all $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$ ($1 \le p < \infty$); in particular, $\widehat{\mathbb{E}} \equiv 0$ in the sense of Lemma 3.2.1, and 0 is the unique continuous extension of $\mathbb{E}|_{\mathcal{S}}$ to $\mathcal{B}(\mathrm{L}^p(\mathbf{R}))$ (or equivalently, to $\mathrm{TI}^p(\mathbf{C})$), for any $p \in [1,\infty)$. ◁

For $\mathbb{E} \in \mathrm{TI}^p$, $p < \infty$ this cannot happen since $\mathbb{E}|_{\mathcal{S}}$ determines $\mathbb{E}$ uniquely.

In the above example, $\mathbb{E}f$ coincides with the unique extension (namely 0) of $\mathbb{E}|_{\mathcal{S}}$ to $\mathrm{TI}^p(\mathbf{C})$ for all $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$, but we do not know whether this is the case in general. At least the same cannot happen for $\mathbb{E} \in \mathrm{TI}^p$, $p < \infty$: when $p, q \in [1,\infty)$, $\mathbb{E} \in \mathrm{TI}^p(X,Y)$ and $f \in \mathrm{L}^p \cap \mathrm{L}^q(\mathbf{R};X)$, there is $\{f_n\} \subset \mathcal{C}_c^\infty \subset \mathcal{S}$ s.t. $f_n \to f$ in both L^p and L^q, by Theorem B.3.11, hence then the unique extension (if any) of $\mathbb{E}|_{\mathcal{S}}$, to TI^q necessarily coincides with $\mathbb{E}$ on $\mathrm{L}^p \cap \mathrm{L}^q$.

(Also when extending an element of TI^p_ω to TI^p_α for some $\omega, \alpha \in \mathbf{R}$, we may have similar problems only in the case that $p = \infty$, due to same density arguments as above. See also Proposition E.1.8.)

Proof of Example 3.2.3: Define $\Lambda_n \in \mathrm{L}^\infty(\mathbf{R})^*$ by $\Lambda_n f := n^{-1}\int_{-n}^0 f\,dm$, for $n \in \mathbf{N}+1$, and set $\Lambda f := \lim_{n\to+\infty}\Lambda_n f$ for $f \in X$, where $X \subset \mathrm{L}^\infty(\mathbf{R})$ is the set of those $f \in \mathrm{L}^\infty(\mathbf{R})$ for which the limit exists. Use the Hahn–Banach theorem to extend Λ to $\mathrm{L}^\infty(\mathbf{R})^*$ with $\|\Lambda\|_{\mathcal{B}(\mathrm{L}^\infty)} = 1$ (since $\Lambda 1 = 1$). Obviously, $\tau^t f - f \in X$

and $\Lambda(\tau^t f - f) = 0$ for all f, hence Λ is time-invariant, hence so is $\mathbb{E}$. (Note that $\mathbb{E}f = \Lambda * \text{Я} f$ if we use the standard definition $\Lambda * g := \Lambda(\tau^{-t}\text{Я} g)$.)

Since $\mathbb{E}f \equiv 0$ whenever $\pi_{(-\infty,T)} f = 0$ for some $T \in \mathbf{R}$, we have $\mathbb{E}|_{\mathcal{C}_0} = 0$, by continuity, and $\pi_-\mathbb{E}\pi_+ = 0$, hence $\mathbb{E} \in \mathrm{TIC}^\infty$. By the Hölder Inequality, we have $f \in X$ and $\Lambda f = 0$ for any $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$, $p \in [1,\infty)$.

Remark: One could also obtain an analogous operator in $\mathrm{TIC}^\infty(X,Y)$ for general Banach spaces X and Y by starting with $\Lambda_n f := n^{-1}\int_{-n}^0 Lf\,dm$ for some $L \in X^*$. □

Now we present a weak "generalization" of "$\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$" (Theorem 3.1.3) for Banach spaces; recall that Theorem 3.1.7 is an application of this theorem:

Theorem 3.2.4 ($\mathbb{E} \in \mathrm{TI}^p(X,Y) \implies \widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{weak}^*}(i\mathbf{R};\mathcal{B}(X,S^*))$) *Let X and $Y \neq \{0\}$ be separable Banach spaces and $1 \le p < \infty$.*

By Lemma A.3.9, we can choose a sequence $\{\Lambda_k\} \subset Y^*$ s.t. $\|\Lambda_k\| = 1$ for all $k \in \mathbf{N}$ and $\|y\|_Y = \sup_{k\in\mathbf{N}} |\Lambda_k y|$.

Set $S := \mathrm{span}(\{\Lambda_k\}) \subset Y^*$. Then $(Iy)\Lambda := \Lambda y$ defines a (natural) linear isometry $I : Y \to S^*$, hence we can consider Y as a subspace of S^* and I as the inclusion $Y \subset S^*$.

If $\mathbb{E} \in \mathrm{TI}^p(X,Y)$, then there is $\widehat{\mathbb{E}} : i\mathbf{R} \to \mathcal{B}(X,S^*)$ s.t. $\|\widehat{\mathbb{E}}(it)\|_{\mathcal{B}(X,S^*)} \le \|\mathbb{E}\|_{\mathrm{TI}^p(X,Y)}$ for all $t \in \mathbf{R}$ and

$$(\widehat{\mathbb{E}}\widehat{f})\Lambda = \mathcal{L}\Lambda\mathbb{E}f \text{ a.e. } \textit{on } i\mathbf{R} \textit{ for all } \Lambda \in S \textit{ and all finite-dimensional } f \in \mathrm{L}^2(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X). \tag{3.18}$$

Moreover, the following hold:

(a) Furthermore, in (3.18) we can allow Λ to be any $\Lambda \in \bar{S} \subset Y^*$ (recall that $(\overline{S})^* = S^*$); if $p \le 2$, then, simultaneously, any $f \in \mathrm{L}^1(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X)$ can be allowed.

(b1) For a fixed $\widehat{\mathbb{E}}$, equation (3.18) characterizes $\mathbb{E} \in \mathrm{TI}^p(X,Y)$ uniquely.

(b2) For a fixed $\mathbb{E} \in \mathrm{TI}^p(X,Y)$, equation (3.18) characterizes $\widehat{\mathbb{E}}$ uniquely in the sense that if $\widehat{\mathbb{E}}, \widehat{\mathbb{F}} : i\mathbf{R} \to \mathcal{B}(X,S^*)$ satisfy (3.18), then $\widehat{\mathbb{E}}x = \widehat{\mathbb{F}}x$ a.e. for all $x \in X$.

(c) For each $x \in X$ and $\Lambda \in S$, we have $(\widehat{\mathbb{E}}x)\Lambda \in \mathrm{L}^\infty(i\mathbf{R})$.

(d) For a fixed $x \in X$, the function $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is (Bochner) measurable iff $\widehat{\mathbb{E}}x \in Y$ a.e. However, if $\widehat{\mathbb{E}}x \in Y$ a.e. for all $x \in X$, then $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(X,Y))$.

As noted in the second remark of Example 3.3.4, the Plancherel Theorem does not hold for general Banach spaces. However, if $f \in \mathrm{L}^2(\mathbf{R};X)$ and f has a finite-dimensional range, or in $f \in \mathrm{L}^1(\mathbf{R};X)$, then f has a well-defined Fourier transformation (in L^2 or $\mathcal{C}_0$, respectively), by Lemma D.1.11(a1) or Lemma A.3.4(Q1).

By Corollary 3.2.2, we have $\Lambda\mathbb{E}f \in \mathrm{L}^2(\mathbf{R})$ for each $\Lambda \in Y^*$ and each finite-dimensional $f \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R};X)$, hence $\widehat{\Lambda\mathbb{E}f} \in \mathrm{L}^2(i\mathbf{R})$ and $\widehat{f} \in \mathrm{L}^2(\mathbf{R};X)$ are well defined in (3.18). If $p \le 2$ and $f \in \mathrm{L}^1 \cap \mathrm{L}^p$, then $\widehat{f} \in \mathcal{C}_0(i\mathbf{R};X)$ and $\Lambda\mathbb{E}f \in \mathrm{L}^p(\mathbf{R})$, hence then $\widehat{\Lambda\mathbb{E}f} \in \mathrm{L}^q(i\mathbf{R};Y)$ is well defined in (3.18), where $p^{-1}+q^{-1}=1$ (see Theorem E.1.7).

Naturally, we can shift the above theorem to obtain a result on TI^p_ω for any $\omega \in \mathbf{R}$. By slightly modifying the proof, we obtain an analogous claim for TI^{C_0} too (with a density argument similar to that in 4° of the proof of Lemma 3.2.1).

Proof of Theorem 3.2.4: (N.B. one can deduce from the proof and Lemma 3.2.1 that the operator $\Lambda\mathbb{E}$ has a unique continuous extension to $\mathrm{L}^2(\mathbf{R};X_0)$ for any finite-dimensional subspace X_0 of X; and (3.18) holds for all elements $\mathrm{L}^2(\mathbf{R};X_0)$ (for this extended $\Lambda\mathbb{E}$).)

The construction of the normed space $S \subset Y^*$ is straightforward. Clearly $\|Iy\| \le \|y\|_{Y^{**}} = \|y\|_Y$; the converse follows from $\|Iy\|_{S^*} \ge \sup_k |\Lambda_k y| = \|y\|_Y$. Moreover, any functional $T \in S^*$ has a (necessarily norm-preserving) unique extension $\overline{T} \in (\overline{S})^*$, by, e.g., Lemma A.3.10. Thus only the claims concerning $\widehat{\mathbb{E}}$ are left to be proved.

1° *The functions $T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$:* We denote $\|\mathbb{E}\|_{\mathrm{TI}^p(X,Y)}$ by $\|\mathbb{E}\|$. If $x \in X$ and $\Lambda \in Y^*$, then $\|g \mapsto \Lambda\mathbb{E}gx\|_{\mathrm{TI}^p(\mathbf{C})} \le \|\mathbb{E}\|\,\|\Lambda\|\,\|x\|$, hence, by Lemma 3.2.1 and Theorem 3.1.3(a1), there is $T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. (we choose the representative $T_{\Lambda,x} := LT_{\Lambda,x}$ from Lemma B.5.3)

$$\sup_{i\mathbf{R}} |T_{\Lambda,x}| \le \|\mathbb{E}\|\,\|\Lambda\|\,\|x\| \quad \text{and} \tag{3.19}$$

$$T_{\Lambda,x}\widehat{g} = \mathcal{L}\Lambda\mathbb{E}gx \text{ (a.e.) for all } g \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R}). \tag{3.20}$$

The mapping $Y^* \times X \ni (\Lambda,x) \mapsto T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ is bilinear (because $\mathcal{L}$ is linear), and its norm is at most $\|\mathbb{E}\|$, by (3.19). It follows from (B.56) (and the choice $T_{\Lambda,x} := LT_{\Lambda,x}$) that

$$it \in \mathrm{Leb}(T_{\Lambda,x}) \cap \mathrm{Leb}(T_{\Lambda,x'}) \implies T_{\Lambda,\alpha x+\beta x'}(it) = \alpha T_{\Lambda,x}(it) + \beta T_{\Lambda,x'}(it) \tag{3.21}$$

whenever $t \in \mathbf{R},\ \Lambda \in Y^*,\ x,x' \in X,\ \alpha,\beta \in \mathbf{C}$.

2° *The construction of $\widehat{\mathbb{E}}$:* Let $\{x_j\} \subset X$ be dense. Set

$$A := \cap_{j,k\in\mathbf{N}} \mathrm{Leb}(T_{\Lambda_k,x_j}), \qquad X_0 := \mathrm{span}(\{x_j\}_{j\in\mathbf{N}}) \subset X. \tag{3.22}$$

Then $m(i\mathbf{R} \setminus A) = 0$. For $it \in i\mathbf{R} \setminus A$, we set $\widehat{\mathbb{E}}(it) = 0$. For $it \in A$, $x \in X_0$ and $\Lambda \in S$, we define $(\widehat{\mathbb{E}}(it)x)\Lambda := T_{\Lambda,x}(it)$. It follows from (B.56) that $\widehat{\mathbb{E}}(it)x : S \to \mathbf{C}$ is linear; it is bounded by $\|\mathbb{E}\|\|x\|$, by (3.19), hence $\widehat{\mathbb{E}}(it)x \in S^*$ (for fixed $x \in X_0$ and $it \in A$).

By (3.21), mapping $x \mapsto \widehat{\mathbb{E}}(it)x \in S^*$ is linear, and by (3.19) it is bounded by $\|\mathbb{E}\|$, hence $\widehat{\mathbb{E}}(it) \in \mathcal{B}(X_0,S^*)$ (for fixed $it \in A$). By Lemma A.3.10, $\widehat{\mathbb{E}}(it)$ can be extended to an element of $\mathcal{B}(X,S^*)$ without affecting its norm, hence $\|\widehat{\mathbb{E}}(it)\|_{\mathcal{B}(X,S^*)} \le \|\mathbb{E}\|$ for all $it \in A$, hence for all $it \in i\mathbf{R}$.

3° *The verification of (3.18) for finite-dimensional $f \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R};X)$:* Let $\Lambda \in S$ and $g \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R})$. Then

$$(\widehat{\mathbb{E}}\widehat{g}x)\Lambda = \widehat{g}(\widehat{\mathbb{E}}x)\Lambda = \widehat{g}T_{\Lambda,x} = \mathcal{L}\Lambda\mathbb{E}gx \quad \text{a.e. on } i\mathbf{R} \tag{3.23}$$

(the first equality holds everywhere, the second on A; the third equality holds in L^2, by (3.20), hence also pointwise a.e.) when $x \in X_0$; by continuity, this holds whenever $x \in X$ (the right-hand-side converges in L^2, hence a.e.; the left-hand-side converges pointwise everywhere when $x \to x'$ for some $x' \in X$). Thus, (3.18) holds for $f = gx$ with $g \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R})$ and $x \in X$ arbitrary, hence whenever $f \in \mathrm{L}^2(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X)$ is finite-dimensional, by linearity.

