Chapter 4

Corona Theorems and Inverses

If I have not seen so far it is because I stood in giant's footsteps.

In Theorem 4.1.1, we show that MTI, CTI (resp. MTIC, CTIC) and most of their subclasses are inverse closed in TI (resp. in TIC), and provide several equivalent conditions for the invertibility of such maps. We also give some extensions and related results. Then we show that these classes are adjoint closed (Lemma 4.1.3).

Thereafter, we study the Corona Theorem and its consequences, giving equivalent conditions for left-invertibility (use duality for right-invertibility) in $\mathcal{A}(U,Y)$, where $\dim U < \infty$ and $\mathcal{A}$ equals TIC, $\mathrm{MTIC}^{\mathrm{L}^1}$, CTIC or some of certain other classes. We also list some consequences of these results to coprime factorization, following M. Vidyasagar [Vid].

The Corona Theorem does not extend to infinite-dimensional U (see Lemma 4.1.10), but we give several partial results for the infinite-dimensional case. A casual reader probably wants to just read (main) Theorems 4.1.1 and 4.1.6 and then go on to the next section.

Recall that U, H and Y denote Hilbert spaces of arbitrary dimensions unless something else is indicated.

We start by showing that several useful subclasses of TIC (and those of TI) are inverse closed (this means the equivalence (ii)$\Leftrightarrow$(i) in (b) and (a) below):

Theorem 4.1.1 (Inverse-closed classes) *Let $p \in [1,\infty]$. $\mathcal{A}$ be one of the classes*

$$\text{TI, CTI, CTI}^{\mathcal{BC}}, \tag{4.1}$$

$$\text{MTI, MTI}^{\mathcal{BC}},\ \text{MTI}_{\mathrm{d}},\ \text{MTI}_{\mathrm{d}}^{\mathcal{BC}},\ \text{MTI}^{\mathrm{L}^1},\ \text{MTI}^{\mathrm{L}^1,\mathcal{BC}},\ \mathcal{B}+(\mathrm{L}^1\cap\mathrm{L}^p)*. \tag{4.2}$$

Then $\mathcal{A}$ is inverse-closed in TI*; in particular, $\widetilde{\mathcal{A}} := \mathcal{A}\cap\mathrm{TIC}$ is inverse-closed in* TIC. *In fact, we can say more:*

(a) For $\mathbb{E} \in \mathcal{A}(U,Y)$ the conditions (i)–(iv) are equivalent (and they are equivalent to (iv'), unless $\mathcal{A} = \mathrm{TI}$):

(i) $\mathbb{E} \in \mathcal{G}\mathcal{A}$;
(ii) $\mathbb{E} \in \mathcal{G}\mathrm{TI}$;
(iii) $\mathbb{E} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$;

(iv) $\widehat{\mathbb{E}} \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$;

(iv') $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(U,Y))$, *i.e.,* $\widehat{\mathbb{E}}^{-1}$ *exists and is bounded on* $i\mathbf{R}$.

If $\dim U = \dim Y < \infty$, *then also the left-invertibility conditions in Lemma 4.1.9 are equivalent to (i) as well as to (v) (and to (v'), unless* $\mathcal{A} = \mathrm{TI}$*):*

(v) $\operatorname{ess\,inf}_{i\mathbf{R}} |\det(\widehat{\mathbb{E}})| > 0$.

(v') $\inf_{i\mathbf{R}} |\det(\widehat{\mathbb{E}})| > 0$;

(b) Let $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$. *For* $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$ *the conditions (i)–(iv) are equivalent, where*

(i) $\mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}$;

(ii) $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$;

(iii) $\pi_+\mathbb{D}\pi_+ \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$;

(iv) $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty$, *i.e.,* $\widehat{\mathbb{D}}^{-1}$ *exists and is bounded on* $\mathbf{C}^+$.

See Lemma 2.2.3 for further equivalent conditions. If $\dim U = \dim Y < \infty$, *then also the left-invertibility conditions in Theorem 4.1.6(a) are equivalent to (i) as well as to (v):*

(v) $\inf_{\mathbf{C}^+} |\det(\widehat{\mathbb{D}})| > 0$.

(c) If $\mathbb{E} \in \mathcal{G}\mathrm{MTI}$, *then the discrete part of* $\mathbb{E}^{-1}$ *is the inverse of the discrete part of* $\mathbb{E}$. *Moreover, if* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S} \subset \mathbf{R}$ *and* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *then* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}^{-1}) \subset \mathbf{S}$.

In particular, classes $\mathrm{MTI}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ *(resp.* $\mathrm{MTIC}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$*) are inverse closed in* TI *(resp.* TIC*).*

(d) Let $\mathcal{D}(U,Y)$ *be either* $\ell^1 := \ell^1(\mathbf{Z};\mathcal{B}(U,Y))$ *or* $\ell^1_{\mathcal{BC}} := \{a \in \ell^1 \,|\, a_j \in \mathcal{BC}(U,Y)$ *for all* $j \neq 0\}$, *with norm* $\|(a_j)_{j\in\mathbf{Z}}\|_{\ell^1} := \sum_{j\in\mathbf{Z}} \|a_j\|_{\mathcal{B}(U,Y)}$ *and convolution as the group operation. Set* $\widehat{\mathcal{D}}(U,Y) := \{\widehat{a} := \sum_{j\in\mathbf{Z}} a_j z^j \,\big|\, (a_j) \in \mathcal{D}\}$ *and* $\widehat{\mathcal{D}^+}(U,Y) := \{\widehat{a} \in \widehat{\mathcal{D}} \,\big|\, a_j = 0$ *for* $j < 0\}$ *(these are the* Z-transforms *of* $\mathcal{D}$ *and* $\mathcal{D}^+ := \mathcal{D} \cap \mathrm{tic}$, *cf. Theorem 5.1.3).*

Let $a \in \mathcal{D}(U,Y)$ *and* $\mathbb{E} := \sum_{j\in\mathbf{Z}} a_j \tau^j$. *Then the following are equivalent:*

(i) $a \in \mathcal{G}\mathcal{D}(U,Y)$;

(iii) $(a*) \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{Z};\mathrm{L}^2([0,1);U)))$;

(iv) $\widehat{a} \in \mathcal{G}\mathcal{C}(\partial\mathbf{D};\mathcal{B}(U,Y))$;

(v) $\mathbb{E} \in \mathcal{G}\mathrm{MTI}_{\mathrm{d},\mathbf{Z}}$;

(v') $\mathbb{E} \in \mathcal{G}\mathrm{MTI}$.

If, in addition, $a \in \mathcal{D}^+(U,Y)$, *then the following are equivalent:*

(i) $a \in \mathcal{G}\mathcal{D}^+(U,Y)$;

(iii) $\pi_{\mathbf{N}}(a*)\pi_{\mathbf{N}} \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{N};\mathrm{L}^2([0,1);U)))$;

(iv) $\widehat{a} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U,Y))$;

(iv') $\widehat{a} \in \mathcal{GC}(\overline{\mathbf{D}};\mathcal{B}(U,Y))$;
(v) $\mathbb{E} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},\mathbf{Z}}$;
(v') $\mathbb{E} \in \mathcal{G}\mathrm{MTIC}$.

(e) **(Banach spaces)** *Assume, in addition, that $\mathcal{A} \neq \mathrm{TI}$, $\mathcal{A} \neq \mathrm{CTI}$ and $\mathcal{A} \neq \mathrm{CTI}^{\mathcal{BC}}$. If U and Y are arbitrary Banach spaces, then (a)–(d) still hold (with TI and its subspaces defined exactly as in the Hilbert space case).*

(f) **(Banach algebras)** *Assume, in addition, that $\mathcal{A} \neq \mathrm{TI}$, $\mathcal{A} \neq \mathrm{CTI}$ and $\mathcal{A} \neq \mathrm{CTI}^{\mathcal{BC}}$. If we replace everywhere $\mathcal{B}(U,Y)$ by an arbitrary Banach algebra A, then parts (a)–(d) hold to the following extent:*

(a) (i)⇔(iv');
(b) (i)⇔(iv);
(c) Completely as stated;
(d) (i)⇔(iv)⇔(v)⇔(v') for $\mathcal{D}$;
(i)⇔(iv)⇔(iv')⇔(v)⇔(v') for $\mathcal{D}^+$.

The above replacement means, e.g., that

$$\mathrm{MTI} := \{\mathbb{E} = \sum_j a_j \delta_{t_j} * + f * \,\big|\, \|\mathbb{E}\|_{\mathrm{MTI}} := \sum_j \|a_j\|_A + \|f\|_{\mathrm{L}^1(\mathbf{R};A)} < \infty\}, \tag{4.3}$$

$$\mathrm{MTIC} := \{\mathbb{E} = \sum_j a_j \delta_{t_j} * + f * \in \mathrm{MTI} \,\big|\, f \in \mathrm{L}^1(\mathbf{R}_+;A) \;\&\; t_j \geq 0 \text{ for all } j\}, \tag{4.4}$$

and their subclasses are defined analogously, $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{MTIC}$ in (b), etc.