(a) 4° *The verification of (3.18) for finite-dimensional* $f \in \mathrm{L}^p(\mathbf{R};X)$*:* Assume that $1 \le p \le 2$. Then (3.20) also holds for any $g \in \mathrm{L}^p(\mathbf{R})$, by Lemma 3.2.1, hence (3.18) holds whenever $f \in \mathrm{L}^p(\mathbf{R};X)$ is finite-dimensional, by linearity, as in 3°.

For a general $f \in \mathrm{L}^1(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X)$, there are finite-dimensional $f_n \in \mathcal{C}_c^\infty(\mathbf{R};X)$ $(n \in \mathbf{N})$ s.t. $f_n \to f$ in L^1 and in L^p, as $n \to +\infty$, by Theorem B.3.11(b1). It follows that $\Lambda\mathbb{E}f_n \to \Lambda\mathbb{E}f$ in L^p, hence $\mathcal{L}\Lambda\mathbb{E}f_n \to \mathcal{L}\Lambda\mathbb{E}f$ in L^q, hence a.e., where $p^{-1} + q^{-1} = 1$. But $\widehat{f_n} \to \widehat{f}$ in $\mathcal{C}_0$, hence $\widehat{\mathbb{E}}\widehat{f_n}\Lambda \to \widehat{\mathbb{E}}\widehat{f}\Lambda$ everywhere, hence $\mathcal{L}\Lambda\mathbb{E}f = \widehat{\mathbb{E}}\widehat{f}\Lambda$ a.e.

(Note that if X is a Hilbert space, then $\mathcal{L} \in \mathcal{B}(\mathrm{L}^p(\mathbf{R};X), \mathrm{L}^q(\mathbf{R};X))$, by Theorem E.1.7, hence then any $f \in \mathrm{L}^p(\mathbf{R};X)$ will do (then we can have $\widehat{f_n} \to \widehat{f}$ in L^q, hence a.e. on $i\mathbf{R}$ in the above proof).)

5° *Case* $\Lambda \in \bar{S}$*:* The extension for $\Lambda \in \bar{S}$ follows by continuity (because S and $\bar{S} \subset Y^*$ have the same dual, by Lemma A.3.10).

(c) 6° Let $g := \phi$, where $\phi \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R})$ is as in Lemma D.1.25. Divide (3.23) by $\widehat{\phi}^{-1}$ to obtain that $(\widehat{\mathbb{E}}x)\Lambda = T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ for any $x \in X$ (hence $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is "weakly*-measurable").

(b1) 7° If $\widehat{\mathbb{E}} = 0$, then $\Lambda\mathbb{E}f = 0$ a.e. for all $\Lambda \in S$ and all simple $f \in \mathrm{L}^p(\mathbf{R};X)$, by (3.18), hence then $\mathbb{E}f = 0$ a.e. for all simple $f \in \mathrm{L}^p(\mathbf{R};X)$ by Lemma B.2.6, hence $\mathbb{E}f = 0$ for all $f \in \mathrm{L}^p(\mathbf{R};X)$, by density. Thus, if $\mathbb{E}$ and $\mathbb{F}$ correspond to some $\widehat{\mathbb{E}}$ as in the theorem, then $(\mathbb{F} - \mathbb{E}) = 0$ as elements of $\mathrm{TI}^p(X,Y)$.

(b2) 8° If $\mathbb{E} = 0$, then $(\widehat{\mathbb{E}}x)\Lambda = \widehat{\phi}^{-1}\mathcal{L}\Lambda\mathbb{E}\phi x = 0$ a.e. on $i\mathbf{R}$ for all $x \in X$ and $\Lambda \in S$, where ϕ is as in Lemma D.1.25; thus, then $\widehat{\mathbb{E}}x = 0$ a.e. for each $x \in X$ (choose a null set N_k for Λ_k for each k to obtain that $\widehat{\mathbb{E}}x = 0$ on $i\mathbf{R} \setminus \cup_{k \in \mathbf{N}} N_k$).

Thus, $\widehat{\mathbb{E}}, \widehat{\mathbb{F}} : i\mathbf{R} \to \mathcal{B}(X,S^*)$ satisfy (3.18) (at least for one-dimensional L^2 functions f and all $\Lambda \in S$), then $(\widehat{\mathbb{F}} - \widehat{\mathbb{E}})x = 0$ a.e. for each $x \in X$, as required.

(d) 9°

9.1° If $x \in X$ and $\widehat{\mathbb{E}}x \in Y$ a.e., then $\widehat{\mathbb{E}}x$ is almost separably-valued, and from the measurability of $\Lambda\widehat{\mathbb{E}}x$ for each $\Lambda \in S$ (see 6°) one can deduce that $\Lambda\widehat{\mathbb{E}}x$ is measurable for each $\Lambda \in Y^*$ (see, e.g., [Thomas, Corollary 2.9]), hence then $\widehat{\mathbb{E}}x$ is Bochner measurable.

(N.B. Whenever $\widehat{\mathbb{E}}x$ is almost separably-valued, equivalently, whenever $\widehat{\mathbb{E}}x \in Y_1$ a.e., where Y_1 is any separable subspace of S^*, then $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is Bochner-measurable, by the above reasoning.)

9.2° Conversely, assume that $x \in X$ and that $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is measurable. Then $f := \widehat{\mathbb{E}}(i\cdot)x$ is bounded, by 2°, hence $f \in \mathrm{L}^\infty(\mathbf{R};S^*)$. By (3.18) and Lemma D.1.11(e1), we have

$$(\mathcal{L}f\chi_A)\Lambda = \mathcal{L}(f\chi_A\Lambda) = 2\pi\Lambda\mathbb{E}(i\cdot)g_A(i\cdot)x \tag{3.24}$$

a.e. whenever $A \subset \mathbf{R}$, $m(A) < \infty$ and $\widehat{g}_A := \chi_A$ (note that $\mathcal{L}f\chi_A \in \mathcal{C}_0(i\mathbf{R};S^*)$).

Choose $N \subset \mathbf{R}$ s.t. $m(N) = 0$ and equality holds on $\mathbf{R} \setminus N$ in (3.24) when

$\Lambda \in \{\Lambda_k\}$. By linearity, the same holds for all $\Lambda \in S$ on $\mathbf{R}\setminus N$, hence $(\mathcal{L}f\chi_A)(it) = 2\pi\Lambda\mathbb{E}(it)g_A(it)x \in Y$ as elements of S^*, for each $t \in \mathbf{R}\setminus N$, hence a.e. Because A was arbitrary, we have $\widehat{\mathbb{E}}(i\cdot)x = f(\cdot) \in Y$ a.e., by Lemma D.1.22.

9.3° Finally, assume that $\widehat{\mathbb{E}}x \in Y$ a.e. for each $x \in X$ Then $\widehat{\mathbb{E}}x \in \mathrm{L}^\infty(i\mathbf{R};Y)$, as shown above. Let $\{x_k\} \subset X$ be dense, and let $\widehat{\mathbb{E}}x_k \in Y$ on N^c for each $k \in \mathbf{N}$, where $m(N) = 0$. By continuity, then $\widehat{\mathbb{E}}x \in Y$ on N^c for all $x \in X$, so we can redefine $\widehat{\mathbb{E}} = 0$ on N to make $\widehat{\mathbb{E}}$ $\mathcal{B}(X,Y)$-valued without affecting its properties stated in the theorem. Thus, $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(X,Y))$. □

As another application of the above theorem, we deduce an implication that will be needed for Theorem 4.1.1:

Lemma 3.2.5 ($\mathbb{E} \in \mathrm{MTI}(X,Y) \cap \mathcal{G}\mathrm{TI}(X,Y) \implies \widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(X,Y))$) *Let X and Y be Banach spaces. Let $\mathbb{E} \in \mathrm{TI}(X,Y)$ and $\mathbb{V} \in \mathrm{TI}(Y,X)$. Assume, in addition, that $\mathbb{E} \in \mathrm{MTI}(X,Y)$.*

If $\mathbb{V}\mathbb{E} = I$, then $\|\widehat{\mathbb{E}}x\| \ge \|x\|/\|\mathbb{V}\|$ a.e. for all $x \in X$. If, in addition, $\mathbb{E}\mathbb{V} = I$ (i.e., $\mathbb{V} = \mathbb{E}^{-1}$), then $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(X,Y))$ and $\|\widehat{\mathbb{E}}^{-1}\| \le \|\mathbb{E}^{-1}\|$.

(In fact, $\mathbb{E}$ could be replaced by a more general measure.)

Proof: W.l.o.g. we assume that $X \ne \{0\} \ne Y$.

1° *The separable case, $\mathbb{V}\mathbb{E} = I$:*
Choose $S \subset X^*$ and $\widehat{\mathbb{V}} : i\mathbf{R} \to \mathcal{B}(Y,S^*)$ for $\mathbb{V} \in \mathrm{TI}(Y,X)$ as in Theorem 3.2.4. Let $g \in \mathrm{L}^1(\mathbf{R};X) \cap \mathrm{L}^2(\mathbf{R};X)$, so that $f := \mathbb{E}g \in \mathrm{L}^1 \cap \mathrm{L}^2$ and $\widehat{f} = \widehat{\mathbb{E}}\widehat{g}$, by Lemma D.1.12(c2)&(c1). Equation (3.18) with $f = \mathbb{E}g$ in place of an arbitrary $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ becomes (use Theorem 3.2.4(a))

$$(\widehat{\mathbb{V}}\widehat{\mathbb{E}}\widehat{g})\Lambda = \mathcal{L}\Lambda\mathbb{V}\mathbb{E}g = \mathcal{L}\Lambda I g \text{ a.e. for all } \Lambda \in S. \tag{3.25}$$

Because $g \in \mathrm{L}^1 \cap \mathrm{L}^2$ was arbitrary, the uniqueness claim of Theorem 3.2.4 applied to $I \in \mathrm{TI}(X)$ implies that for an arbitrary $x \in X$ we have $Ix = \widehat{\mathbb{V}}\widehat{\mathbb{E}}x$ a.e. (in S^*, hence in X), in particular,

$$\|\widehat{\mathbb{E}}x\| \le \|x\|/\|\widehat{\mathbb{V}}\| \le \|x\|/\|\mathbb{V}\| \tag{3.26}$$

a.e., hence everywhere, by the continuity of $\widehat{\mathbb{E}}$.

2° *The separable case, $\mathbb{V} = \mathbb{E}^{-1}$:*
It is enough to show that the range of $\widehat{\mathbb{E}}(it) \in \mathcal{B}(X,Y)$ is dense for all $t \in \mathbf{R}$, because then $\widehat{\mathbb{E}}(it) \in \mathcal{G}\mathcal{B}(X,Y)$ for all $t \in \mathbf{R}$, by 1° and Lemma A.3.4(D1), consequently, $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}(i\mathbf{R};\mathcal{B}(X,Y))$ (the inverse is continuous by Lemma A.3.3(A)), and the bound $\|\widehat{\mathbb{E}}^{-1}\| \le \|\mathbb{V}\|$ ($\mathbb{V} = \mathbb{E}^{-1}$) follows from Lemma A.3.4(D1) (a.e., hence everywhere), thus, $\widehat{\mathbb{E}}^{-1} \in \mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(Y,X))$ (note also that $\widehat{\mathbb{V}}y = \widehat{\mathbb{V}}\widehat{\mathbb{E}}\widehat{\mathbb{E}}^{-1}y = \widehat{\mathbb{E}}^{-1}y \in X$ a.e. for each $y \in Y$).

Therefore, it is sufficient to assume that there is $a \in \mathbf{R}$ s.t. the range $Y_a := \widehat{\mathbb{E}}(ia)[X]$ is not dense in Y and derive a contradiction — that shall we do.

By Lemma A.3.14, there are $y_0 \in Y$ and $\Lambda_0 \in Y^*$ s.t. $\|y_0\| = 1 = \|\Lambda_0\|$, $\Lambda_0 Y_a = \{0\}$, and $\|\Lambda_0 y_0\| > 1/2$. Because $\Lambda_0\widehat{\mathbb{E}} : i\mathbf{R} \to X^*$ is continuous and

$\Lambda_0\widehat{\mathbb{E}}(ia) = 0$, there is $\delta > 0$ s.t.

$$\|\Lambda_0\widehat{\mathbb{E}}(it)\|_{X^*} < \delta' := 1/999(1+\|\mathbb{E}^{-1}\|) \text{ when } |t-a| < \delta^2/2, \tag{3.27}$$

i.e, when $it \in J := i(a-\delta^2/2, a+\delta^2/2) \subset i\mathbf{R}$.

Set $g := (\Lambda_0 y_0)^{-1}(\mathcal{L}^{-1}\chi_J)y_0 \in \mathrm{L}^2(\mathbf{R};Y)$. Then $\mathcal{L}\Lambda_0 g = \chi_J$, so we can choose a simple function $f \in \mathrm{L}^2(\mathbf{R};X)$ so that $\|f - \mathbb{E}^{-1}g\|_2$ is $< \delta$ and small enough to guarantee that $\delta/2 > \|\mathcal{L}\Lambda_0\mathbb{E}(f - \mathbb{E}^{-1}g)\|_2$, i.e., that

$$\delta/2 > \|\mathcal{L}\Lambda_0\mathbb{E}f - \chi_J\|_2 = \|\Lambda_0\widehat{\mathbb{E}}\widehat{f} - \chi_J\|_2. \tag{3.28}$$

From $\|\chi_J\|_2 = \delta$ it follows that

$$\|\mathbb{E}^{-1}g\|_2 \le \|\mathbb{E}^{-1}\||\Lambda_0 y_0|^{-1}(2\pi)^{-1/2}\|\chi_J\|_2 \le \delta\|\mathbb{E}^{-1}\|, \tag{3.29}$$

hence $\|f\|_2 \le \delta(1+\|\mathbb{E}^{-1}\|)$, so

$$\delta^2/99 > (\delta')^2 2\pi\|f\|_2^2 = (\delta')^2\int_J \|\widehat{f}\|_X^2\,dm \ge \int_J |\Lambda_0\widehat{\mathbb{E}}\widehat{f}|^2\,dm. \tag{3.30}$$

But from (3.28) we obtain that

$$\|\Lambda_0\widehat{\mathbb{E}}\widehat{f}\|_2 \ge \|\chi_J\| - \delta/2 = \delta/2, \tag{3.31}$$

which together with (3.30) leads to a contradiction, as desired.