(g1) **(Exponential stability)** *Let $\omega \in \mathbf{R}$. By Remark 2.1.6 (see also Lemma D.1.12(d)), the (stability-shifted) class $\mathcal{A}_\omega := e^{\omega\cdot}\mathcal{A}e^{-\omega\cdot}$ (resp. $\widetilde{\mathcal{A}}_\omega := e^{\omega\cdot}\widetilde{\mathcal{A}}e^{-\omega\cdot}$) is inverse-closed in TI_ω (resp. TIC_ω), and claims analogous to (a)–(f), (i) and (j) apply (with $i\mathbf{R}$ replaced by $\omega + i\mathbf{R}$ etc.).*

(g2) We can replace $\widetilde{\mathcal{A}}$ in (b) by $\widetilde{\mathcal{A}}_{\exp} := \cup_{\omega<0}\widetilde{\mathcal{A}}_\omega$. Thus, $\mathbb{D} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}_{\exp}$ for $\mathbb{D} \in \widetilde{\mathcal{A}}_{\exp}$.

(h) Everywhere in (a)–(d), we have $\widehat{\widehat{\mathbb{E}}^*} = \widehat{\mathbb{E}}^*$, $(a*)^* = (\text{Я}a^*)*$, $\widehat{\text{Я}a^*} = \widehat{a}^*$, $(\pi_+\mathbb{E}\pi_+)^* = \pi_+\mathbb{E}^*\pi_+$ and $(\pi_{\mathbf{N}}(a*)\pi_{\mathbf{N}})^* = \pi_{\mathbf{N}}(\text{Я}a^*)\pi_{\mathbf{N}}$, and the adjoint in TI is the same as the adjoint in $\mathcal{B}(\mathrm{L}^2)$).

Consequently, if $\mathbb{E} = \mathbb{E}^*$, then $\mathbb{E}$ is invertible iff any of its forms is left-invertible (or right-invertible).

(i) **($\dim U < \infty$)** *Assume that $\dim U = \dim Y < \infty$ or that the operator under study is self-adjoint.*

Then, in (a)–(e), we may replace $\mathcal{G}$ by $\mathcal{G}_{\mathrm{left}}$ or by $\mathcal{G}_{\mathrm{right}}$ where $\mathcal{G}_{\mathrm{left}}\mathcal{X} := \{x \in \mathcal{X} \,\big|\, yx = I \text{ for some } y \in \mathcal{X}\}$, $\mathcal{G}_{\mathrm{right}}\mathcal{X} := \{x \in \mathcal{X} \,\big|\, xy = I \text{ for some } y \in \mathcal{X}\}$, except that in (b)(iii) and latter (d)(iii) only $\mathcal{G}_{\mathrm{right}}$ can be allowed.

In particular, Theorem 4.1.6 and Lemma 4.1.9 provide additional equivalent conditions.

(j) $(\mathcal{B}+(\mathrm{H}^p_{[\mathrm{strong}]}\cap\mathrm{H}^\infty))$ *The classes* $\mathcal{B}+(\mathrm{H}^p\cap\mathrm{H}^\infty)(\mathbf{C}^+;\mathcal{B})$ *and* $\mathcal{B}+(\mathrm{H}^p_{\mathrm{strong}}\cap\mathrm{H}^\infty)(\mathbf{C}^+;\mathcal{B})$ *are inverse-closed in* H^∞.

Analogously, one can prove the inverse-closedness of several other similar classes. Note from (g2) that $\mathrm{TIC}_{\mathrm{exp}}$, $\mathrm{MTIC}_{\mathrm{exp}}$ etc. are inverse-closed in TIC.

For maps $\mathbb{E}$ form $\mathrm{MTI}^{\mathrm{L}^1}$ or CTI (resp. $\mathrm{MTIC}^{\mathrm{L}^1}$ or CTIC), a pointwise inverse of $\widehat{\mathbb{E}}$ on $i\mathbf{R}\cup\{\infty\}$ (resp. on $\overline{\mathbf{C}^+}\cup\{\infty\}$) is necessarily bounded (because it is continuous on a compact set); hence in that case pointwise invertibility is one more equivalent condition in (a) (resp. in (b)).

Parts (e) and (f) are not needed in this monograph, they are stated only for future reference.

Proof: We start by some preparations (1°–3°):

1° *Any of the conditions in (a)–(d) implies that* $\dim U=\dim Y$*:* For (a)(ii), this follows from Lemma 2.2.1(c4). By Lemma 2.1.5 and Theorem 3.1.3(a1), conditions (a)(ii)–(a)(iv) are equivalent. The other conditions in (a)–(c) obviously imply some of (a)(ii)–(a)(iv). For (d), the situation is analogous.

2° *In (a)–(d), we can w.l.o.g. assume that* $U=Y$ *(*$=\mathbf{C}^n$ *if* $\dim U<\infty$*):* The set $\mathcal{GB}(Y,U)$ is nonempty, by 1°. Each of the conditions in (a)–(d), is invariant under the (left) multiplication by an operator in $\mathcal{GB}(Y,U)$. For the same reason, we can take $U=\mathbf{C}^n$ if $\dim U<\infty$ (by Lemma A.3.4(Q1)).

3° We shall also use the fact that if $\dim U<\infty$, then the left invertibility of $\mathbb{E}$ or $\mathbb{D}$ is equivalent to its invertibility, by Lemma 2.2.1(b).

Case TI: For $\mathcal{A}=\mathrm{TI}$, claims (i) and (ii) coincide. In 1° we observed that (ii)–(iv) are equivalent.

"(i)⇔(v)": If (i) holds and $\dim U<\infty$, then $\operatorname{ess\,inf}|\det(\widehat{\mathbb{E}})|=\operatorname{ess\,sup}|1/\det(\widehat{\mathbb{E}}^{-1})|<\infty$; the converse follows from the determinant formula of an inverse matrix.

Other cases: If $\widehat{\mathbb{E}}\in\mathcal{C}_{\mathrm{b}}$ and $\widehat{\mathbb{E}}$ exists, then it is continuous, by Lemma A.3.4(A2), so the equivalence ("i.e.") stated in (iv') holds.

For $\mathbb{E}\in\mathrm{MTI}\cup\mathrm{CTI}$, we have $\widehat{\mathbb{E}}\in\mathcal{C}_{\mathrm{bu}}$, by Theorem 2.6.4(e1), hence (i)⇒(iv') and (v)⇔(v'). Therefore, for $\mathcal{A}\neq\mathrm{TI}$, it is enough to assume all the other (equivalent) conditions (ii)–(iv') and prove (i). That shall be done below.

Case CTI: For $\mathcal{A}=\mathrm{CTI}$ we have (iv')⇒(i), by Lemma A.3.4(A2).

Case $\mathrm{MTI}_{\mathbf{d}}$*:* For $\mathcal{A}=\mathrm{MTI}_{\mathrm{d}}$, (iv) implies (i) by [Gri, Theorem 4] (to be exact, the theorem says that (iv) implies that $\mathbb{E}^{-1}$ is a measure. One easily verifies that the discrete part of $\mathbb{E}^{-1}$ is also an inverse of $\mathbb{E}$, hence equal to $\mathbb{E}^{-1}$).

Case MTI: Let $\mathbb{E}\in\mathrm{MTI}\cap\mathcal{G}\mathrm{TI}$, so that $\widehat{\mathbb{E}}\in\mathcal{GC}_{\mathrm{bu}}$. Let $\widehat{\mathbb{E}}_{\mathrm{d}}$ be the discrete part of $\widehat{\mathbb{E}}$ and set $\widehat{f}:=\widehat{\mathbb{E}}-\widehat{\mathbb{E}}_{\mathrm{d}}\in\mathcal{L}\mathrm{L}^1$.

Because $\widehat{f}$ vanishes at infinity, by Lemma D.1.11(b), $\|\widehat{\mathbb{E}}^{-1}\widehat{\mathbb{E}}_{\mathrm{d}}-I\|=\|\widehat{\mathbb{E}}^{-1}\widehat{f}\|<1/2$ outside $i[-T,T]$ for some $T>0$, in particular, $\widehat{\mathbb{E}}_{\mathrm{d}}$ is boundedly invertible outside $i[-T,T]$. We fix such a $T>0$ and let $M:=\sup_{|t|>T}\|\widehat{\mathbb{E}}(it)\|$

By the almost-periodicity of $\widehat{\mathbb{E}}_{\mathrm{d}}$ [Lemma C.1.2(h2)], there is $R>T$ s.t. $\|\widehat{\mathbb{E}}_{\mathrm{d}}(it)-\widehat{\mathbb{E}}_{\mathrm{d}}(i(t-R))\|<1/2M$ for all $t\in\mathbf{R}$, hence for $t\in[-T,T]$ the operator $\widehat{\mathbb{E}}_{\mathrm{d}}^{-1}(i(t-R))\widehat{\mathbb{E}}_{\mathrm{d}}(it)\in\mathcal{B}$ has an inverse of norm≤ 2, so the operator $\widehat{\mathbb{E}}_{\mathrm{d}}(it)$ is boundedly invertible; $\mathbb{E}_{\mathrm{d}}^{-1}\in\mathrm{MTI}_{\mathrm{d}}$ by case $\mathrm{MTI}_{\mathrm{d}}$ above.

Now $\widehat{\mathbb{E}}_{\rm d}^{-1}\widehat{\mathbb{E}} = I + \widehat{\mathbb{E}}_{\rm d}^{-1}\widehat{f}$ is boundedly invertible, hence $\mathbb{E}^{-1}$ is a bounded Borel measure, by [Gri, Theorem 5]. Because L^1 is an ideal of bounded Borel measures (see, e.g., [Gri, p. 159]), we have $\widehat{g} = \widehat{\mathbb{E}}_{\rm d}^{-1}\widehat{f}$ for some $g \in \mathrm{L}^1$.

By [Gri, Theorem 5], $\mu := (I + g*)^{-1}$ is a bounded Borel measure, so $I = \mu + g * \mu$ implies that $\mu - I = -g * \mu \in \mathrm{L}^1$, hence $\mu \in \mathrm{MTI}^{\mathrm{L}^1}$.

Case WTI: By case MTI, the inverse of $E + g* \in \mathrm{MTI}^{\mathrm{L}^1} \cap \mathcal{G}\mathrm{TI}$ is $E^{-1} + f*$ for some $f \in \mathrm{L}^1$. (Alternatively, use Lemma 4.1.2(a1)&(b).)

Case $\mathcal{B} + (\mathrm{L}^1 \cap \mathrm{L}^p)*$: Because $\mathrm{L}^1 * (\mathrm{L}^1 \cap \mathrm{L}^p) \subset \mathrm{L}^1 \cap \mathrm{L}^p$, by Lemma D.1.7, $\mathcal{A}$ is a subclass of $\mathrm{MTI}^{\mathrm{L}^1}$. By Lemma 4.1.2(a1)&(b), class $\mathcal{A}$ is inverse-closed in $\mathrm{MTI}^{\mathrm{L}^1}$, hence in TI.

$\mathcal{BC}$ cases: Use again Lemma 4.1.2(a1)&(b) (with $\mathcal{X} \mapsto \{\mathbb{E} - \Pi_{\{0\}}\mathbb{E} \,|\, \mathbb{E} \in \mathrm{MTI}^{\mathcal{BC}}\}$, $\mathcal{A} \mapsto \{\mathbb{E} - \Pi_{\{0\}}\mathbb{E} \,|\, \mathbb{E} \in \mathrm{MTI}\}$ in case $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}$, and analogously for $\mathrm{MTI}_{\rm d}^{\mathcal{BC}}$ and $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$).

(b) *Case TI:* Conditions (i) and (ii) again coincide, "(ii)⇔(iii)" is contained in Lemma 2.2.3 and "(ii)⇔(iv)" follows from Theorem 6.2.1. Equivalence "(iv)⇔(v)" follows again from the determinant formula of an inverse matrix.

Other cases: As the rest is proved above, it is enough to assume (ii)–(iv) and prove (i), because (i)⇒(ii) follows from $\widetilde{\mathcal{A}} \subset \mathrm{TIC}$. But if $\mathbb{D} \in \mathcal{A} \cap \mathcal{G}\mathrm{TIC}$, then $\mathbb{D}^{-1} \in \mathcal{A}$, by (a), hence $\mathbb{D}^{-1} \in \mathcal{A} \cap \mathrm{TIC}$.

(c) The first claim was proved in case MTI of the proof of (a). For the second claim, let $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. Define the projection $\Pi_{\mathbf{S}} \in \mathcal{B}(\mathrm{MTI}, \mathrm{MTI}_{\rm d})$ by

$$\Pi_{\mathbf{S}}((\sum_k T_k \delta_{t_k} + f)*) := (\sum_{t_k \in \mathbf{S}} T_k \delta_{t_k}) * . \tag{4.5}$$

One easily verifies that $\mathbb{E} \in \Pi_{\mathbf{S}}\mathrm{MTI}$ implies $\mathbb{E}(\Pi_{\mathbf{S}}\mathbb{F}) = \Pi_{\mathbf{S}}(\mathbb{E}\mathbb{F})$ for any $\mathbb{F} \in \mathrm{MTI}$. Thus, if $\mathbb{E} \in \Pi_{\mathbf{S}}\mathrm{MTI} \cap \mathcal{G}\mathrm{MTI}$ and $\mathbb{F} = \mathbb{E}^{-1}$, then $\mathbb{E}\mathbb{F} = \Pi_{\mathbf{S}}(\mathbb{E}\mathbb{F}) = \mathbb{E}\Pi_{\mathbf{S}}\mathbb{F}$, hence $\Pi_{\mathbf{S}}\mathbb{F} = \mathbb{E}^{-1} = \mathbb{F}$, so the last claim of the theorem holds for $\mathbb{E} \in \mathcal{G}\mathrm{MTI}_{\rm d}$; in the general case follows by applying this for the (invertible, by Lemma 5.2.3(a)) discrete part of $\mathbb{E}$.

(d) (Note that $\mathcal{D}(U)$ is a Banach algebra with respect to convolution; see Section 13.1 for details. In the definition of $\mathbb{E}$ we refer to $\tau^t \in \mathrm{TI}(U,Y)$ (not $\mathrm{ti}(U,Y)$), hence $\mathbb{E} \in \mathrm{TI}(U,Y)$.) As above, we assume that $Y = U$.

We prove the claims on $\mathcal{D}$; those for $\mathcal{D}_+$ can be proved analogously (recall that $\mathbb{E} \in \mathrm{MTIC} \Leftrightarrow \mathrm{supp}_{\rm d}(\mathbb{E}) \subset \mathbf{R}_+$).

Clearly $\widehat{\mathcal{D}}(U,Y) \subset \mathcal{C}(\partial\mathbf{D}; \mathcal{B}(U,Y))$. The equivalence (i)⇔(v), follows from Theorem 13.4.5(m), (v)⇔(v') from (c) (with $S = \mathbf{Z}$), and the other equivalences by applying (a) to $\mathbb{E}$ and then using Theorem 13.4.5(m) to convert the conditions on $\mathbb{E}$ to those on a.

(e) (Example 3.3.4 shows an $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C}, \ell^\infty))$ (even "$\widehat{\mathrm{CTIC}}$") function that does not correspond to a TI operator; it does not even map $\mathcal{L}\chi_{[0,1]}$ into $\mathcal{L}\mathrm{L}^2$. Moreover, we do not know whether $\mathcal{C}_{\rm b}$ is inverse-closed in $\mathrm{L}^\infty_{\rm strong}$ when U is not a Hilbert space nor separable (cf. Lemma F.1.3(f1)&(f2)). Therefore we have concentrated on MTI only (MTI maps L^2 into L^2, by Lemma D.1.12(c2)).)

The proof goes as in the Hilbert space case above, but it is most easily obtained as follows:

(e) on (a): The implication (i)⇔(iv') is given in (f). The implication (i)⇒(ii) holds because $\mathcal{A} \subset$ TI, and equivalences (ii)⇔(iii) and (iv')⇔(v')⇔(v) can be proved as above. Implication (ii)⇒(iv') follows from Lemma 3.2.5, and implication (iv)⇔(iv') follows from Theorem F.1.9(s4).

(e) on (b): The implication (i)⇔(iv) is given in (f). The implication (ii)⇒(iv) follows from Theorem 2.3 of [W91a]. The proof of Lemma 2.2.2(a2) again establishes (ii)⇔(iii), and equivalence (iv)⇔(v) follows again from the determinant formula of an inverse matrix.

(f) *(f) on (a):* We have (i)⇒(iv'), by Lemma D.1.12(a1)&(c)&(a3). The proof of the major part of the proof of (a) (starting from "case $\mathrm{MTI_d}$") shows that (iv') implies (i).

(f) on (b): We have (i)⇒(iv), by Lemma D.1.12(a1')&(c')&(a3'). The converse can be deduced from part (a) as in part "Other cases" of the proof of (b).

(f) on (c) and (d): The proof of part (c) applies without changes. The proof (d) also applies, because the last paragraph of Theorem 13.4.5(m) is valid in the Banach algebra case too, with the same proof (*mutatis mutandis*).

(g1) This is obvious, because the stability shift is an isometric isomorphism and commutes with compositions (and shifts the Laplace and Fourier transforms by the formula $\widehat{e^{\omega\cdot}\mathbb{E}e^{-\omega\cdot}}(\cdot) = \widehat{\mathbb{E}}(\omega+\cdot)$).