3° *The general case,* $\mathbb{V}\mathbb{E} = I$*:*
Let $x_0 \in X$ be arbitrary. Set $X_0 := \mathrm{span}(x_0)$, $Y_0 := \{0\}$, and find closed, separable subspaces $X' \subset X$ and $Y' \subset Y$ as in Lemma 3.2.6.

Set $\mathbb{E}' := \mathbb{E}_{|\mathrm{L}^2(\mathbf{R};X')} \in \mathrm{TI}(X',Y')$, $\mathbb{V}' := \mathbb{V}_{|\mathrm{L}^2(\mathbf{R};Y')} \in \mathrm{TI}(Y',X')$. Clearly $\mathbb{V}'\mathbb{E}' = I \in \mathrm{TI}(X')$ and $\mathbb{E}'\mathbb{V}' = I \in \mathrm{TI}(Y')$, hence, by part I,

$$\|\widehat{\mathbb{E}'}x_0\| \ge \|x_0\|/\|\mathbb{V}'\| \ge \|x_0\|/\|\mathbb{V}\|, \tag{3.32}$$

as required (if $A \mapsto \mu(A) \in \mathcal{B}(X,Y)$ is the measure generating $\mathbb{E}$, then by the definition of convolution, $A \mapsto \mu(A)_{|X'} \in \mathcal{B}(X',Y')$ generates $\mathbb{E}'$, in particular $\widehat{\mathbb{E}'}x = \widehat{\mathbb{E}}x$). Because $x_0 \in X$ was arbitrary, the claim follows.

4° *The general case,* $\mathbb{V} = \mathbb{E}^{-1}$*:*
Let $y_0 \in Y$ and $t \in \mathbf{R}$ be arbitrary. Choose $X_0 = \{0\}$ and $Y_0 = \mathrm{span}(y_0)$, and proceed as in 3°. By part I, $\widehat{\mathbb{E}'}(it) \in \mathcal{GB}(X',Y')$, hence there is $x_0 \in X$ s.t. $y_0 = \widehat{\mathbb{E}'}(it)x_0 = \widehat{\mathbb{E}}(it)x_0$.

Because $y_0 \in Y$ was arbitrary, $\widehat{\mathbb{E}}(it)$ is onto, hence $\widehat{\mathbb{E}}(it)$ invertible and $\|\widehat{\mathbb{E}}(it)^{-1}\| \le \|\mathbb{V}'\| \le \|\mathbb{V}\|$, by 3° and Lemma A.3.4(D1). Because $t \in \mathbf{R}$ was arbitrary, the proof is complete. □

We have already used the following lemma several times to reduce certain results to the separable case:

Lemma 3.2.6 ($\mathbf{TI}^p(X,Y) \to \mathbf{TI}^p(X_0,Y_0)$) *Let X and Y be Banach spaces and $1 \le p < \infty$. Let $\mathbb{E} \in \mathrm{TI}^p(X,Y)$.*

Then, for each closed separable subspace X_0 of X, there is a closed separable subspace Y_0 of Y s.t. $\mathbb{E}\mathrm{L}^p(\mathbf{R};X_0) \subset \mathrm{L}^p(\mathbf{R};Y_0)$, i.e., that $\mathbb{E}_{|\mathrm{L}^p(\mathbf{R};X_0)} \subset \mathrm{TI}^p(X_0,Y_0)$.

If, in addition, $\mathbb{V} \in \mathrm{TI}^p(Y,X)$, then any separable subsets $X_0 \subset X$ and $Y_0 \subset Y$ are contained, respectively, in closed separable subspaces $X' \subset X$ and $Y' \subset Y$, s.t. $\mathbb{E}|_{\mathrm{L}^p(\mathbf{R};X')} \subset \mathrm{TI}^p(X',Y')$ and $\mathbb{V}|_{\mathrm{L}^p(\mathbf{R};Y')} \subset \mathrm{TI}^p(Y',X')$.

Analogously, if $\mathbb{E}_k \in \mathrm{TI}^p(X_k,X_{k+1})$ for $k = 1,\dots,n$, $n \in \mathbf{N}+1$, $X_{n+1} = X_1$, and the sets $X_k' \subset X_k$ are separable ($k = 1,\dots,n$), then there are closed separable subspaces $X_k'' \subset X_k$ ($k = 1,\dots,n$) s.t. $X_k' \subset X_k''$ and $\mathbb{E}_k|_{\mathrm{L}^p(\mathbf{R};X_k')} \in \mathrm{TI}^p(X_k',X_{k+1}')$ ($k = 1,\dots,n$). Moreover, we can require that $X_k'' = X_j''$ whenever $X_k = X_j$ for some k,j.

We might as well have stated the lemma for a general $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p(\mathbf{R};X),\mathrm{L}^q(\mathbf{R};Y))$, where $p,q \in [1,\infty)$ (finding Y_0 s.t. $\mathbb{E}\mathrm{L}^p(\mathbf{R};X_0) \subset \mathrm{L}^q(\mathbf{R};Y_0)$; the latter claims can be generalized analogously), with the same proof (with only slight changes if $q \neq p$).

Furthermore, we could instead of separability require the density of a subset a some greater cardinality than that of $\mathbf{N}$ (in X_0 and in Y_0).

Proof: 1° *Finding Y_0:*
Let $\{f_k\} \subset \mathrm{L}^p(\mathbf{R})$ and $\{x_k\} \subset X$ be dense subsets (where k ranges over $\mathbf{N}$). Then the set D of finite linear combinations of functions of the form $f_k x_j$ is dense in $\mathrm{L}^p(\mathbf{R};X)$, because simple functions can obviously be approximated by such functions.

Choose a separably-valued representative of each $\mathbb{E}f$ $(f \in D)$, and let Y_0 be the closed span of $\cup_{f\in D}(\mathbb{E}f)[\mathbf{R}]$. Then Y_0 is separable as a countable union of separable sets, and $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y)$ (as an equivalence class) for all $f \in D$, hence, by the continuity of $\mathbb{E}$, for all $f \in \mathrm{L}^p(\mathbf{R};X_0)$.

2° *Finding X' and Y':*
Replace first X_0 and Y_0 by their closed spans. Choose then a closed separable subspace Y_0'' of Y, as Y_0 was chosen in 1° for the pair $(\mathbb{E},X_0)$, then set $Y_1 := \operatorname{span} Y_0 \cup Y_0''$.

For each $k \in \{2,3,4,\dots\}$ choose $X_k \subset X$ for the pair $(\mathbb{V},Y_{k-1})$, and then choose $Y_k \subset Y$ for the pair $(\mathbb{E},X_k)$, as in 1°.

Set $X'' := \cup_k X_k$, $Y'' := \cup_k Y_k$, $X' := \overline{X''}$, $Y' = \overline{Y''}$. If $f \in \mathrm{L}^p(\mathbf{R};X')$ is simple and has its values in X'', then $f \in \mathrm{L}^p(\mathbf{R};X_k)$ for some $k \in \mathbf{N}$, hence then $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y_k) \subset \mathrm{L}^p(\mathbf{R};Y')$. But such functions are dense in $\mathrm{L}^p(\mathbf{R};X'')$, by Theorem B.3.11(a1)&(a3), hence, by continuity, $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y')$ for any $f \in \mathrm{L}^p(\mathbf{R};Y')$ (since $\mathrm{L}^p(\mathbf{R};Y')$ is closed in $\mathrm{L}^p(\mathbf{R};Y)$). Similarly, $\mathbb{V}[\mathrm{L}^p(\mathbf{R};Y')] \subset \mathrm{L}^p(\mathbf{R};X')$.

By Lemma B.2.3(a)&(c), X' and Y' are separable.

3° *Requiring that $X' = Y'$ (assuming that $X = Y$):* When choosing X_k in 2°, replace the one from 1° with its union with Y_{k-1}. Analogously, when choosing Y_k in 2°, replace the one from 1° with its union with X_k.

4° *Finding X_k' $(k = 1,\dots,n)$:* Use the method of 2°–3° suitably modified. □

Notes

The classical scalar Fourier multiplier result, Lemma 3.2.1, is essentially contained in Theorem 6.1.2 of [BL] (which also gives further results). When U and Y are Hilbert spaces of arbitrary dimensions, we have $M_2(U,Y) =$

$\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)$, by Theorem 3.1.3(a1), $M_\infty(U,Y) = \mathcal{B}(\mathcal{C}_0(i\mathbf{R};U),Y)$, by Lemma D.1.14, and $M_p(U,Y)^* = M_q(Y^*,U^*)$ (this last claim also holds when U and Y are Banach spaces), where $p^{-1}+q^{-1} = 1$, by a proof analogous to that in the scalar case. This gives us no information on M_p for $p \neq 2,\infty$, since interpolation would require that, e.g., $M_p(U,Y) = M_q(Y^*,U^*)$. (Here $M_p(U,Y) := \widehat{\mathrm{TI}^p}(U,Y)$ for $p < \infty$, $M_\infty(U,Y) := \widehat{\mathrm{TI}^{\mathcal{C}}_0}(U,Y)$; the elements of these sets are called *Fourier multipliers*.)

On p. 135 of [BL], it is claimed that Theorem 6.1.2 has an obvious analogy in this general setting and that "the proofs are the same with trivial changes", but we cannot share this view for the following reasons: 1. the case $p = 2$ is far from obvious (in fact, Theorem 3.1.3(a1) seems to be a new result); 2. the proof of $M_p = M_{p'}$ on p. 133 of [BL] would only yield $M_p^* = M_{p'}$ even for finite-dimensional Hilbert spaces, hence the original proof seems to cover only the case $p = \infty$; 3. we do not know any similar results from the literature (even in the finite-dimensional result mentioned below Lemma 3.2.1 one would require a different proof for a sharp norm bound).

There are several results on vector-valued Fourier multipliers in the literature; see, e.g., [BL], [Prüss93] or [Zimmermann] for sufficient conditions of a bounded function $i\mathbf{R} \to \mathcal{B}$ to be a Fourier multiplier.

Example 3.2.3 resembles the example of Section 3 of [W91a]. Another Banach limit is given in Exercise 4 of [Rud73].

3.3 H^2 and H^∞ boundary functions in L^2 and $\mathrm{L}^\infty_{\mathrm{strong}}$

Boundary, *n.:*
In political geography, an imaginary line between two nations, separating the imaginary rights of one from the imaginary rights of the other.
— Ambroce Bierce (1842–1914), "The Devil's Dictionary"

In this section, we establish several results on the boundary functions of holomorphic functions, the most important of which is the connection $\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y)) \to \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. We also give other related results that will be needed for WPLS theory of Parts II and III, and we construct counter-examples for analogous results for more general settings.

At the end of this section, we shall show that the set of singular points ("poles") of the pointwise inverse of a transfer function may have limit points unlike in the case of finite-dimensional input and output spaces; we then use this to construct a "completely unstable" transfer function.

As before, H, U and Y are assumed to be arbitrary Hilbert spaces unless something else is indicated.

Recall the transfer functions of TIC_∞ operators from Theorem 2.1.2; recall also that we use the Lebesgue measure (of $\mathbf{R}^1$) on the imaginary axis $i\mathbf{R} := \{ir \mid r \in \mathbf{R}\}$ and on its translations $\omega + i\mathbf{R}$ ($\omega \in \mathbf{R}$). For any $r > 0$, the circle $\partial r\mathbf{D}$ is identified with $[0,2\pi)$, hence $m(\partial r\mathbf{D}) = 2\pi$, where $\partial\mathbf{D} := \{e^{it} \mid t \in [0,2\pi)\} = \{s \in \mathbf{C} \mid |s| = 1\}$.