(g2) Combine (b) with Lemma 2.2.7.

(h) For (a)(iv), this follows from Theorem 3.1.3(a). The TI and $\mathcal{A}$ adjoints mean L^2 adjoints, by definition. Part (c) follows from (a). See Lemmas 3.3.8 and 13.1.8 for $\widehat{\mathbb{E}}^*$ and $\widehat{a}^*$ and p. 782 for $(a*)^*$.

(i) 1° If $\mathbb{E} = \mathbb{E}^*$, then it follows from (h) that $\mathbb{V}\mathbb{E} = I$ implies that $\mathbb{E}\mathbb{V}^* = (\mathbb{E}\mathbb{V})^* = I$; consequently, $\mathbb{V}^* = \mathbb{V}\mathbb{E}\mathbb{V}^* = \mathbb{V}$ is the inverse of $\mathbb{E}$ (by (h) $\mathbb{E} = \mathbb{E}^*$ iff $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}^*$).

2° Let $\dim U = \dim Y < \infty$. We have $\mathcal{G}_{\mathrm{left}}\mathrm{TI}(U) = \mathcal{G}\mathrm{TI}(U)$, by Lemma 2.2.1(b), hence $\mathcal{G}_{\mathrm{left}}\mathrm{TIC}(U) = \mathcal{G}\mathrm{TIC}(U)$. Obviously, (i)–(iv') of (a) imply (ii) (for (iii) this follows from Lemma 2.2.1(a)); the converses follows from the original (a). The noncausal part of (d) is obtained analogously (alternatively, apply Theorem 13.4.5(m)).

The claims on $\mathcal{G}_{\mathrm{right}}$ follow analogously (alternatively, by taking adjoints).

Part (b) is analogous to (a) except for (iii), which follows from Lemma 2.2.3. Parts (c) and (d) can deduced from (a) and (b).

(j) For $p = \infty$ this is trivial, so assume that $p < \infty$. Set $\mathcal{A} := \{\mathbb{D} \in \mathrm{H}^\infty \cap \mathrm{ULR} \,\big|\, D = 0\}$. Then $\mathcal{B} + \mathcal{A} = \mathrm{H}^\infty \cap \mathrm{ULR}$ is inverse-closed in H^∞, by (c); $\mathrm{H}^p_{\mathrm{strong}} \subset \mathcal{A}$, by Proposition 6.3.3(a), and $\mathrm{H}^\infty \cdot (\mathrm{H}^\infty \cap \mathrm{H}^p_{\mathrm{strong}}) \subset (\mathrm{H}^\infty \cap \mathrm{H}^p_{\mathrm{strong}})$, by Lemma F.3.5(c). Consequently, $\mathcal{B} + \mathrm{H}^p_{\mathrm{strong}} \cap \mathrm{H}^\infty$ is inverse-closed in $\mathcal{B} + \mathcal{A}$, hence in H^∞, by (a1)&(a2)&(b) of Lemma 4.1.2. For H^p, the proof is analogous and hence omitted. □

We list here a few basic results on inverse-closedness, some of which were already used:

Lemma 4.1.2 (Inverse-closedness) *Let* $\omega \in \mathbf{R}$.

(a1) Assume that $\mathcal{B}+\mathcal{X} \underset{a}{\subset} \mathcal{B}+\mathcal{A} \underset{a}{\subset} \mathrm{TI}_\omega$ *(recall that this requires closedness under composition), and that* $\mathcal{B}\cap\mathcal{A}=\{0\}=\mathcal{B}\cap\mathcal{X}$.

If $ax \in \mathcal{X}$ *for all* $a \in \mathcal{A}$, $x \in \mathcal{X}$, *then* $\mathcal{B}+\mathcal{X}$ *is inverse-closed in* $\mathcal{B}+\mathcal{A}$.

(a2) Part (a1) also holds with TIC_ω, H^∞_ω, L^∞_ω *or* $\mathrm{L}^\infty_{\mathrm{strong},\omega}$ *in place of* TI_ω *and/or with xa in place of ax.*

(b) If $\mathcal{P}$ *is inverse-closed in* $\mathcal{Q}$ *and* $\mathcal{Q}$ *is inverse-closed in* $\mathcal{R}$, *then* $\mathcal{P}$ *is inverse-closed in* $\mathcal{R}$.

(c) If $\mathcal{Z}$ *is inverse-closed in* TI_ω, *then so is* $\mathcal{Z}^*$. *If* $\mathcal{Z}$ *is inverse-closed in* TIC_ω, *then so is* $\mathcal{Z}^{\mathrm{d}}$.

Proof: (Recall that $\mathcal{P}$ is inverse-closed in $\mathcal{Q}$ iff $\mathcal{P}$ is a subgroup of $\mathcal{Q}$ and $\mathcal{P}\cap\mathcal{G}\mathcal{Q}=\mathcal{G}\mathcal{P}$, i.e., $x\in\mathcal{P}\,\&\,x^{-1}\in\mathcal{Q}\Rightarrow x^{-1}\in\mathcal{P}$ for all $x\in\mathcal{P}$.)

(a1) Let $X+x \in \mathcal{B}+\mathcal{X}$ have the inverse $A+a\in\mathcal{B}+\mathcal{A}$. Then $I=(A+a)(X+x)=AX+aX+Ax+ax$, hence $\mathcal{B}\ni I-AX=aX+Ax+ax\in\mathcal{A}$, hence $AX=I$. Analogously, $XA=I$, hence $a=-(A+a)xX^{-1}\in\mathcal{X}$.

(N.B. usually this formula also provides a norm estimate for the inverse, e.g., $\|(X+x)^{-1}-X^{-1}\|_{\mathrm{H}^2}\le\|(X+x)^{-1}\|_{\mathrm{H}^\infty}\|x\|_{\mathrm{H}^2}\|X^{-1}\|_{\mathcal{B}}$ for any $X\in\mathcal{B}$, $x\in\mathrm{H}^\infty\cap\mathrm{H}^2$.)

(a2) Same proof applies to TI, H^∞_ω etc. too.

(b) This is obvious (if $p\in\mathcal{P}\cap\mathcal{G}\mathcal{R}$, then $p^{-1}\in\mathcal{Q}$, hence then $p^{-1}\in\mathcal{P}$).

(c) If $\mathcal{Z}\cap\mathcal{G}\mathrm{TI}_\omega=\mathcal{G}\mathcal{Z}$ and $y\in\mathcal{Z}^*\cap\mathcal{G}\mathrm{TI}_\omega$, then $y^*\in\mathcal{Z}\cap\mathcal{G}\mathrm{TI}=\mathcal{G}\mathcal{Z}$, hence $y^{-1}=((y^*)^{-1})^*\in\mathcal{Z}^*$. The case for $\mathcal{Z}^{\mathrm{d}}$ is analogous. □

Now we will show the rather obvious fact that most (noncausal) classes mentioned in Theorem 4.1.1 are also adjoint-closed (see Definition 2.1.4 for the definition of the adjoint):

Lemma 4.1.3 (Adjoint-closed classes) *Let* $\mathcal{A}$ *be one of the classes* TI, CTI, $\mathrm{CTI}^{\mathcal{BC}}$, MTI, $\mathrm{MTI}^{\mathcal{BC}}$, $\mathrm{MTI_d}$, $\mathrm{MTI}_{\mathrm{d}}^{\mathcal{BC}}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$ *and* $\mathcal{B}+(\mathrm{L}^1\cap\mathrm{L}^p)*$, *and let U and Y be Hilbert spaces of arbitrary dimensions. Then we have the following:*

(a) The class $\mathcal{A}$ *is adjoint-closed in* TI*: if* $\mathbb{E}\in\mathcal{A}(U,Y)$, *then* $\mathbb{E}^*,\mathbb{E}^{\mathrm{d}}\in\mathcal{A}(Y,U)$.

(b) If $\omega\in\mathbf{R}$ *and* $\mathbb{E}\in\mathcal{A}_\omega(U,Y):=e^{\omega\cdot}\mathcal{A}(U,Y)e^{-\omega\cdot}$, *then* $\mathbb{E}^*\in\mathcal{A}_{-\omega}(Y,U)$ *and* $\mathbb{E}^{\mathrm{d}}\in\mathcal{A}_\omega(Y,U)$.

(c) If $\mathbb{E}\in\mathcal{G}\mathrm{MTI}$, *then the discrete part of* $\mathbb{E}^*$ *is the adjoint of the discrete part of* $\mathbb{E}$. *Furthermore,* $\mathrm{supp_d}(\mathbb{E}^*)=-\mathrm{supp_d}(\mathbb{E})$ *and* $\mathrm{supp_d}(\mathbb{E}^{\mathrm{d}})=\mathrm{supp_d}(\mathbb{E})$, *and classes such as* $\mathrm{MTI}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ *are adjoint closed in* TI *when* $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$.

(d) Set $\widetilde{\mathcal{A}}:=\mathcal{A}\cap\mathrm{TIC}$, $\widetilde{\mathcal{A}}_\omega:=e^{\omega\cdot}\widetilde{\mathcal{A}}e^{-\omega\cdot}$, *where* $\omega\in\mathbf{R}$. *Then* $\mathbb{D}\in\widetilde{\mathcal{A}}_\omega(U,Y)\Leftrightarrow\mathbb{D}^{\mathrm{d}}\in\widetilde{\mathcal{A}}_\omega(Y,U)$.

Set $\widetilde{\mathcal{A}}:=\mathcal{A}_{\mathbf{S}}\cap\mathrm{TIC}$, $\widetilde{\mathcal{A}}_{\omega,S}:=e^{\omega\cdot}\widetilde{\mathcal{A}}_Se^{-\omega\cdot}$, *where* $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$. *Then* $\mathbb{D}\in\widetilde{\mathcal{A}}_{\omega,S}(U,Y)\Leftrightarrow\mathbb{D}^{\mathrm{d}}\in\widetilde{\mathcal{A}}_{\omega,S}(Y,U)$.

One could, of course, extend this result for Banach spaces with ease; although the Banach adjoint of a $\mathrm{TI}(U,Y)$ operator is an element of $\mathrm{TI}(Y^*,U^*)$.

Proof: (a) 1° *We have* $\mathbb{E}^* \in \mathcal{A}(Y,U)$*:* Because $\tau(t)^* = \tau(-t)$, the claim holds for $\mathcal{A} = \mathrm{TI}$. For CTI this is given in Lemma 2.6.2; the case $\mathcal{A} = \mathrm{CTI}^{\mathcal{B}C}$ follows. For MTI and its subclasses the claim follows from the fact that $(\mu *)^* = (\text{Я}\mu^*)*$, by Lemma D.1.12(d).

2° *We have* $\mathbb{E}^{\mathrm{d}} \in \mathcal{A}(Y,U)$*:* By Lemma D.1.12(d), $(\mu *)^{\mathrm{d}} = (\mu^*)*$, hence this holds for MTI and its subclasses; for TI this is obvious.

(b) Because $(\mathrm{e}^{\omega\cdot})^* = \mathrm{e}^{\omega\cdot}$, it follows from (a) that

$$(\mathrm{e}^{\omega\cdot}\mathcal{A}(U,Y)\mathrm{e}^{-\omega\cdot})^* = \mathrm{e}^{-\omega\cdot}\mathcal{A}(Y,U)\mathrm{e}^{\omega\cdot} =: \mathcal{A}_{-\omega}(Y,U). \tag{4.6}$$

(c) This follows from Lemma D.1.12(d).

(d) This follows from (b) and Lemma D.1.12(d). □

The Carleson Corona Theorem states that for $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^{n\times 1})$ to have a left inverse $\widehat{\mathbb{V}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^{1\times n})$, the (clearly necessary) condition $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ (for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$) is also sufficient.

For general $\mathbb{D} \in \mathrm{TIC}(U,Y)$, the left-inverse $(\widehat{\mathbb{D}}^*\widehat{\mathbb{D}})^{-1}\widehat{\mathbb{D}}^* \in \mathcal{C}_{\mathrm{b}}(\mathbf{C}^+;\mathcal{B}(Y,U))$ of $\widehat{\mathbb{D}}$ is not holomorphic, the left-inverse $(\mathbb{D}^*\mathbb{D})^{-1}\mathbb{D}^* \in \mathrm{TI}(Y,U)$ of $\mathbb{D}$ is not causal, and, surprisingly, there need not be any (H^∞, i.e., TIC) inverse in general, as Serge Treil has shown (see Lemma 4.1.10). However, assuming $\dim U < \infty$, such an inverse is guaranteed, as shown in (Corona) Theorem 4.1.6.

We will use the Corona Theorem to find certain left inverses and coprime factorizations. We start by showing that the theorem (part (ii) below) is equivalent to two other problems (see, e.g., Chapters 10 & 11 of [Rud73] for basic results on Banach algebras and maximal ideal spaces).

Lemma 4.1.4 *Assume that $\mathcal{A}$ is a commutative (complex) Banach algebra, $1_{\mathcal{A}}$ be its identity, $\mathfrak{M}$ be its maximal ideal space, and $\mathfrak{M}_0 \subset \mathfrak{M}$. Then the following are equivalent:*

(i) $\mathfrak{M}_0$ is dense in $\mathfrak{M}$.

(ii) **(Corona Theorem)** *Let $f_1,\dots,f_n \in \mathcal{A}$. There are vectors $g_1,\dots,g_n \in \mathcal{A}$ s.t.*

$$f_1g_1 + \cdots + f_ng_n = 1_{\mathcal{A}} \tag{4.7}$$

iff there is $\varepsilon > 0$ s.t. $|\widehat{f}_1(M)| + \cdots + |\widehat{f}_n(M)| \geq \varepsilon$ for all $M \in \mathfrak{M}_0$.

(iii) Let $\mathbb{D} \in \mathcal{A}^{n\times m}$. Then there is $\mathbb{V} \in \mathcal{A}^{m\times n}$ s.t. $\mathbb{V}\mathbb{D} = I$ iff

$$\widehat{\mathbb{D}}(M)^*\widehat{\mathbb{D}}(M) \geq \varepsilon I \ \text{ for all } \ M \in \mathfrak{M}_0. \tag{4.8}$$

In (ii) and (iii), we might write "if" instead of "iff", because the converse holds for any $\mathfrak{M}_0 \subset \mathfrak{M}$ (take $\varepsilon := 1/\sup\|\widehat{\mathbb{V}}\|^2_{\mathcal{B}(\mathbf{C}^n,\mathbf{C}^m)}$).

The Corona Theorem means in general a proof that the *corona* $\mathfrak{M}\setminus\overline{\mathbb{M}_0}$ is empty. By the lemma, this is the case iff (iii) holds, hence we can call results of form (iii) corona theorems.

Proof: By Lemmas 8.1.28 and 8.1.34 of [Vid, pp. 339–340], claim (i) implies (ii) and (iii). Claim (ii) is the case $m = 1$ of (iii), and (ii)⇒(i) is established in, e.g., pp. 201–203 of [Duren]. $\square$

Now we list certain Banach algebras for which the set $\mathfrak{M}_0 := \mathbf{C}^+$ is dense in their maximal ideal spaces (with the standard identification $\mathbf{C}^+ \ni s \mapsto \{f \in \mathcal{A} \,|\, \hat{f}(s) = 0\} \in \mathfrak{M}_0$):

Lemma 4.1.5 (Maximal ideals) *The extended closed half-plane $\overline{\mathbf{C}^+} \cup \{\infty\}$ with the one-point-compactification topology is the maximal ideal space of* $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$ *and* $\mathrm{CTIC}(\mathbf{C})$*, and $\mathbf{C}^+$ is dense also in the maximal ideal spaces of* $\mathrm{TIC}(\mathbf{C})$, $\mathrm{MTIC}(\mathbf{C})$ *and* $\mathrm{MTIC}_{\mathrm{d}}(\mathbf{C})$.

The maximal ideal space of $\mathrm{CTI}(\mathbf{C})$, $\ell^1(\mathbf{N})$ *and* $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ *(for $T > 0$) is the closed unit disc $\overline{\mathbf{D}}$; the latter space through identification $\sum_{k=0}^{\infty} \alpha_k \delta_{Tk} \mapsto \sum_{k=0}^{\infty} \alpha_k z^k$. Similarly, the maximal ideal space of $\ell^1(\mathbf{Z})$ and* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ *is the unit circle $\{z \in \mathbf{C} \,|\, |z| = 1\}$.*

The extended imaginary axis $i\mathbf{R} \cup \{\infty\}$ with the one-point-compactification topology is the maximal ideal space of $\mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C})$*, and $i\mathbf{R} \cup \{\infty\}$ dense also in the maximal ideal spaces of* $\mathrm{MTI}(\mathbf{C})$ *and* $\mathrm{MTI}_{\mathrm{d}}(\mathbf{C})$.

Theorem 4.1.6(a) will show that $\overline{\mathbf{C}^+} \cup \{\infty\}$ (resp. $i\mathbf{R} \cup \{\infty\}$) is dense in the maximal ideal space of $\mathrm{MTIC}_{\mathbf{S}}$ (resp. $\mathrm{MTI}_{\mathbf{S}}$), whenever $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. The same holds for $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ and $\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$ unless $\mathbf{S} = T\mathbf{Z}$ for some $T \in \mathbf{R}$ (case $T = 0$ is trivial, case $T \neq 0$ is given in the above lemma).