The following theorem is the main result of this section. We use often claims (a2)&(a1) for the boundary functions of H^2 functions and claim (c1) for the boundary functions of operator-valued H^∞ functions, whereas the others are used just a few times:

Theorem 3.3.1 ($\mathbf{H}^p$ boundary functions) *Let $\omega \in \mathbf{R}$ and $1 \le p \le \infty$. Let B be a Banach space and let H, U and Y be Hilbert spaces. Then the following hold:*

(a1) Let $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B)$. Assume that there is $f_0 \in \mathrm{L}^p(\omega + i\mathbf{R};Y)$ s.t. any one of (1.)–(6.) holds:

(1.) $\lim_{t\to\omega+} \Lambda f(ir+t) = \Lambda f_0(ir+\omega)$, for almost every $r \in \mathbf{R}$, whenever $\Lambda \in B^*$;

(2.) f converges to f_0 nontangentially at every Lebesgue point of f_0, hence a.e.;

(3.) f is the Poisson integral *of f_0, i.e.,*

$$f(\omega+t+ir) = \frac{t}{\pi}\int_{\mathbf{R}} \frac{f(\omega+i\rho)\,d\rho}{t^2+(r-\rho)^2} \quad (t>0,\ r \in \mathbf{R}). \tag{3.33}$$

(4.) $\int_E f(i\cdot+t)\,dm \to \int_E f_0(i\cdot+\omega)\,dm$ for all bounded measurable $E \subset \mathbf{R}$;

(5.) $\int_{-R}^R g f(i\cdot+t)\,dm \to \int_{-R}^R g f_0(i\cdot+\omega)\,dm$ for all $R > 0$ and $g \in \mathrm{L}^\infty(\mathbf{R};\mathcal{B}(B,*))$;

(6.) $f(i\cdot+t) \to f_0(i\cdot+\omega)$ *in* L^p*, as* $t \to \omega+$*;*

Then f_0 *is unique, (1.)–(5.) hold (and (6.) if* $p<\infty$*), and*

$$\|f_0\|_p = \|f\|_{\mathrm{H}^p_\omega} = \lim_{r\to\omega+} \|f(i\cdot+r)\|_p \ge \|f(i\cdot+t)\|_p \quad (t>\omega). \tag{3.34}$$

If this is the case, then we call f_0 *the* (vector) L^p boundary function *of* f *and denote* $f(ir+\omega) := f_0(ir+\omega)$ *(*$r\in\mathbf{R}$*) and write* $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B)\cap\mathrm{L}^p(\omega+i\mathbf{R};B)$*. Any of (1.)–(6.) characterizes* f_0 *uniquely (in* L^p*, that is, a.e.).*

(a2) Every $f\in\mathrm{H}^p(\mathbf{C}^+_\omega;H)$ *has a* L^p *boundary function (recall that* U*,* H *and* Y *are assumed to be Hilbert spaces).*

(a3) If $f\in\mathrm{H}^p(\mathbf{C}^+_\omega;B)$*, then* $f|_{\mathbf{C}^+_{\omega'}} \in \mathrm{H}^p(\mathbf{C}^+_{\omega'};B)$ *has the* L^p *boundary function* $f|_{\omega'+i\mathbf{R}}$ *for any* $\omega'>\omega$*.*

(a4) If $f\in\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y))$*,* $n\in\mathbf{N}$*, then* $f\in\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y))\cap\mathrm{L}^p(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n,Y))$*; thus, then* f *has a* L^p *boundary function.*

(b) **(Paley–Wiener Theorem)** *The Laplace transform* $\mathrm{L}^2_\omega(\mathbf{R}_+;Y)\ni h\mapsto\hat h\in\mathrm{H}^2(\mathbf{C}^+_\omega;Y)$ *and the Fourier transform* $\mathrm{L}^2_\omega(\mathbf{R};Y)\ni g\mapsto\hat g\in\mathrm{L}^2(\omega+i\mathbf{R};Y)$ *are isomorphisms times* $\sqrt{2\pi}$*, and the former can be considered as the restriction of the latter to* $\pi_+\mathrm{L}^2_\omega$*. Finally, for all* $h\in\mathrm{L}^2_\omega(\mathbf{R}_+;Y)$ *and* $f,g\in\mathrm{L}^2(\mathbf{R};Y)$ *we have that*

$$\langle\hat f,\hat g\rangle_{\mathrm{L}^2} = 2\pi\langle f,g\rangle_{\mathrm{L}^2}, \quad \|\hat g\|_{\mathrm{L}^2} = \sqrt{2\pi}\|g\|_{\mathrm{L}^2}, \quad \|\hat h\|_{\mathrm{H}^2_\omega} = \sqrt{2\pi}\|h\|_{\mathrm{L}^2_\omega}. \tag{3.35}$$

(c1) For every $f\in\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *there is a unique (in* $\mathrm{L}^\infty_{\mathrm{strong}}$*)* (operator) boundary function $f_0\in\mathrm{L}^\infty_{\mathrm{strong}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))$ *s.t. for all* $u_0\in U$ *the function* f_0u_0 *is the boundary function (see (a1)) of* $fu_0\in\mathrm{H}^\infty(\mathbf{C}^+_\omega;Y)$*. It follows that* f *is the strong Poisson integral ("*$\oint$*") of* f_0 *and* $\|f_0\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|f\|_{\mathrm{H}^\infty_\omega}$ *(we can choose* f_0 *s.t.* $\sup\|f_0\| = \|f\|_{\mathrm{H}^\infty_\omega}$*).*

We denote $f(\omega+ir) := f_0(\omega+ir)$*, where* f_0 *is the function constructed in the proof. For each* $\hat u\in\mathrm{H}^2(\mathbf{C}^+_\omega;U)$ *we have* $(f\hat u)(\omega+ir) = f(\omega+ir)\hat u(\omega+ir)$ a.e. *on* $\omega+i\mathbf{R}$*; in particular,*

$$\mathbb{D}\in\mathrm{TIC}_\omega(U,Y) \implies \widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat u \text{ a.e. on } \omega+i\mathbf{R}. \tag{3.36}$$

The mapping $\mathrm{H}^\infty_\omega\ni f\mapsto f_0\in\mathrm{L}^\infty_{\mathrm{strong}}$ *is an isometry to a closed subspace of* $\mathrm{L}^\infty_{\mathrm{strong}}$*.*

Finally, for any $\mathbb{D}\in\mathrm{TIC}_\omega$*, the boundary function of* $\widehat{\mathbb{D}}\in\mathrm{H}^\infty$ *coincides with the Fourier transform of* $\mathbb{D}$ *(from Theorem 3.1.3).*

(c2) If U *is separable, then we can choose* f_0 *in (c1) s.t.* $f(\omega+ir+\cdot)\to f_0(\omega+ir)$ *nontangentially in the strong topology of* $\mathcal{B}(U,Y)$*, for a.e.* $r\in i\mathbf{R}$*, and* $\|f(\omega+ir+\cdot)\|\to\|f_0(\omega+ir)\|$ *nontangentially for a.e.* $r\in i\mathbf{R}$*.*

(c3) Let $\mathbb{D}\in\mathrm{TIC}_\infty(\mathbf{C}^n,Y)$ *and* $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}]\subset\mathrm{L}^2_\omega$*. Then* $f:=\widehat{\mathbb{D}}\in\mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y))$ *and* $\widehat{\mathbb{D}}$ *has a "*$\mathrm{L}^2_{\mathrm{loc}}$ *boundary function"* $f_0 =: \widehat{\mathbb{D}}$ *s.t.* $f(i\cdot+t)\to f_0(i\cdot+\omega)$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};B)$*, as* $t\to\omega+$*, and* $(\cdot+1-\omega)^{-1}\widehat{\mathbb{D}}\in\mathrm{L}^2(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n,Y))$*, and*

(1.), (2.), (4.) and (5.) of (a1) are satisfied. Consequently, $\widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ *a.e. on* $i\mathbf{R}$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;\mathbf{C}^n)$*,* $\alpha < \omega$*.*

If $\omega = 0$*, then* $\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(\mathbf{C}^n,Y))$*.*

(d1) For every $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *there is a* boundary function f_0 *whose values are (possibly unbounded) operators with domains in* U *and ranges in* Y*, s.t.* $f_0u_0 \in \mathrm{L}^p(\omega+i\mathbf{R},Y)$ *is the boundary function of* fu_0 *(see (a1)) for any* $u_0 \in U$*.*

In particular, $\sup_{u_0\in U}\|f_0u\|_p = \|f\|_{\mathrm{H}^p_{\mathrm{strong}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))}$*. If* U *is separable, then*
$\mathrm{Dom}(f_0(\omega+ir))$ *is dense for a.e.* $r \in \mathbf{R}$*.*

(d2) Let $f \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$*. Then* $f \in \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *iff the boundary function* $f_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ *(see (c1)) is in* $\mathrm{L}^p_{\mathrm{weak}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))$ *too.*

If this is the case, then $\|f_0\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|f\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B})}$*. Claim (d2) also holds with "strong" in place of "weak".*

(d3) If $f \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))\cap\mathrm{L}^p(\omega+i\mathbf{R};\mathcal{B}(U,Y))$*, then* $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$*.*

(d4) Conversely, if $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *and* U *is separable, then* f *has a unique "strong operator boundary function"* $f_0 \in \mathrm{L}^p_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *s.t.* f_0u_0 *is the boundary function of* fu_0 *for any* $u_0 \in U$ *(in the sense of (a1), and the same null set applies for all* u_0*). Moreover,* $\|\|f_0\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^p(i\mathbf{R})} = \|f\|_{\mathrm{H}^p}$*.*

(e) Results analogous to (a1)–(d4) (except that (c3) must be replaced by Lemma 13.1.3(d)) hold for $R\mathbf{D} := \{z \in \mathbf{C} \mid |z| < R\}$ *(*$R > 0$*) in place of* $\mathbf{C}^+_\omega$ *(write the convergence "(1.)" as* $f_0(z) = \lim_{r\to R-} f(rz)$ *a.e., where* $|z| = R$*). The Poisson integral on* $R\mathbf{D}$ *is given by*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2} f(Re^{it})\,dt \quad (r > 0,\ \theta \in \mathbf{R}). \tag{3.37}$$

See Proposition D.1.21(c) and Theorem 3.1.6 for the boundary functions of $\mathrm{H}^2(\mathbf{C}_{a,b};U)$ and $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$ functions, respectively.

Note that the operator boundary function of (c1) need not be a vector boundary function (i.e., the limits need not converge in the operator norm), not even in the separable case of (d3), by Example 1. on p. 92 of [RR], where $f(z) := (x_k)_{k\in\mathbf{N}} \mapsto (z^kx_k)_{k\in\mathbf{N}}$, so that $f \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(\ell^2(\mathbf{N})))$.

Part (c1) (which seems to be new in the unseparable case) is the best we can say for unseparable U; e.g., it may be that the strong or weak limit of $f(ir+t)$ exists for no $ir \in i\mathbf{R}$ as $t \to 0+$ (i.e., there are U and $f \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ s.t. for each ir, there is $u_0 \in U$ s.t. $f(ir+t)u_0$ does not have even a weak limit as $t \to 0+$), as shown in p. 133 [Thomas]). See [Thomas] for further results for the separable case and a counter-example for those results in the unseparable case.

The proof of Theorem 3.1.6 also shows how we could reduce (c1) to Theorem 3.1.3(a1), but we have preferred to give a direct proof below, because this proof is much simpler than that of Theorem 3.1.3(a1). The proof of (c1) lies heavily on the boundedness of the function, thus it cannot be used for H^p, $p < \infty$.

The equation $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ in (3.36) shows that the boundary function coincides with the Fourier transform of $\mathbb{D}$ given in Theorem 3.1.3. Naturally, the boundary function is in L^∞ if $\dim U < \infty$.

The result (c1) seems to be useful only for proving results such as those in Lemma 6.3.6, because many important properties of a $\mathbb{D} \in \mathrm{TIC}$ are not shared pointwise (a.e.) by an arbitrary representative of the Fourier transform $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. (E.g., if $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$, then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$, but $\widehat{\mathbb{D}}(ir)$ may be noninvertible for all $r \in \mathbf{R}$.) An analogous remark applies to Theorem 3.1.3(a1).

The boundary function in (d1) in not $\mathcal{B}(U,Y)$-valued in general (see Example 3.3.6), unless $p = \infty$ (note that $\mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$, by the Closed Graph Theorem, hence (c1) applies to $p = \infty$).

For general $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ (or H^p), we have $f \in \mathrm{H}^p_{\mathrm{strong}} \cap \mathrm{H}^\infty(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(U,Y))$ ($\varepsilon > 0$), by Lemma F.3.2(a), hence we can apply (d2) with $\omega+\varepsilon$ in place of ω.

Proof of Theorem 3.3.1: W.l.o.g., we state the proofs for the case $\omega = 0$ (because $\mathrm{L}^2_\omega \ni u \mapsto e^{-\omega t}u \in \mathrm{L}^2$ is an isometric isomorphism and $\widehat{e^{-\omega t}u}(s) = \widehat{u}(s+\omega)$ for all $s \in \mathbf{C}^+$).

(a1)&(a2) (See p. 967 for nontangential limits.)

1° *Implications "(3.)⇒(2.)⇒(1.)", "(3.)⇒(5.)", and "(6.)⇒(5.)⇒(4.)" (any p) and "(3.)⇒(6.)" ($p < \infty$), and (3.34) for general B:* These follow from Lemma D.1.8(a3)&(a1) ("C_0" of (a1)) except for "(6.)⇒(5.)" (note that $\chi_{[-R,R]}g \in \mathrm{L}^q$, where $q^{-1} + p^{-1} = 1$, and use Hölder) and "(5.)⇒(4.)" (take $g := \chi_E$, $R := \sup|E|$).

2° *Uniqueness:* By Lemma B.2.6, (1.) characterizes f_0 uniquely (note that we may have a different null set for each Λ); so does also (4.) by Theorem B.4.12(e). By 1°, so do (2.), (3.), (5.) and (6.) too.

3° *(a2) when H is a separable Hilbert space:* Now $f_0 \in \mathrm{L}^p$ satisfying (2.) and (3.) (hence (1.)–(5.), by 1°) exists, by pp. 81, 85 and 90 of [RR]. By Lemma D.1.8(a1), we have $\|f_0\|_p = \|f\|_{\mathrm{H}^p}$

4° *(a2) when H is a Hilbert space:* Replace H by the closed span of $f[\mathbf{C}^+]$, which is a separable Hilbert space, by Lemma B.2.3(f)&(a). By 3°, $f_0 \in \mathrm{L}^p$ satisfying (1.)–(5.) (and (6.) for $p < \infty$) and $\|f_0\|_p = \|f\|_{\mathrm{H}^p}$ exists.

(We note one could prove (a2) whenever Y is a reflexive Banach space (an alternative condition is that Y^{**} is separable) and $1 < p \leq \infty$, by using the (scalar) techniques of Chapter 11 of [Rud86], and the fact that Y is a Radon–Nikodym space [DU].)

5° *(a1) when B is a Banach space:* Assume any of (1.)–(6.). Let $g \in \mathrm{H}^p$ be the Poisson integral of f_0. Now Λf_0 is the boundary function of Λf, hence $\Lambda g = \Lambda f$, for any $\Lambda \in B^*$, by 3°. Having thus established (3.), we get the claim from 1° and 2°.

(a3) By the continuity of f, condition (1.) of (a1) is satisfied.

(a4) By Lemma A.1.1(a4), $f = \sum_{k=1}^n f_k P_k$ for some $f_1,\dots,f_n \in \mathrm{H}^p(\mathbf{C}^+_\omega;Y)$, where P_k is the kth canonical projection $\mathbf{C}^n \to \mathbf{C}$. Let $\widetilde{f_k} \in \mathrm{H}^p \cap \mathrm{L}^p$ be the boundary function of f_k ($k = 1,\dots,n$) (see (a2)). Obviously, $\|\sum_{k=1}^n f_k(ir+t)P_k - \sum_{k=1}^n \widetilde{f_k}(ir)P_k\|_{\mathcal{B}(\mathbf{C}^n;Y)} \to 0$ for a.e. $r \in \mathbf{R}$, as $t \to \omega+$, hence $\sum_{k=1}^n \widetilde{f_k}P_k \in$

$\mathrm{L}^p(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n;Y))$ is the boundary function of f.