Proof: The maximal ideal spaces of $\mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C})$ and $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$ are given on pages 107 and 112 of [GRS], respectively, and those of $\mathrm{CTIC}(\mathbf{C})$ and $\mathrm{CTI}(\mathbf{C})$ in [Rud73, Example 11.13(c)&(a)] (to be exact, [Rud73] shows that the closed unit disc is the maximal ideal space of the disc algebra (via the Z transform); $\mathrm{CTIC}(\mathbf{C})$ is isomorphic to the disc algebra through the Cayley transform).

For TIC (i.e., H^∞) this is the Corona theorem [Duren, Chapter 12].

Case MTIC is shown on p. 145–150 (cf. [Vid, p 342]).

Cases $\ell^1(\mathbf{N})$ and $\ell^1(\mathbf{Z})$ are given on [GRS, p. 118] and in [Rud73, Example 11.13(b)], respectively. The cases $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ and $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$ follow from the isomorphism $\sum_k \alpha_k \delta_{Tk} \mapsto \{\alpha_k\}$.

For $\mathrm{MTIC}_{\mathrm{d}}$ this is follows from Theorem 5.2 of [BKRS] as follows: Clearly $\mathfrak{M}_0 := \mathbf{C}^+$ is a subset of $\mathfrak{M} := \mathfrak{M}(\mathrm{MTIC}_{\mathrm{d}}(\mathbf{C}))$. Let now $\mathbb{D} \in \mathrm{MTIC}_{\mathrm{d}}(\mathbf{C}^{n \times m})$ and $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq \varepsilon I$ (this condition is necessary, because a left $\mathrm{MTIC}_{\mathrm{d}}$ inverse is a left TIC inverse). Then there is $\mathbb{V} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ s.t. $\mathbb{V}\mathbb{D} = I$, hence $\|\pi_- \mathbb{D} u\|_2 \geq \varepsilon_1 \|\pi_- u\|_2$, for all $u \in \mathrm{L}^2$, where $\varepsilon_1 := \|\pi_- \mathbb{V} \pi_-\|$, i.e., $\pi_- \mathbb{D}^* \pi_- \mathbb{D} \pi_- \geq \varepsilon_1^2 \pi_-$. By the $*$-isomorphism introduced in Theorem 8 of [Karlovich93], this is equivalent to the condition (iii) of Theorem 5.2 of [BKRS] (for $A := \varepsilon_1^{-1} \mathbb{D}^*$), hence Theorem 5.2(viii) provides $\mathbb{V} \in \mathrm{MTI}_{\mathrm{d}}(\mathbf{C}^{n \times m})$ s.t. $A\mathbb{V}^* = I$, i.e., $\varepsilon_1^{-1} \mathbb{V}\mathbb{D} = I$, so we obtain the density of $\mathbf{C}^+$ from Lemma 4.1.4.

Cases $\mathrm{MTI}(\mathbf{C})$ and $\mathrm{MTI}_{\mathrm{d}}(\mathbf{C})$ follow from Lemma 4.1.4, because $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq \varepsilon I$ on $i\mathbf{R}$ implies that $\mathbb{D}^* \mathbb{D} \geq \varepsilon I$, hence the left inverse $\mathbb{V} := (\mathbb{D}^* \mathbb{D})^{-1} \mathbb{D}^*$ belongs to $\mathcal{A}$, by Theorem 4.1.1. $\square$

We are now ready to state several equivalent conditions for left-invertibility:

Theorem 4.1.6 (Corona Theorem) *Let $\mathcal{A}$ be one of the classes* TIC*,* CTIC*,* MTIC*,* $\mathrm{MTIC_d}$*,* $\mathrm{MTIC}^{\mathrm{L}^1}$*,* $\mathrm{MTIC}_{\mathbf{S}}$ *and* $\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$*, where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$*. Let* $\dim U < \infty$*. Then (a) and (b) hold:*

(a) For $\mathbb{D} \in \mathcal{A}(U,Y)$ *the following are equivalent:*

> *(i)* $\mathbb{V}\mathbb{D} = I$ *for some* $\mathbb{V} \in \mathcal{A}(Y,U)$*;*
> *(ii)* $\mathbb{V}\mathbb{D} = I$ *for some* $\mathbb{V} \in \mathrm{TIC}(Y,U)$*;*
> *(iii)* $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ *for all* $s \in \mathbf{C}^+$ *and some* $\varepsilon > 0$*;*
> *(iv)* $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*,* $\omega > 0$ *and some* $\varepsilon > 0$*;*
> *(v)* $\mathbb{D}^*\pi_-\mathbb{D} \geq \varepsilon\pi_-$ *on* L^2 *for some* $\varepsilon > 0$*;*
> *(vi)* $\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon\pi_{[0,t)}$ *for all* $t > 0$ *and some* $\varepsilon > 0$*.*

(See Proposition 4.1.7 for additional equivalent conditions.)

(b) Let $\mathbb{N} \in \mathcal{A}(U,Y)$*,* $\mathbb{M} \in \mathcal{A}(U)$*. Then* $\mathbb{N}$ *and* $\mathbb{M}$ *are r.c. over* $\mathcal{A}$ *iff* $\widehat{\mathbb{N}}(s)^*\widehat{\mathbb{N}}(s) + \widehat{\mathbb{M}}(s)^*\widehat{\mathbb{M}}(s) \geq \varepsilon I$ *for all* $s \in \mathbf{C}^+$ *and some* $\varepsilon > 0$*.*

Let $\mathcal{A}$ *be* TIC*,* CTIC*,* $\mathrm{MTIC}^{\mathrm{L}^1}$ *,* $\mathrm{MTIC}_{T\mathbf{Z}}$ *or* $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$*, where* $T \in \mathbf{R}$*. Then also (c) and (d) hold:*

(c) Let $\mathbb{D} \in \mathcal{A}(U,Y)$*. Then the conditions (i)–(vi) are equivalent to*

> *(vii)* $\mathbb{D}$ *can be complemented to* $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix} \in \mathcal{GA}(U \times Y_0, Y)$*, where* Y_0 *is a closed subspace of* Y*.*

If $Y = U \times Z$*, where also* Z *is a Hilbert space, we may require* $Y_0 = Z$ *in (vii).*

(d) Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *have a r.c.f.* $\mathbb{N}\mathbb{M}^{-1}$ *(or a l.c.f.* $\mathbb{M}^{-1}\mathbb{N}$*) with* $\mathbb{N},\mathbb{M} \in \mathcal{A}$*. Then* $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$*.*

(See Definitions 6.4.4 and 6.4.1 for *.c.f. and *.c.) By taking (causal) adjoints, one gets the dual claims, e.g., the equation $\mathbb{D}\mathbb{V} = I$ has a solution $\mathbb{V} \in \mathcal{A}$ iff $\widehat{\mathbb{D}}\widehat{\mathbb{D}}^* \geq \varepsilon I$ on $\mathbf{C}^+$. If $\mathbb{D} = \mathbb{D}^*$ or $\dim U = \dim Y < \infty$, then $\mathbb{D}$ is left-invertible iff $\mathbb{D}$ is invertible. by Theorem 4.1.1(h).

Part (a) of the theorem fails (at least for $\mathcal{A} = \mathrm{TIC}$) when $\dim U = \infty$ and $\dim Y \geq \dim U$, by Lemma 4.1.10.

Condition (iii) can obviously be rephrased as $\|\widehat{\mathbb{D}}u_0\| \geq \varepsilon^{1/2}\|u_0\|$ on $\mathbf{C}^+$ for all $u_0 \in U$. Similarly, $\mathbb{M}^*\mathbb{M} + \mathbb{N}^*\mathbb{N} \geq \varepsilon I$ on $\mathbf{C}^+$ for some $\varepsilon > 0$ iff $\|\mathbb{M}u_0\| + \|\mathbb{N}u_0\| \geq \varepsilon'\|u_0\|$ on $\mathbf{C}^+$ for all $u_0 \in U$ and some $\varepsilon' > 0$.

If $\mathbb{N}$ and $\mathbb{M}$ in (d) are exponentially r.c., i.e., $\mathbb{N}_\alpha := e^{\alpha\cdot}\mathbb{N}e^{-\alpha\cdot}$ and $\mathbb{M}_\alpha$ are r.c. over $\mathcal{A}$ for some $\alpha < 0$ (equivalently, $\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix}^*\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix} \geq \varepsilon I$ on $\mathbf{C}^+_\alpha$), then we can take all maps in the d.c.f. to be exponentially stable (by choosing a d.c.f. for $\mathbb{N}_\alpha\mathbb{M}_\alpha^{-1}$ and shifting it back to TIC_α).

Proof: (Note that $\dim Y \geq \dim U$ is necessary for (ii), hence for all equivalent conditions.)

I The case $\dim Y < \infty$ *of parts (a) and (c):* W.l.o.g. (the statements are invariant under isomorphisms (i.e., $\mathcal{GB}(\mathbf{C}^m, U)$ and $\mathcal{GB}(\mathbf{C}^n, Y)$ mappings) we set $U = \mathbf{C}^m$ and $Y = \mathbf{C}^n$.