(b) See p. 91 of [RR]. (And alternative reference to (a) and (b) is A.6.18–21 of [CZ].)

(c1) (We take $\omega=0$ w.l.o.g.; cf. Remark 2.1.6.) To be exact, we construct here a *function* $f_0 : i\mathbf{R}\to\mathcal{B}(U,Y)$ satisfying the conditions in the lemma, and then we show that the equivalence class of f_0 is a unique member of $\mathrm{L}^\infty_{\mathrm{strong}}$.

For each $r\in\mathbf{R}$, we define $U_r := \{u\in U \,|\, \exists \lim_{t\to 0+} f(ir+t)u =: y_{r,u}\}$. For a fixed r, the map $U_r\ni u\mapsto y_{r,u}$ is clearly linear and $\|u\mapsto y_{r,u}\|_{\mathcal{B}(U_r,Y)}\le\|f\|_\infty$, hence $u\mapsto y_{r,u}$ has a norm-preserving extension $f_0(ir)\in\mathcal{B}(U,Y)$ (note that $\|f_0(ir)\|\le\|f\|_{\mathrm{H}^\infty}$; we extend it to $\overline{U_r}$ by continuity (see Lemma A.3.10) and set $f_0(ir)=0$ on $U_r^\perp$). For any $u\in U$, we have $f_0(ir)u=\lim_{n\to\infty} f(ir+1/n)u$ a.e. r (because $u\in U_r$ a.e. $r\in\mathbf{R}$ by (b)), hence $f_0(i\cdot)u$ is measurable. Since u was arbitrary and f_0 is bounded, $f_0\in\mathrm{L}^\infty_{\mathrm{strong}}$.

Let $\widehat{u}\in\mathrm{H}^2$. Because the closed span U_u of $\widehat{u}[\mathbf{C}^+]$ is separable, we can take a null set $N_u\subset\mathbf{R}$ s.t. $f(ir+t)u_0\to f_0(ir)u_0$ for all $u_0\in U_u$ and all $r\notin N_u$ (choose a null set for each u_0 in a countable dense subset of U_u, and let N_u be the union of these sets; by Lemma A.3.4(H1) with "$F(s)=f(ir+1/s)-f_0(ir)$" we get the convergence for all $u_0\in U_u$ and any fixed $r\notin N_u$). Choosing null set N for $f\widehat{u}\in\mathrm{H}^2$ as in (a), we now have that

$$\begin{aligned}
(f\widehat{u})(ir) &= \lim_{t\to 0+} f(ir+t)\widehat{u}(ir+t)\\
&= \lim_{t\to 0+} [f(ir+t)[\widehat{u}(ir+t)-\widehat{u}(ir)]+f(ir+t)\widehat{u}(ir)]\\
&= 0+f(ir)\widehat{u}(ir)
\end{aligned}\tag{3.38}$$

for $r\notin N\cup N_u$. The equality $\|f_0\|_\infty=\|f\|_{\mathrm{H}^\infty}$ follows from the fact that $\|f_0u_0\|_\infty=\|fu_0\|_{\mathrm{H}^\infty}$ for all $u_0\in U$ by (b).

Moreover, if $f_1 : i\mathbf{R}\to\mathcal{B}(U,Y)$ also satisfies $f_1(i\cdot)u_0=\lim_{t\to0+}f(i\cdot+t)u_0$ a.e. for $u_0\in U$, then $(f_1-f_0)u_0=0$ a.e. for $u_0\in U$, hence $f_1=f_0$ as a member of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. However, considered as functions, they may differ everywhere (and we may have $\|f_1(ir)\|_{\mathcal{B}(U,Y)}\ge r$ for all $r\in\mathbf{R}$ even though $f_1=0$ in $\mathrm{L}^\infty_{\mathrm{strong}}$ (i.e., $f_1u_0=0$ a.e. on $i\mathbf{R}$, for all $u_0\in U$)). The mapping $f\mapsto f_0$ is clearly linear, hence it is an isometry to a subspace of $\mathrm{L}^\infty_{\mathrm{strong}}$. The claims about TIC_ω now follow from Theorem 6.2.1.

Finally, if $G\in\mathrm{L}^\infty_{\mathrm{strong}}$ is the Fourier transform of $\mathbb{D}\in\mathrm{TIC}$ (see Theorem 3.1.3), then $F\widehat{f}u_0=\widehat{\mathbb{D}fu_0}=G\widehat{f}u_0=\widehat{f}Gu_0$ a.e. on $i\mathbf{R}$ for all $f\in\mathrm{L}^2$ and all $u_0\in U$, hence $Fu_0=Gu_0$ a.e. for all $u_0\in U$, i.e., $F=G$ in $\mathrm{L}^\infty_{\mathrm{strong}}$.

(c2) This is Theorem B on p. 85 of [RR], provided that also Y is separable. In the general case, the closed span Y_0 of $\{\widehat{\mathbb{D}}(z)u_0 \,|\, u_0\in U, z\in\mathbf{C}^+_\omega\}$ is separable, hence $\widehat{\mathbb{D}}\in\mathrm{H}^\infty(U,Y_0)$, and we can apply the result mentioned above.

(c3) (Take $\omega=0$ w.l.o.g. Set $U:=\mathbf{C}^n$.) By Lemma 2.1.13, we have $g:=(\cdot+1)^{-1}\widehat{\mathbb{D}}\in\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$. By Lemma F.3.2(e), $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))=\mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))$. By (a4), we have $g\in\mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))\cap\mathrm{L}^2(i\mathbf{R};\mathcal{B}(U,Y))$, in particular, g has a L^2 boundary function g_0. Set $f_0:=(\cdot+1)g_0$, so that $f_0\in(\cdot+1)\mathrm{L}^2(i\mathbf{R};\mathcal{B}(U,Y))\subset\mathrm{L}^2_{\mathrm{loc}}(i\mathbf{R};\mathcal{B}(U,Y))$ and f and f_0 inherit (1.), (2.)

and (5.) (hence also (4.)) from g and g_0; in particular, $\widehat{\mathbb{D}}$ converges to f_0 nontangentially at every $ir \in \mathrm{Leb}(f_0) = \mathrm{Leb}(g_0)$ (see (B.54)).

Let $\alpha < 0$ and $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$. Since $\mathbb{D} \in \mathcal{B}(\pi_+\mathrm{L}^2_\alpha, \pi_+\mathrm{L}^2)$, by (2.19), the function $\widehat{\mathbb{D}}\widehat{u}$ converges to $\widehat{\mathbb{D}u}$ nontangentially a.e., by (a2). Since $\widehat{\mathbb{D}} \to f_0$ and $\widehat{u} \to \widehat{u}$ a.e. nontangentially, we must have $f_0\widehat{u} = \widehat{\mathbb{D}u}$ a.e.

The claim on $\omega = 0$ follows from Lemma 2.1.13.

(d1) (We set $\omega = 0$ w.l.o.g.) For $ir \in i\mathbf{R}$ we set

$$\mathrm{Dom}(f_0(ir)) := \{u_0 \in U \mid \lim_{t \to 0+} f(ir+t)u_0 =: f_0(ir)u_0 \text{ exists }\}. \tag{3.39}$$

Obviously, $\mathrm{Dom}(f_0(ir))$ is a subspace of U and $f_0(ir)$ is linear. Let $u_0 \in U$ be arbitrary. Then $fu_0 \in \mathrm{H}^p(\mathbf{C}^+;U)$, hence $f_0(ir)u_0$ is defined a.e. and equal to the boundary function described in (a). In particular,

$$\sup_{u_0 \in U} \|f_0u_0\|_p = \sup_{u_0 \in U} \|fu_0\|_{\mathrm{H}^p} =: \|f\|_{\mathrm{H}^p_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))}. \tag{3.40}$$

(Note that f_0 is unique in the sense that f_0u_0 is uniquely defined a.e. for each $u_0 \in U$.)

Let U be separable. Let $\{u_k\} \subset U$ be dense. For each k, there is a null set $N_k \subset \mathbf{R}$ s.t. $u_k \in \mathrm{Dom}(f_0(ir))$ for $r \in \mathbf{R} \setminus N_k$; consequently, $\{u_k\} \subset \mathrm{Dom}(f_0(ir))$ for all $r \in \mathbf{R} \setminus N$, where $N := \cup_k N_k$.

(d2) We prove the $\mathrm{H}^p_{\mathrm{strong}}$ claim; add $\Lambda \in Y^*$ [$\|\Lambda\| \le 1$] everywhere to obtain the weak result.

1° *"Only if":* Let $u_0 \in U$. Then $fu_0 \in \mathrm{H}^p_{\mathrm{strong}}$, hence it converges a.e. to a boundary function $f_{u_0} \in \mathrm{L}^p(\omega + i\mathbf{R};Y)$, by (a), with $\|f_{u_0}\|_p = \|fu_0\|_{\mathrm{H}^p}$. But $f_{u_0} = f_0u_0$ a.e., by (c1), hence

$$\|f\|_{\mathrm{L}^p_{\mathrm{strong}}} = \sup_{\|u_0\| \le 1} \|f_0u_0\|_p = \sup_{\|u_0\| \le 1} \|fu_0\|_{\mathrm{H}^p} = \|f_0\|_{\mathrm{H}^p_{\mathrm{strong}}}. \tag{3.41}$$

By (a), fu_0 is the Poisson integral of f_0u_0 for each $u_0 \in U$, i.e., f is the strong Poisson integral of f_0.

2° *"If":* This follows from Lemma D.1.8(a1)&(a3).

(d3) This follows from (d2) and Lemma D.1.8(a1)&(a3).

(d4) By Theorem B on p. 85 of [RR], f has a boundary function f_0 in the strong operator topology (any f in the Nevanlinna class N has, and H^p functions belong to the Nevanlinna class, by (4-1) on p. 75 of [RR]). By the theorem, there is a null set $N \subset i\mathbf{R}$ s.t. $fu_0 \to f_0u_0$ nontangentially at every point of $i\mathbf{R} \setminus N$ (i.e., $f \to f_0$ nontangentially "in the strong operator topology" at every point of $i\mathbf{R} \setminus N$).

By Theorem C on p. 90, $f \mapsto f_0$ is an isometry $\mathrm{H}^p \to$"$\mathrm{L}^p_{\mathcal{B}}(i\mathbf{R})$", and "$\mathrm{L}^p_{\mathcal{B}}(i\mathbf{R})$" means "weakly measurable functions" with finite norm $\|\|\cdot\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^p(i\mathbf{R})}$; the strong measurability of f_0 follows from that of f_0u_0 (by (a1)) for each $u_0 \in H$, hence $f_0 \in \mathrm{L}^p_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

(In [RR], we should have $U = Y$, but we may replace f by $\left[\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right] \in \mathrm{H}^p(\mathbf{C}^+;\mathcal{B}(U \times Y))$. Replace Y first by its separable subspace if necessary (see Lemma B.2.4).

(e) This can be proved in the same way as (a)–(d) were proved (use the

$\mathrm{H}(\mathbf{D};*)$ results of [RR]). □

If a boundary function is zero on a set of positive measure, then both this function and the original function are zero almost everywhere:

Lemma 3.3.2 *Let f be a holomorphic function, and let f_0 be its boundary function in the sense of some of (a1)–(d4) of Theorem 3.3.1. Then the following are equivalent:*

(i) $f \not\equiv 0$ on $\mathbf{C}^+_\omega$;

(ii) $f \neq 0$ a.e. on $\mathbf{C}^+_\omega$;

(iii) $f_0 \neq 0$ (on a subset of positive measure of $\omega + i\mathbf{R}$);

(iv) $f_0 \neq 0$ a.e. (on $\omega + i\mathbf{R}$).

For (e), the above claims hold with substitutions $\mathbf{C}^+_\omega \mapsto R\mathbf{D}$ and $\omega + i\mathbf{R} \mapsto \partial R\mathbf{D} = \{z \in \mathbf{C} \,\big|\, |z| = R\}$.

Proof: Trivially (iv)⇒(iii). By Lemma D.1.2(e), we have (i)⇔(ii) (if $f \in \mathrm{H}(\Omega;B)$ and $E := f^{-1}[\{0\}]$ has a positive measure, then $K \cap E$ is infinite for some compact $K \subset \Omega$, by Lemma A.2.3, hence Ω (and K) contains a limit point of $E(\cap K)$, hence $f = 0$). Thus, it suffices to show establish (i)⇒(iv), (iii)⇒(i).

1° *Case $f \in \mathrm{H}^p(\mathbf{C}^+)$, $p \in [1,\infty]$:* This is well known (use e.g., Theorem 17.18 of [Rud86] and Cayley transform (see Lemma 13.2.1(e2)&(d)) for (i)⇒(iv), and the Poisson formula for (iii)⇒(i)).

2° *Case $f \in \mathrm{H}^p(\mathbf{C}^+;B)$, $p \in [1,\infty]$:* Now Λf_0 is the boundary function of Λf for each $\Lambda \in B^*$, hence this follows from 1° and Lemma B.2.6.

3° *Case $f \in \mathrm{H}^p(\mathbf{C}^+;\mathcal{B}(U,Y))$, $p \in [1,\infty]$:* Now $f_0 u_0$ is the boundary function of $f u_0$ for each $u_0 \in U$, hence this follows from 2°.

4° *Other cases:* For $r\mathbf{D}$, $r > 0$, in place of $\mathbf{C}^+$, the above proof applies mutatis mutandis. For $\mathbf{C}^+_\omega$ in place of $\mathbf{C}^+$, we obtain this by shifting f and f_0. □

We shall later need the following lemma:

Lemma 3.3.3 ($\langle \widehat{v}, F\widehat{u}\rangle_{\mathrm{L}^2(i\mathbf{R};\mathbf{C}^n)} = 0$ for all $u,v \implies F = 0$) *Let $\mathbb{D},\mathbb{N} \in \mathrm{TIC}_\infty(\mathbf{C}^n, Y)$, $\mathbb{D}[\mathrm{L}^2_c], \mathbb{N}[\mathrm{L}^2_c] \subset \mathrm{L}^2$ and $J \in \mathcal{B}(Y)$. Set $F := \widehat{\mathbb{N}}^* J \widehat{\mathbb{D}}$. Then $(1+\cdot)^{-2} F \in \mathrm{L}^1(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$.*

If $\langle \widehat{v}, F\widehat{u}\rangle_{\mathrm{L}^2(i\mathbf{R};\mathbf{C}^n)} = 0$ for all $u,v \in \mathrm{L}^2_c(\mathbf{R}_+;\mathbf{C}^n)$, then $F = 0$ a.e.

Proof: 1° Set $U := \mathbf{C}^n$. By Theorem 3.3.1(c3), we have $\widehat{\mathbb{D}}, \widehat{\mathbb{N}} \in (1+\cdot)\mathrm{L}^2(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$, hence $F \in (1+\cdot)^2 \mathrm{L}^1(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$.

2° *Assumptions:* To obtain a contradiction, we assume that $\langle \widehat{v}, F\widehat{u}\rangle_{\mathrm{L}^2(i\mathbf{R};U)} = 0$ for all $u,v \in \mathrm{L}^2_c(\mathbf{R}_+;U)$ (hence for all $u,v \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ and all $\omega < 0$, by Lemma 2.1.13), but F is not zero a.e. on $i\mathbf{R}$. Then there is $r \in \mathbf{R}$ s.t. $ir \in \mathrm{Leb}(F)$ and $F(ir) \neq 0$.

W.l.o.g. we assume that $U = \mathbf{C}$, $0 \in \mathrm{Leb}(F)$ and $a := F(0)/2 > 0$ (choose $v_0, u_0 \in U$ and $\alpha \in \partial\mathbf{D}$ s.t. $\alpha v_0^* F(ir)u_0 > 0$ and replace F by $\alpha v_0^* F(\cdot - ir)u_0$; replace then u by $e^{ir\cdot}uu_0$ and v by $e^{ir\cdot}vv_0$ at the end of the proof).

3° Obviously, $|(1+ir)\widehat{f}_{t,0}(ir)| < 4t$ when $|r| > 1$ and $t \in (0,1)$, where $\widehat{f}_{t,0}(s) := 2t^{3/2}(s+t)^{-2}$. Consequently,

$$\int_{\pm[1,\infty)i} |f_{t,0}(ir)|^2 |F(ir)|\,dr < a/5 \tag{3.42}$$

for all $t \in (0, \delta_1)$, where $\delta_1 := a\|(\cdot+1)^{-2}F\|_1/20 > 0$.

4° $\varepsilon, \delta > 0$: Choose $\varepsilon > 0$ s.t. $\mathrm{Re}(2R)^{-1}\int_{-R}^R F(ir)\,dr > a$ for all $R \in (0,\varepsilon)$. Choose $\delta > 0$ s.t. $\int_{r\in(\varepsilon,1)} \|\widehat{f}_{t,0}(ir)\|^2 \|F(ir)\|\,dr < a/3$ for all $t \in (0,\delta]$ (see Lemma D.1.24(a) and note that $F \in \mathrm{L}^1([-1,1]i)$).

5° Choose $t \in (0, \max\{\delta, a/2\})$ s.t. $\int_{(-\varepsilon,\varepsilon)i} |f_{t,0}|^2\,dm > 2/3$ (see Lemma D.1.24(a)). From, e.g., Theorem 1.19 of [Rud86] we obtain functions

$$g_n := \sum_{k=1}^{m_n} R_{n,k}\chi_{(-r_{n,k},r_{n,k})i} \quad (n \in \mathbf{N}) \tag{3.43}$$

s.t. $R_{n,k} > 0$, $r_{n,k} \in (0,\varepsilon)$ $(n,k \in \mathbf{N})$, $0 \le g_1 \le g_2 \le \cdots \le g_n \le \cdots$ and $g_n(ir) \to g := |f_{t,0}(ir)|^2\chi_{i(-\varepsilon,\varepsilon)} = (t^2+r^2)^{-1}\chi_{i(-\varepsilon,\varepsilon)}$ monotonously, for each $r \in \mathbf{R}$. Consequently, $\|g_n\|_1 = \sum_k R_{n,k}(2r_{n,k}) \to \|g\| > 2/3$. By The Monotone Convergence Theorem and 2°, we have

$$\mathrm{Re}\int_{-\varepsilon}^{\varepsilon} |f_{t,0}(ir)|^2 F(ir)\,dr = \lim_{n\to\infty}\int_{-\varepsilon}^{\varepsilon} g_n(ir)\,\mathrm{Re}\,F(ir)\,dr \ge \limsup_{n\to\infty}\sum_k R_{n,k}(2r_{n,k})a > 2a/3. \tag{3.44}$$

By this, (3.42) and 4°, we have $\mathrm{Re}\int_{i\mathbf{R}} |f_{t,0}|^2 F\,dm = \mathrm{Re}\langle f_{t,0}, Ff_{t,0}\rangle > a(2/3 - 1/5 - 1/3) > 0$, a contradiction, as required. □

According to Theorem 2.3 of [W91a] (or to Theorem 3.1.7), the first part of Theorem 2.1.2 holds for arbitrary Banach spaces U and Y and for any $p \in [1,\infty)$. Even in that case, $\mathcal{L}\colon \mathrm{TIC} \to \mathrm{H}^\infty$ is a contractive algebra isomorphism of $\mathrm{TIC}(U,Y)$ *into* $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$. However, the isomorphism is not isometric, nor onto:

Example 3.3.4 (A counter-example for Banach spaces: $\mathbf{H}^\infty \not\subset \widehat{\mathbf{TI}}$) Let $U = \mathbf{C}$, $Y = \ell^\infty(\mathbf{N})$, $e_k := \chi_{\{k\}}$ $(k \in \mathbf{N})$ (i.e., $e_0 = \{1,0,0,\ldots\}$, $e_1 := \{0,1,0,0,\ldots\}$ etc.) and $\widehat{\mathbb{D}}(s)u_0 := (e^{-ks}u_0)_{k\in\mathbf{N}}$, so that $\|\widehat{\mathbb{D}}(s)\| \le 1$ for all $s \in \overline{\mathbf{C}^+}$.

By Lemma D.1.1, $\widehat{\mathbb{D}}$ is holomorphic, because $e^{-k\cdot}$ is holomorphic for each $k \in \mathbf{N}$. Thus, $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C},Y))$. However, $\widehat{\mathbb{D}}\widehat{f} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R}_+;Y)]$ for $f := \chi_{[0,1)} \in \mathrm{L}^2(\mathbf{R}_+)$, hence $\widehat{\mathbb{D}}$ is not the transfer function of any $\mathbb{D} \in \mathrm{TI}(\mathbf{C},Y)$.

Furthermore, $\widehat{\mathbb{V}}\colon s \mapsto (k^{-1/2}(s+1)^{-1}\widehat{\mathbb{D}}(s))_{k\in\mathbf{N}}$ satisfies $\widehat{\mathbb{V}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C},Y))$ and $\widehat{\mathbb{V}}$ is continuous on $\overline{\mathbf{C}^+} \cup \{\infty\}$ ("$\widehat{\mathbb{V}} \in \widehat{\mathrm{CTIC}}$") but still $\widehat{\mathbb{V}}$ is not the transfer function of any $\mathbb{V} \in \mathrm{TI}(\mathbf{C},Y)$ (thus, "CTIC $\not\subset$ TIC" in this sense; cf. Definition 2.6.1).

Remarks: 1. We can use $\mathrm{TIC}(Y)$ in the example in place of $\mathrm{TIC}(\mathbf{C},Y)$.
2. Fourier transform does not map $\mathrm{L}^2(\mathbf{R};Y)$ into (nor onto) $\mathrm{L}^2(i\mathbf{R};Y)$.

3. TIC $\to \mathrm{H}^\infty$ is not an isometry (nor onto). $\triangleleft$

Proof: (Note that we just use the Hilbert space definitions of Chapter 2 extended to allow for Banach spaces as input and output spaces.)

1° $\widehat{\mathbb{D}}\widehat{f} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R}_+;Y)]$ *for* $f := \chi_{[0,1)}$*:* By $P_k \in Y^*$ we denote the projection of $P_k : (y_k)_{k\in\mathbf{N}} \mapsto y_k$, for any $k \in \mathbf{N}$. Then P_k commutes with the Laplace transform (because $P_k \in Y^*$).

Choose $f \in \mathrm{L}^2([0,1])$ s.t. $\|f\|_2 = 1$ (e.g., $f = \chi_{[0,1]}$). If $g \in \mathrm{L}^2(\mathbf{R};Y)$ were s.t. $\widehat{g} = \widehat{\mathbb{D}}\widehat{f}$ on $\mathbf{C}^+$, then $P_k\widehat{g} = P_k(\widehat{\mathbb{D}}\widehat{f}) = \mathrm{e}^{-ks}\widehat{f}$, hence we would have $P_k g = \tau(-k)f$, and, consequently,

$$\|g\|_2^2 = \sum_k \int_k^{k+1} \|g\|_Y^2\, dm = \sum_k \int_k^{k+1} |g_k|^2\, dm = \sum_k \|f\|_2^2 = \infty. \tag{3.45}$$

Therefore, $\widehat{\mathbb{D}}$ does not map $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;U)$ into $\mathcal{L}\mathrm{L}^2(\mathbf{R};Y)$, hence $\widehat{\mathbb{D}}$ does not define a $\mathrm{TI}(U,Y)$ map.

2° *Constructing* $\widehat{\mathbb{V}}$ *with required properties:* Let $(a_k)_{k\in\mathbf{N}}$ be a sequence in $\mathbf{C}$ s.t. $a_k \to 0$ as $k \to \infty$. Define $T \in \mathcal{B}(\ell^\infty)$ by $(y_k)_{k\in\mathbf{N}} := (a_k y_k)_{k\in\mathbf{N}}$. Then also $\widehat{\mathbb{F}}(s) = T\widehat{\mathbb{D}}$ is in $\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, but, unlike for $\widehat{\mathbb{D}}$, we have $\widehat{\mathbb{F}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y))$, and $\widehat{\mathbb{F}}(ir+t) \to (a_k \mathrm{e}^{-irk})_{k\in\mathbf{N}} =: \widehat{\mathbb{F}}(ir) \in \mathcal{C}_{\mathrm{bu}}(i\mathbf{R};\mathcal{B}(U,Y))$ as $t \to 0+$, which will be shown in 3° below.

The function $\widehat{\mathbb{V}}(s) := \frac{1}{s+1}\widehat{\mathbb{F}}(s)$ is continuous on $\overline{\mathbf{C}^+} \cup \{\infty\}$ (we use the symbol $\widehat{\mathrm{CTIC}}$ of this kind of functions in Section 2.6), because it has the limit 0 at ∞ (because $\|\widehat{\mathbb{V}}(s)\| \le \frac{1}{|s|+1} \to 0$ as $|s| \to \infty$ on $\overline{\mathbf{C}^+}$). Let $F(t) := \mathrm{e}^{-t}\chi_{\mathbf{R}_+}(t)$, so that $\widehat{F}(s) = \frac{1}{s+1}$. If $\widehat{h} = \widehat{\mathbb{V}}\widehat{f} = \widehat{F}T\widehat{g}$, then $h = F * Tg$, in particular $P_k h = F * P_k T g = a_k F * g_k = a_k F * \tau(-k) f = \tau(-k) a_k F * f$, hence

$$\|h\|_2^2 = \sum_k \int_k^{k+1} \|h\|_Y^2\, dm \ge \sum_k \int_k^{k+1} |h_k|^2\, dm \ge \sum_k \int_0^1 |a_k F * f|^2\, dm \ge \sum_k |a_k|^2 r = \infty, \tag{3.46}$$

if $\sum_k |a_k|^2 = \infty$ (e.g., $a_k = k^{-1/2}$) and $r := \int_0^1 |F * f|^2 > 0$ (e.g., $f = \chi_{[0,1]}$). Thus, also $\widehat{\mathbb{V}}$ does not map $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;U)$ into $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;Y)$.

3° $\widehat{\mathbb{F}}$ is continuous on $\overline{\mathbf{C}^+}$: Given $s_0 \in \overline{\mathbf{C}^+}$ and $\varepsilon > 0$, choose first K s.t. $|a_k| < \varepsilon/2$ for $k \ge K$. Use then continuity to choose $\delta > 0$ s.t. $|\mathrm{e}^{-sk} - \mathrm{e}^{-s_0 k}| < \varepsilon/\sup_k |a_k|$ when $s \in \overline{\mathbf{C}^+}$ and $|s - s_0| < \delta$. Now $\|\widehat{\mathbb{F}}(s) - \widehat{\mathbb{F}}(s_0)\|_Y := \sup_k |\widehat{\mathbb{F}}(s)_k - \widehat{\mathbb{F}}(s_0)_k| < \varepsilon$ for such s, QED.

Remarks: 1. *We can use* $\mathrm{TIC}(Y)$ *in the example in place of* $\mathrm{TIC}(\mathbf{C},Y)$*:* just use $\widehat{\mathbb{D}}P_0$ and $\widehat{\mathbb{V}}P_0$ in place of $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{V}}$, where $P_0(\alpha_k)_{k\in\mathbf{N}} := \alpha_0$.