(a) 1° "(iii)⇒(i)": This follows from Lemmas 4.1.4 and 4.1.5, except for the case $\mathcal{A} = \mathrm{MTIC}_{T\mathbf{Z}}$, which is deduced from the case $\mathcal{A} = \mathrm{MTIC}$ as follows:

Let $\mathbb{D} = (\mu + f)* \in \mathrm{MTIC}_{\mathbf{S}}$. The equivalence $r + s \in \mathbf{S} \Leftrightarrow r \in \mathbf{S}$, valid for $s \in \mathbf{S}$, implies that for $\nu \in \mathrm{MTI}$ we have

$$\Pi_{\mathbf{S}}\nu = 0 \Longrightarrow \Pi_{\mathbf{S}}(\nu' * \mu) = 0. \tag{4.9}$$

If $(\nu + g)* \in \mathrm{MTIC}$ is a left-inverse of $\mu + f$, and we set $\nu' := \Pi_{\mathbf{S}}\nu$, $\nu'' := \nu - \nu'$, then

$$I\delta_0 = (\nu' * \mu + \nu'' * \mu) + (\nu' * f + \nu'' * f + g * \mu), \tag{4.10}$$

hence $\nu' * \mu = I\delta_0 \in \Pi_{\mathbf{S}}\mathrm{MTIC}$. Thus, $(\nu' + h) * (\mu + f) = I\delta_0$, if we set $h := -(\nu' * f) * (\nu + g) \in \mathrm{L}^1$.

2° The other implications: "(i)⇒(ii)" is trivial, because $\mathcal{A} \subset \mathrm{TIC}$, "(ii)⇒(v)" follows from equation $\|\pi_-\mathbb{V}\pi_-\|\|\pi_-\mathbb{D}u\|_2 \geq \|\pi_-u\|_2$ (i.e., we can take $\varepsilon := \|\pi_-\mathbb{V}\pi_-\|^{-1/2}$), "(v)⇒(vi)⇒(iv)⇒(iii)" is given in Proposition 4.1.7(C)&(A).

(c) The implication (vii)⇒(i) is trivial, so we study the converse.

The case $\mathcal{A} = \mathrm{TIC}$ is contained in the Tolokonnikov's lemma [Nikolsky, App. 3.10, p. 293].

By [Vid, Theorem 8.1.68], a complex Banach algebra with a contractible maximal ideal space is Hermite ($\overline{\mathbf{C}^+} \cup \{\infty\}$ is homeomorphic to $\overline{\mathbf{D}}$, hence contractible); see [Vid, p. 348] or [Lin] for details. Thus, $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$, $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ and $\mathrm{CTIC}(\mathbf{C})$ are Hermite rings, by Lemma 4.1.5.

By [Vid, Theorem 8.1.59], the ring $\mathcal{A}(\mathbf{C})$ is Hermite iff every left-invertible $\mathbb{D} \in \mathcal{A}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented, so cases $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1}$, $\mathcal{A} = \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$ and $\mathcal{A} = \mathrm{CTIC}$ follow from this (hence also $\mathrm{TIC}(\mathbf{C})$ (and $\mathrm{H}^\infty(\mathbf{C})$) is Hermite).

Finally, let $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2 \in \mathrm{MTIC}_{T\mathbf{Z}}$, where $\mathbb{D}_1 = \Pi_{\mathbf{R}}\mathbb{D}$ is the discrete part of $\mathbb{D}$, be left-invertible over $\mathrm{MTIC}_{T\mathbf{Z}}$. Then so is $\mathbb{D}_1 \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, hence $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1\end{bmatrix}^{-1} = \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C}^n)$ for some $\mathbb{E}_1, \mathbb{F}_1, \mathbb{G}_1 \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, by (c). Therefore,

$$\mathbb{H} := \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}\mathbb{D} = \begin{bmatrix}I + \mathbb{F}_1\mathbb{D}_2 \\ \mathbb{G}_1\mathbb{D}_2\end{bmatrix} \in \mathrm{MTIC}^{\mathrm{L}^1}, \tag{4.11}$$

is left-invertible over $\mathrm{MTIC}_{T\mathbf{Z}}$, hence over TIC, hence over $\mathrm{MTIC}^{\mathrm{L}^1}$, by (a), so $\mathbb{H}$ can be complemented $\begin{bmatrix}\mathbb{H} & \mathbb{K}\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$. Thus,

$$\begin{bmatrix}\mathbb{D} & \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}^{-1}\mathbb{K}\end{bmatrix} = \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}^{-1}\begin{bmatrix}\mathbb{H} & \mathbb{K}\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}_{T\mathbf{Z}}. \tag{4.12}$$

If $Y = U \times Z$, then $\dim U \times Y_0 = \dim U \times Z$, hence $\dim Y_0 = \dim Z$ (because $\dim U < \infty$), equivalently, there is a map (isomorphism) $T \in \mathcal{GB}(Z, Y_0)$. Just replace $\mathbb{E}$ by $\mathbb{E}T$ (i.e., $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix}$ by $\begin{bmatrix}I & 0 \\ 0 & T\end{bmatrix} \in \mathcal{GB}$) to the right to replace Y_0 by Z.

II The general case of parts (a) and (c) for $\mathcal{A} = \mathrm{TIC}$*:*

The implication "(iii)⇒(vii)" (with norm estimates for $\mathbb{E}$ and $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix}^{-1}$) is Tolokonnikov's Lemma [Nikolsky, Lemma A.3.10, p. 293] (which uses the solution of "(iii)⇒(ii)", i.e., the Fuhrmann–Vasyunin Theorem [Nikolsky, Theorem A.3.11, p. 293]). The above proofs of the other implications apply to the general case too (note that (vii) implies (i)).

In [Nikolsky], Y was assumed to be separable, but that is no loss of generality:

Because $\widehat{\mathbb{D}} : \mathbf{C}^+ \times U \to Y$ is continuous and $\mathbf{C}^+ \times U$ is separable, so is $\widehat{\mathbb{D}}[\mathbf{C}^+][U]$. Let Y_1 be the (separable) closed span of $\widehat{\mathbb{D}}[\mathbf{C}^+][U]$, so that we can write $\mathbb{D} =: \begin{bmatrix}\widetilde{\mathbb{D}} \\ 0\end{bmatrix} \in \mathrm{TIC}(U, Y_1 \times Y_1^\perp)$.

Take $Y_0 \subset Y_1$ and $\widetilde{\mathbb{E}}$ s.t. $\begin{bmatrix}\widetilde{\mathbb{D}} & \widetilde{\mathbb{E}}\end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times Y_0, Y_1)$. Then $\begin{bmatrix}\widetilde{\mathbb{D}} & \widetilde{\mathbb{E}} & 0 \\ 0 & 0 & I\end{bmatrix} \in \mathrm{TIC}(U \times Y_0 \times Y_1^\perp, Y)$ is an invertible (by Lemma A.1.1(b1)) complementation of $\mathbb{D}$.

III The general case of parts (a) and (c) for $\mathcal{A} \neq$ TIC:

(a) "(iii)⇒(i)": Let $y_1, \ldots, y_m$ span DU. Let $y_{m+1}, y_{m+2}, \ldots$ be chosen as the sequence "$y_1, y_2, \ldots$" from Lemma 2.6.5 (applied to $\mathbb{D} - D$; use also Theorem 2.6.4(j)). Then $P_n'\mathbb{D} \to \mathbb{D}$ in $\mathcal{A}(U,Y)$ (hence in $\mathrm{TIC}(U,Y)$ too, hence $P_n'\widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ in $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$), as $n \to \infty$; here P_n' is the orthogonal projection of Y onto $Y_n := \mathrm{span}\{y_1, \ldots, y_m\}$. The rest depends on the statement:

Take n large enough to have $(P_n'\widehat{\mathbb{D}})^*(P_n'\widehat{\mathbb{D}}) \geq \varepsilon/2I$ on $\mathbf{C}^+$. Let $\mathbb{V}_n \in \mathcal{A}(Y_n, U)$ s.t. $\mathbb{V}_n(P_n'\widehat{\mathbb{D}}) = I$. Choose $\mathbb{V} := \mathbb{V}_n P_n' \in \mathcal{A}(Y,U)$ to get (i). The other implications are obtained as in 2° of I(a) above.

(c) "(iii)⇒(vii)": Take P_n' as above and write $\mathbb{D}$ as $\begin{bmatrix}\mathbb{D}_1 \\ \mathbb{D}_2\end{bmatrix} \in \mathcal{A}(U, Y_n \times Y_n^\perp)$. Choose then $\mathbb{E}_1 \in \mathcal{A}(Y_n, U)$ s.t. $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1\end{bmatrix} \in \mathcal{G}\mathcal{A}(Y_n, U \times Y_0)$, where Y_0 is a subspace of Y_n. Now $\mathbb{E} = \begin{bmatrix}\mathbb{E}_1 & 0 \\ 0 & I\end{bmatrix} \in \mathcal{A}(Y_0 \times Y_n^\perp)$ complements $\mathbb{D}$ to $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1 & 0 \\ \mathbb{D}_2 & 0 & I\end{bmatrix}$, which is invertible, by Lemma A.1.1(b1).