2. *The Fourier transform does not map* $\mathrm{L}^2(\mathbf{R};Y)$ *into (nor onto)* $\mathrm{L}^2(i\mathbf{R};Y)$*; in fact, it does not even map* $\mathrm{L}^1 \cap \mathrm{L}^\infty \to \mathrm{L}^2$*:* Since $\|\widehat{g}\|_2 = \|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2 = \sqrt{2\pi}$ and $\|g\|_2 = \infty$, we observe that the Fourier transform does not map $\mathrm{L}^2(\mathbf{R}_+;Y)$ onto $\mathrm{L}^2(i\mathbf{R};Y)$ (since g is the only possible inverse transform of $\widehat{g}$, by 1°). By replacing the roles of g and $\widehat{g}$, one can show that the Fourier transform does not map $\mathrm{L}^2(\mathbf{R}_+;Y)$ into $\mathrm{L}^2(i\mathbf{R};Y)$; not even $\mathrm{L}^1 \cap \mathrm{L}^\infty$ into $\mathrm{L}^2(i\mathbf{R};Y)$. (Indeed, if $f \in \mathcal{S}(\mathbf{R})$, then $\widehat{f} \in \mathcal{S}(i\mathbf{R})$, hence then $\widehat{g} := \widehat{\mathbb{D}}\widehat{f} \in \mathrm{L}^2(i\mathbf{R};Y)$ is "rapidly decreasing" (though not differentiable), hence $\widehat{g} \in \mathrm{L}^p$ for all $p \in [1,\infty]$.)

3. $\mathrm{TIC} \to \mathrm{H}^\infty$ *is not an isometry (nor onto):* The operator $\widehat{\mathbb{V}}_n := \sum_{k=0}^n P_k \widehat{\mathbb{D}} \in \mathrm{H}^\infty$ is obviously the transfer function of a $\mathrm{TIC}(\mathbf{C}, Y)$ map and $\|\widehat{\mathbb{V}}_n\| = \|\widehat{\mathbb{D}}\| = 1$ for $n \in \mathbf{N}$, but $\|\mathbb{V}_n\|_{\mathrm{TIC}} \to \infty$ (by the above computations), hence the Laplace transform $\mathrm{TIC} \to \mathrm{H}^\infty$ is not an isometry (nor onto, by 1°). □

In fact, the above example also shows that Theorem 3.3.1(c1) does not hold for Banach spaces:

Example 3.3.5 (A counter-example for Banach spaces: $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ has no strong nor $\mathrm{L}^\infty_{\mathrm{strong}}$ boundary function) Let $U = \mathbf{C}$, $Y = \ell^\infty(\mathbf{N})$, and let $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U, Y))$ be as in Example 3.3.4, i.e., $\widehat{\mathbb{D}}(s) = (\mathrm{e}^{-ks})_{k \in \mathbf{N}}$.

Then $\widehat{\mathbb{D}}$ does not have a boundary function in the sense of Theorem 3.3.1(c1); in fact, $\widehat{\mathbb{D}}(ir + t)u_0$ does not converge in Y, as $t \to 0+$, for any nonzero $u_0 \in U$. Moreover, the componentwise boundary function of $\widehat{\mathbb{D}}$ is not an element of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U; Y))$. ◁

Recall from Example 3.3.4 that the above $\widehat{\mathbb{D}}$ does not correspond to any TIC (not even to any TI) operator.

Proof: 1° Clearly $(\widehat{\mathbb{D}}(ir+t)u_0)_k \to \mathrm{e}^{-irk} u_0$ as $t \to 0+$, for every $u_0 \in \mathbf{C} = U$, $k \in \mathbf{N}$, so the componentwise boundary function $ir \mapsto \widehat{\mathbb{D}}(ir) := (\mathrm{e}^{-irk})_{k \in \mathbf{N}}$ is the only possible boundary function.

However, $\sup_{t \in (0,\delta)} \sup_{k \in \mathbf{N}} |\mathrm{e}^{-(ir+t)k} u_0 - \mathrm{e}^{-irk} u_0| = |u_0| > 0$ for any $\delta > 0$ (take $k > 2/\delta$ and $t = \pi/2k$), hence there is no limit of $\widehat{\mathbb{D}}(ir+t)u_0$ as $t \to 0+$, i.e., there is no strong boundary function.

2° Moreover, $\widehat{\mathbb{D}}(ir) \notin \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U, Y))$: if $\widehat{\mathbb{D}}(ir)$ were in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U, Y))$, then $\widehat{\mathbb{D}}(i\cdot)$ would be an L^∞ function by Theorem 3.1.3(c) (because $U = \mathbf{C}$), hence then we would have $\widehat{\mathbb{D}}(i\cdot)P_0 \in \mathrm{L}^\infty(i\mathbf{R}; \mathcal{B}(Y))$, where P_0 is the projection to the 0th component ($P_0 : (y_k)_{k \in \mathbf{N}} \mapsto y_0$).

However, $r \mapsto \widehat{\mathbb{D}}(ir)P_0$ is a (semi)group of bounded linear operators on $\mathcal{B}(Y)$. Because it is not uniformly continuous, it is not uniformly measurable, by [HP, Theorem 10.2.1]. (A second proof: if $\widehat{\mathbb{D}}$ were $\mathrm{L}^\infty(i\mathbf{R}; Y)$, its Poisson integral would converge pointwise a.e. A third proof: the function $\widehat{g} = \widehat{\mathbb{D}}\widehat{f}$ in Example 3.3.4, would be L^1, hence g would be $\mathcal{C}_0$, but it does not vanish at infinity.) □

Output maps (hence also the causal adjoints of input maps) of WPLSs correspond to strong H^2 functions. Unfortunately, such functions need not have boundary functions with values in $\mathcal{B}$ (cf. Theorem 3.3.1(d)), not even for separable U and Y. For simplicity, we give our counter-example on the unit disc $\mathbf{D}$:

Example 3.3.6 (A counter-example: $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}; \mathcal{B}(\ell^2))$ has no $\mathcal{B}(\ell^2)$-valued boundary function) Let $\{z_n\} \subset \partial\mathbf{D}$ be dense. Choose $r \in (0, 1/2)$. Define $z \mapsto F_n(z) := (z - z_n)^{-r} \in \mathrm{H}^2(\mathbf{D}; \mathbf{C})$ ($n \in \mathbf{N}$). Define the "diagonal operator" $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}; \mathcal{B}(\ell^2))$ by $Fe_n := F_n e_n$ ($n \in \mathbf{N}$), where the vectors $e_n = \chi_{\{n\}}$ form the canonical base of $\ell^2(\mathbf{N}) =: U =: Y$. Indeed, $\|F_n\|_{\mathrm{H}^2} = \|F_0\|_{\mathrm{H}^2} =: M$ $(n \in \mathbf{N})$, hence

$$\|F\sum_{k=0}^{n} \alpha_k e_k\|^2_{\mathrm{H}^2} \le M^2 \sum_{k=0}^{n} |\alpha_k|^2. \tag{3.47}$$

Thus, F can be extended to the whole ℓ^2 so that $\|F\|_{\mathrm{H}^2_{\mathrm{strong}}} \le M$. Moreover, if F is defined on $\partial\mathbf{D}$ so that $F_{|\partial\mathbf{D}}u$ is a.e. the boundary function of Fu for each $u \in U$, then $Fe_n = F_n$ on $\overline{\mathbf{D}} \setminus N$ for each $n \in \mathbf{N}$, where N is a null set.

Given $z \in \partial\mathbf{D} \setminus N$ and $M' > 0$, there is n s.t. $M' < |F_n(z)| = \|F(z)e_n\|_U$ (take n s.t. $|z - z_n|$ is small enough), hence $\|F(z)\|_{\mathcal{B}(U)} \ge M'$. Consequently, the operator $F(z)$ is unbounded (and possibly not defined for all $u \in U$) on $\partial\mathbf{D} \setminus N$.

(Note that Fu has a boundary function a.e. for each $u \in U$, the problem is that these boundary functions for $u \in U$ are not due to any (single) $\mathcal{B}(U)$*-valued* function; the values of the boundary function must be unbounded operators, as above.) $\triangleleft$

See also Example F.3.6.

In the scalar case the Poisson integral $P * f$ of any $f \in \mathrm{L}^p(i\mathbf{R}; B)$ is a harmonic function on the half-plane. This function is analytic iff $f \in \mathrm{H}^p$; a third equivalent condition for $f \in \mathrm{L}^1$ is that the Fourier transform of f is zero on R_-. All this holds also in the vector-valued case:

Lemma 3.3.7 ($f \in \mathrm{H}^p \Leftrightarrow P * f \in \mathrm{H}$) *Let $f_0 \in \mathrm{L}^p(\omega + i\mathbf{R}; B)$, $p \in [1, \infty]$. Then f_0 is the boundary function of some $f \in \mathrm{H}^p(\omega + i\mathbf{R}; B)$ iff the Poisson integral of f_0 is analytic on $\mathbf{C}^+_\omega$. For $p = 1$ a third equivalent condition is that*

$$\widetilde{f}(t) := \int_{\mathbf{R}} f_0(\omega + ir)\mathrm{e}^{irt}\,dr = 0 \quad \textit{for all } t < 0 \tag{3.48}$$

(if this is the case, then $f(\omega + \cdot) = \mathcal{L}\widetilde{f}/2\pi$).

Analogously, $f_0 \in \mathrm{L}^p(\partial\mathbf{D}; B)$ is the boundary function of some $f \in \mathrm{H}^p(\mathbf{D}; B)$ iff the Poisson integral of f_0 is analytic on $\mathbf{D}$; a third equivalent condition is that $\widehat{f}(n) := \frac{1}{2\pi}\int_{\partial\mathbf{D}} \mathrm{e}^{-inr} f_0(\mathrm{e}^{ir})\,dr = 0$ for $n = -1, -2, \ldots$ (if this is the case, then $f = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$).

Recall that if B is a Hilbert space, then any H^p function has an L^p boundary function. Note also that $\int_{\mathbf{R}} f(\omega + ir)\mathrm{e}^{\cdot ir}\,dr \in \mathcal{C}_0(\mathbf{R}; B)$, by Lemma D.1.11(a1)&(a3).

Proof: We prove the $\omega + i\mathbf{R}$ claims; the $\mathbf{D}$ claims (which can be scaled for $\mathbf{D}_r$) follow analogously from Theorem 17.13 of [Rud86] (note that $\mathrm{L}^p \subset \mathrm{L}^1$ on $\partial\mathbf{D}$).

1° If the Poisson integral f of f_0 is analytic, then $f \in \mathrm{H}^p$ and f_0 is the boundary function of f, by Lemma D.1.8(a3). Conversely, if f_0 is the boundary function of some $f \in \mathrm{H}^p$, then f is the Poisson integral of f_0, by Theorem 3.3.1(a1).

2° *Case $p = 1$:* For $B = \mathbf{C}$, the third condition is equivalent, by Lemma II.3.7 and Theorem II.3.8 of [Garnett]. From this and 1° it follows for general B, that the Poisson integral of Λf is analytic for all $\Lambda \in B^*$ iff

$\widetilde{f}(t) := \int_{\mathbf{R}} \Lambda f(\omega + ir)e^{tir}\,dr = 0$ for all $t < 0$. This and Lemma D.1.1(a) imply that the Poisson integral of f is analytic iff $\int_{\mathbf{R}} f(\omega + ir)e^{irt}\,dr = 0$ for all $t < 0$. By Lemma F.3.7(a3), we have $f(\cdot) = \mathcal{L}e^{\omega\cdot}\widetilde{f}/2\pi$ □

Next we extend the standard formula $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ for causal adjoint transfer functions to the vector-valued case (including noncausal TI maps):

Lemma 3.3.8 ($\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$) *Let $u \in \mathrm{L}^2(\mathbf{R};U)$ and $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then $\widehat{\mathbf{Я}u}(ir) = \widehat{u}(-ir)$ for $r \in \mathbf{R}$. Moreover, $\widehat{\mathbb{E}^*}(ir) = \widehat{\mathbb{E}}^*(ir)$, $\widehat{\mathbf{Я}\mathbb{E}\mathbf{Я}}(ir) = \widehat{\mathbb{E}}(-ir)$ and $\widehat{\mathbb{E}^{\mathrm{d}}}(ir) = \widehat{\mathbb{E}}^*(-ir)$ for $r \in \mathbf{R}$.*

If $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, then $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ for $s \in \mathbf{C}^+_\omega$.

See Definition 3.1.1 (and Theorem 3.1.3(d)) for $\widehat{\mathbb{E}}^*$.

Proof: 1° TI: Obviously, $\widehat{\mathbf{Я}u}(s) = \widehat{u}(-s)$ wherever either integral converges. It follows that

$$\mathcal{L}(\mathbf{Я}\mathbb{E}\mathbf{Я}u)(ir) = \mathcal{L}(\mathbb{E}\mathbf{Я}u)(-ir) = \widehat{\mathbb{E}}(-ir)\widehat{u}(ir) \quad (r \in \mathbf{R}). \tag{3.49}$$

Therefore, $\mathcal{L}(\mathbf{Я}\mathbb{E}\mathbf{Я})(ir) = \widehat{\mathbb{E}}(-ir)$ for all $r \in \mathbf{R}$. By linearity and continuity, we obtain that $\langle \widehat{\mathbb{E}}\widehat{u}, \widehat{v}\rangle = \langle \widehat{u}, \widehat{\mathbb{E}}^*\widehat{v}\rangle$ for all $u, v \in \mathrm{L}^2$ (see Definition 3.1.1), hence $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^*$, from which it follows that $\widehat{\mathbb{E}^{\mathrm{d}}}(ir) = \widehat{\mathbb{E}}^*(-ir)$ for $r \in \mathbf{R}$.

2° TIC: Finally, let $\mathbb{D} \in \mathrm{TIC}$ (the TIC_ω result is obtained by shifting) and $f = \widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, and define $h(\cdot) := f(\bar{\cdot})^* \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(Y,U))$. Let $\mathbb{F} \in \mathrm{TIC}(Y,U)$ be defined by $\widehat{\mathbb{F}} = h$, and let h_0 be the boundary function of h.

Then $\langle f_0(-ir)u_0, y_0\rangle = \lim_{t\to 0+}\langle f(-ir+t)u_0, y_0\rangle = \lim_{t\to 0+}\langle u_0, f(-ir+t)^*y_0\rangle = \langle u_0, h_0(ir)y_0\rangle$ a.e. on $i\mathbf{R}$ for $u_0 \in U$, $y_0 \in Y$, hence $\mathbf{Я}[f_0]^* = [h_0] \in \mathrm{L}^\infty_{\mathrm{strong}}$; in particular, $[h_0] = \widehat{\mathbb{F}}$, by 1°. Therefore, h is an H^∞ function with the boundary function $\widehat{\mathbb{F}}$, but, by Theorem 3.3.1(c1), the transfer function of $\mathbb{F}$ is the only such function. □

During the rest of this section, we shall study the poles of the inverses of transfer functions and use the results to construct a "completely" unstable transfer function.

Let $\Omega \subset \mathbf{C}$ is open and $m, n \in \mathbf{N}$. If $0 \neq f \in \mathrm{H}(\Omega;\mathbf{C})$, then the set of zeros of f (i.e., of poles of f^{-1}) does not have limit points in Ω (see, e.g., Theorem 10.18 of [Rud86]). It follows that if $f \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n))$ is invertible at some $s_0 \in \Omega$, then f is invertible on a set whose complement does not have limit points in Ω (this complement is the set of zeros of $\det f$). The same applies to the left-invertibility of $f \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n,\mathbf{C}^m))$ (because if $L \in \mathbf{C}^{n\times m}$ is s.t. $Lf(s_0) = I$, then $Lf \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n))$).

These facts are extensively used in control theory. However, if $\dim U = \infty$ and $f \in \mathrm{H}(\Omega;\mathcal{B}(U))$, then the set of "poles" of f^{-1} may be any closed subset of Ω, even if f were bounded, when, e.g., $\Omega = \mathbf{C}^+_\omega$ ($\omega \in \mathbf{R}$) or $\Omega = \mathbf{D}_r$ ($r > 0$):

Lemma 3.3.9 (Poles of $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$) *Let $\dim U = \infty$. Let $K \subset \mathbf{C}$ be closed and let $s_0 \in \mathbf{C} \setminus K$. Then there is $\widehat{\mathbb{D}} \in \mathrm{H}(\{s_0\}^c;\mathcal{B}(U))$ s.t. $\widehat{\mathbb{D}}^{-1} \in \mathrm{H}(K^c;\mathcal{B}(U))$, $\widehat{\mathbb{D}} \in$*

$\mathcal{G}\mathrm{H}^\infty(K_\varepsilon^c;\mathcal{B}(U))$ *for any* $\varepsilon > 0$, *where* $K_\varepsilon := \{s \in \mathbf{C} \,\big|\, d(s, K\cup\{s_0\}) \le \varepsilon\}$, *and the set of singularities of* $\widehat{\mathbb{D}}^{-1}$ *is* K *in the sense that* $\lim_{K^c \ni s\to s_1} \|\widehat{\mathbb{D}}^{-1}(s)\| = +\infty$ *for each* $s_1 \in \partial K$.

Thus, when $\omega < \omega'$ and $K \subset \{z \,\big|\, \operatorname{Re} z < \omega'\}$ is closed, there is $\mathbb{D} \in \mathrm{TIC}_\omega(U) \cap \mathcal{G}\mathrm{TIC}_{\omega'}(U)$ s.t. the set of singularities of $\widehat{\mathbb{D}}^{-1}$ is K. One observes from the proof that $\widehat{\mathbb{D}}(s) \notin \mathcal{G}\mathcal{B}(U)$ for any $s \in K$, since the poles of the "components" of $\widehat{\mathbb{D}}$ are dense in K.

We might call these singularities "poles" (at least those on ∂K), since this would be in accordance with the definition "$\lim_{\Omega\ni s\to s_1} \|f(s)^{-1}\| \to +\infty$" (for scalar functions this is equivalent to the standard one, by Theorem 10.21(c) of [Rud86]).

(Unless s_1 is an isolated point of K, there is a sequence $\{z_n\} \subset K^c$ s.t. $z_n \to s_1$ and $\|(z_n - s_1)^N \widehat{\mathbb{D}}(s_1)\| \to +\infty$, as $n \to +\infty$, for any $N \in \mathbf{N}$ (e.g., choose them so that $d(z_n, K) < (z_n - s_1)^N/n$ and use (3.51)), hence one might argue that the points of ∂K (or K) should nevertheless be called essential singularities.

Proof: By Lemma A.3.1(a2), we can assume that $U = \ell^2(A)$ for some infinite set A. Set $K_0 := K \cup \{s_0\}$, $\gamma := d(s_0, K) > 0$.

1° *Functions* $\{f_a\}_{a\in A}$*:* Choose $\phi : A \to K$ s.t. $\phi[A]$ is dense in K. Set

$$f_a(s) := \frac{s - \phi(a)}{s - s_0} = 1 + \frac{s_0 - \phi(a)}{s - s_0} \quad (s \in \mathbf{C} \setminus \{s_0\},\ a \in A). \tag{3.50}$$

Then $|f_a(s)|, |f_a^{-1}(s)| < 1 + \gamma/\varepsilon =: M_\varepsilon$ for all $\varepsilon > 0$ and all $s \in K_\varepsilon^c$. Moreover, $f_a \in \mathrm{H}(\{s_0\}^c)$ and $f_a^{-1} \in \mathrm{H}(K^c)$.

2° *Function* f*:* For each s, we define $f(s) \in \ell^\infty(A)$ by $f(s)_a := f_a(s)$. Now $\Lambda_a f \in \mathrm{H}(\{s_0\}^c)$ for all $a \in A$, where $\Lambda_a u := u_a$, hence $f \in \mathrm{H}(\{s_0\}^c;\ell^\infty(A))$, by Lemma D.1.1(a).

Analogously, $f^{-1} \in \mathrm{H}(K^c;\ell^\infty(A))$, where $f^{-1}(s) = (f_a(s)^{-1})_{a\in A} \in \ell^\infty$ (this is obviously the inverse of $f(s)$ on K_0^c).

But $\ell^\infty(A)$ (as multiplication operators) is a Banach subalgebra of $\mathcal{B}(\ell^2(A))$. Therefore, $f \in \mathrm{H}(\{s_0\}^c;\mathcal{B}(U))$, $f^{-1} \in \mathrm{H}(K^c;\mathcal{B}(U))$, by Lemma D.1.2(b1). By 1°, we have $\|f\|, \|f^{-1}\| \le M_\varepsilon$ on K_ε^c for each $\varepsilon > 0$.

3° *Singularities of* f^{-1}*:* Let $s \in K^c$. For each $\delta > d(s,K)$ there is $a \in A$ s.t. $|s - \phi(a)| < \delta$ and hence

$$|f_a^{-1}(s)| = \frac{|s - s_0|}{|s - \phi(a)|} > \frac{\gamma - \delta}{\delta}, \tag{3.51}$$

therefore $\|f^{-1}(s)\|_{\mathcal{B}(U)} \ge \frac{\gamma - d(s,K)}{d(s,K)}$. Thus, $\|f^{-1}(s)\| \to \infty$ as $d(s,K) \to 0$. Consequently, $\widehat{\mathbb{D}} := f$ has the required properties. □

By using the above techniques, we can construct a "completely unstable" $\mathbb{D} \in \mathrm{TIC}_\infty$ whenever U and Y are infinite-dimensional (note that $\dim Y \ge \dim U$ iff there is a left-invertible $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, by Lemma 2.2.1(c3))

Example 3.3.10 ($\mathbb{D}u \notin \mathrm{L}^2$ for all nonzero $u \in \mathrm{L}^2$) Let $\dim Y \ge \dim U = \infty$. If $\omega > 0$, then there is a left-invertible $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ s.t. $\mathbb{D}u \notin \mathrm{L}^2$ for all nonzero

$u \in \mathrm{L}^2(\mathbf{R}_+;U)$. ◁

In fact, given $\gamma > 0$, we can choose an ULR $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ s.t. $\mathbb{D}$ satisfies the above conditions for any $\omega > \gamma$, and $\widehat{\mathbb{D}} \in \mathrm{H}((\gamma + i/(\mathbf{N}+1))^c;\mathcal{B}(U,Y))$, and $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega_{\omega',\omega};\mathcal{B}(U,Y))$ whenever $\omega' < \gamma < \omega$, where $\Omega_{\omega',\omega} := \{s \in \mathbf{C} \,|\, \mathrm{Re}\, s \in [\omega',\omega]^c\}$ (take $s_n := \gamma + i/(n+2)$ in the proof below).

Therefore, in that case, we have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^-_\alpha;\mathcal{B}(U,Y))$ for each $\alpha < \gamma$, hence this (left) part of $\widehat{\mathbb{D}}$ is the transfer function (Fourier transform) of some $\widetilde{\mathbb{D}} \in \cap_{\alpha<\gamma}\mathrm{TI}_\alpha(U,Y)$, which is strictly anti-causal, i.e., $\pi_+\widetilde{\mathbb{D}}\pi_- = 0 \neq \pi_-\widetilde{\mathbb{D}}\pi_+$ (apply Theorem 3.1.6(b) to $\widehat{\mathbb{D}}$ and Lemma 2.1.11(iv)&(i) to $\boldsymbol{Я}\widetilde{\mathbb{D}}\boldsymbol{Я}$).

Proof: W.l.o.g., we assume that $\dim Y = \dim U$ (replace then $\mathbb{D} \in \mathrm{TIC}_\omega(U)$ by $T\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ for any left-invertible $T \in \mathcal{B}(U,Y)$). By Lemma A.3.1(a2) and Lemma B.2.2, we may assume that $U = \ell^2(A)$ and $Y = \ell^2(A \times \mathbf{N})$ for some infinite set A.

Choose $K_0 := \{s_n\}_{n=-1}^\infty \subset \gamma + i\mathbf{R}$ s.t. $s_n \to s_0$, as $n \to \infty$, and $\gamma := d(s_{-1},K) > 0$, where $K := \{s_n\}_{n=0}^\infty$.

Set $f_j(s) := (s - s_j)/(s - s_{-1})$ $(s \in \mathbf{C} \setminus \{s_{-1}\},\ j \in \mathbf{N})$, so that $|f_j(s)|, |f_j(s)^{-1}| < 1 + \gamma/\varepsilon =: M_\varepsilon$ for all $\varepsilon > 0$ and all $s \in \mathbf{K}^c_\varepsilon$, where $K_\varepsilon := \{s \in \mathbf{C} \,|\, d(s,K_0) \le \varepsilon\}$ (as in 1° of the proof of Lemma 3.3.9).

Define $\widehat{\mathbb{D}}(s) \in \mathcal{B}(U,Y)$ by $(\widehat{\mathbb{D}}(s)u)_{a,j} := 2^{-j} f_j^{-1} u_a$ $(u \in U)$. Then

$$\|\widehat{\mathbb{D}}(s)u\|_Y^2 = \sum_{a\in A,\ j\in\mathbf{N}} \|2^{-j} f_j^{-1}(s) u_a\|^2 \le \sum_{a\in A} 2\|f^{-1}(s)\|_{\ell^\infty} \|u_a\|^2, \tag{3.52}$$

hence $\|\widehat{\mathbb{D}}(s)\|_{\mathcal{B}(U,Y)} \le 2M_\varepsilon$ for $s \in K^c_\varepsilon$.

Define $\widehat{\mathbb{E}}(s) \in \mathcal{B}(Y,U)$ by $(\widehat{\mathbb{E}}(s)y)_a := f_0(s) y_{a,0}$ $(y \in U)$. Then $\|\widehat{\mathbb{E}}(s)\| \le M_\varepsilon$ for $s \in K^c_\varepsilon$. Obviously, $\widehat{\mathbb{E}}(s)\widehat{\mathbb{D}}(s) = I$ for $s \in K^c_0$. It follows from Lemma D.1.1(a) that $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{E}}$ are holomorphic on $\mathbf{C}^+_\omega$, hence $\widehat{\mathbb{D}}, \widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U))$, hence $\mathbb{D}, \mathbb{E} \in \mathrm{TIC}_\omega(U)$.

As in the proof of Lemma 3.3.9, we can verify that $\widehat{\mathbb{D}}$ has the properties claimed above. We proof the claim "$\mathbb{D}u \in \mathrm{L}^2 \Rightarrow u = 0$":

Let $u \in \mathrm{L}^2(\mathbf{R}_+;U) \setminus \{0\}$. Choose $a \in A$ s.t. $u_a \not\equiv 0$. By Lemma D.1.2(e), there is $j \in \mathbf{N}$ s.t. $\widehat{u}_a(s_j) \neq 0$. It follows that $\|f_j \widehat{u}_a\| \to \infty$ as $s \to s_j$ (see 3° of the proof of Lemma 3.3.9).

But $(\widehat{\mathbb{D}}\widehat{u})_{a,j} = 2^{-j} f_j^{-1} \widehat{u}_a$, and $(y \mapsto y_{a,j}) \in \mathcal{B}(Y,\mathbf{C})$, hence $\widehat{\mathbb{D}}\widehat{u} \notin \mathrm{H}(\mathbf{C}^+;Y)$, in particular, $\mathbb{D}u \notin \mathrm{L}^2$. □

On the other hand, if $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ is optimizable (see Definition 6.7.3) and $\dim U < \infty$, then there is $\delta > 0$ s.t. every $\lambda \in \sigma(A) \cap \mathbf{C}^+_{-\delta} =: K$ is an isolated eigenvalue of finite multiplicity, by [JZ99]. In particular, then $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}_{-\delta} \setminus K;\mathcal{B}(U,Y))$

Notes

Most of (a), (b), (c2) and (d4) (and (e)) of Theorem 3.3.1 is well known (see, e.g., [RR] for the separable case). A comprehensive study on the separable case of (c1) is given in [Thomas]. Also Lemmas 3.3.2 and 3.3.8 are probably well known.

The monographs [Duren] and [Hoffman] are classical references for H^p spaces and their boundary functions. The monograph [RR] by Marvin Rosenblum and James Rovnyak is a classical reference on case on separable Hilbert spaces; it also contains further results and extensions to the Nevanlinna class.