(b) This follows from (a) by setting $\mathbb{D} := \begin{bmatrix}\mathbb{N} \\ \mathbb{M}\end{bmatrix}$.

(d) Use (c) to complement $\begin{bmatrix}\mathbb{M} \\ \mathbb{N}\end{bmatrix}$ to an invertible matrix

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X}\end{bmatrix} =: \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}^{-1} \in \mathcal{G}\mathcal{A}(\mathbf{C}^m \times \mathbf{C}^n). \tag{4.13}$$

This gives a d.c.f. of $\mathbb{D}$ over $\mathcal{A}$. □

The above Corona Theorem says that, when $\dim U < \infty$, the condition "$\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I_U$ on $\mathbf{C}^+$" is equivalent to the left-invertibility of any $\mathbb{D} \in \mathrm{TIC}(U, *)$. In the case of an infinite-dimensional U, condition "$\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$ on $\mathbf{C}^+$" is equivalent only to the following kind of *pseudo–left-invertibility* (recall Lemma 4.1.10):

Proposition 4.1.7 (∞-dimensional partial Corona theorem) *Let* $\varepsilon > 0$.

(A) Assume that $\mathbb{D} \in \mathrm{TIC}(U,Y)$. *Then conditions (i)–(v) below are equivalent.*

Furthermore, any of (i')–(v) is invariant under the replacement $\omega > 0 \mapsto \omega \geq 0$. *Finally, if (i) holds, then (a1)–(d) hold for any* $\alpha \geq \omega \geq 0$ *and* $T, \beta \in \mathbf{R}$.

(B) Assume that $\mathbb{D} \in \mathrm{TIC}_{\omega'}(U,Y)$ *for all* $\omega' > 0$. *Then (i)–(v) below are equivalent.*

Furthermore, (iv') and (iv") are invariant under the replacement $\omega > 0 \mapsto \omega \geq 0$. *Finally, if (i) holds, then (a1)–(c2) hold for any* $\alpha \geq \omega > 0$ *and* $T, \beta \in \mathbf{R}$.

(C) Assume that $\mathbb{D} \in \mathrm{TIC}_{\omega'}(U,Y)$ *for all* $\omega' > 0$. *Then (vi) of Theorem 4.1.6 implies (i)–(v). If* $\mathbb{D} \in \mathrm{TIC}$, *then conditions (v) and (vi) of Theorem 4.1.6 are equivalent (and imply (i)–(v) below).*

(i) There is $\widehat{\mathbb{V}} : \mathbf{C}^+ \to \mathcal{B}(Y,U)$ *s.t.* $\widehat{\mathbb{V}}\widehat{\mathbb{D}} \equiv I$ *and* $\|\widehat{\mathbb{V}}\| \leq \varepsilon^{-1}$.

(i') For each $\omega > 0$ *there is* $\mathbb{V} \in \mathrm{TI}_\omega(Y,U)$ *s.t.* $\|\mathbb{V}\|_{\mathrm{TI}_\omega} \leq \varepsilon^{-1}$ *and* $\mathbb{V}\mathbb{D} = I_{\mathrm{TI}_\omega}$.

(iii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon^2 I$ *for all* $s \in \mathbf{C}^+$.

(iv) $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*) for all* $\omega > 0$.

(iv') $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$*) for all* $\omega > 0$.

(iv") $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathcal{C}^\infty_c(\mathbf{R}_+;U)$*) for all* $\omega > 0$.

(v) $(\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega} \geq \varepsilon^2 I$ *for all* $\omega > 0$.

(a) Condition (i) holds with some $\widehat{\mathbb{V}} \in \mathcal{C}_b(\mathbf{C}^+;\mathcal{B}(Y,U))$ *s.t.* $\sup_{s\in\mathbf{C}^+}\|\widehat{\mathbb{V}}(s)\| \leq \varepsilon^{-1}$.

(b1) $\mathbb{D}u \in \mathrm{L}^2_\omega \Leftrightarrow u \in \mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b2) $\mathbb{D}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega \Leftrightarrow u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b3) There is $M < \infty$ *s.t.* $\varepsilon\|u\|_{\mathrm{L}^2_\omega} \leq \|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \leq M\|u\|_{\mathrm{L}^2_\omega}$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(c1) We have $\mathbb{D}\mathbb{F} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{F} \in \mathrm{TIC}_\omega$. *when* $\mathbb{F} \in \mathrm{TIC}_\infty(H,U)$.

(c2) We have $\mathbb{D}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega)$ *when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\beta(\mathbf{R}_+;U))$.

(c3) Let $f \in \mathrm{H}^\infty(\mathbf{C}^+;U)$ *and* $\phi \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}) \setminus \{0\}$. *Then* $\widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty \Leftrightarrow \phi^{-1}f \in \mathrm{H}^\infty$.

(d) $\mathbb{D} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{D}\mathbb{D}^* \gg 0 \Leftrightarrow \widehat{\mathbb{D}}(s) \in \mathcal{G}\mathcal{B}(U,Y)$ *for some* $s \in \mathbf{C}^+$ *(*$\Leftrightarrow DD^* \gg 0$ *provided that* $\mathbb{D} \in \mathrm{UR}$*).*

See also the comments following Lemma 6.5.2. Recall that $\mathrm{L}^2_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{L}^2_\omega$, and note that $\mathbb{D}_{-\omega} := e^{-\omega\cdot}\mathbb{D}e^{\omega\cdot} \in \mathrm{TIC}$ for $\omega \geq 0$.

Proof: (A) The equivalence is given in (B).

Assume that some, hence all of (i)–(v) holds. By (B), (iv') and (iv") hold also after the replacement; trivially, so do (i) and (iii) too. Since the proofs of implications (iv")$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i') also hold for $\omega = 0$, all of (i)–(v) hold after the replacement.

(C) This follows from Lemma 2.2.4(a)&(b).

(B) "(i)$\Leftrightarrow$(iii)$\Rightarrow$(a)": This follows from Lemma A.3.1(c1)(1) when we set $\widehat{\mathbb{V}} := (\widehat{\mathbb{D}}^*\widehat{\mathbb{D}})^{-1}\widehat{\mathbb{D}}^* \in \mathcal{C}_b(\mathbf{C}^+;\mathcal{B}(Y,U))$.

"(i')$\Rightarrow$(v)": This follows from $\|\mathbb{V}\|\|\mathbb{D}u\| \geq \|\mathbb{V}\mathbb{D}u\|$ with $\varepsilon = 1/\|\mathbb{V}\|$.

"(v)$\Rightarrow$(i')": Set $\mathbb{V}' := ((\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega})^{-1}\mathbb{D}_{-\omega}^*$. Then $\|\mathbb{V}'\|_{\mathrm{TI}} \leq \varepsilon^{-1}$, by (c1)(1) of Lemma A.3.1, and $\mathbb{V}'\mathbb{D}_{-\omega} = I$. Set $\mathbb{V} := e^{-\omega\cdot}\mathbb{V}'e^{\omega\cdot}$ to obtain (i').

"(iii)$\Leftrightarrow$(v)": Let $\omega > 0$. Because $\widehat{\mathbb{D}}_{-\omega}(s) = (s+\omega)$ $(s \in \mathbf{C}^+_{-\omega})$, condition (iii) holds iff $(\widehat{\mathbb{D}}_{-\omega})^*\widehat{\mathbb{D}}_{-\omega} - \varepsilon^2 I \geq 0$ on $i\mathbf{R}$ for all $\omega > 0$. Because $\widehat{\mathbb{D}}_{-\omega}$ is continuous on $i\mathbf{R}$, the latter is equivalent to (v), by Theorem 3.1.3(d).

"(iv')$\Rightarrow$(iv")$\Leftrightarrow$(iv)": Assume (iv"). By time-invariance and (2.2), the inequality holds for any $u \in \mathcal{C}_c^\infty$. By density (recall that $\mathbb{D} \in \mathrm{TIC}_\omega$), (iv) holds. The other two implications are trivial.

"(iv)$\Rightarrow$(iv')": Assume (iv). If $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2_\omega$, then, for each $n \in \mathbf{N}$, $\infty > \|u\|_{\mathrm{L}^2_{\alpha_n}} > n$ for some $\alpha_n > \omega$, by the monotone convergence theorem. Consequently, $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \|\mathbb{D}u\|_{\mathrm{L}^2_{\alpha_n}} \geq n\varepsilon$. Because n was arbitrary, we have $\mathbb{D}u \notin \mathrm{L}^2_\omega$. Thus, we have proved (b1) for $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$. Obviously, this and (iv) imply (iv').

"(iv)$\Leftrightarrow$(v)": Now $(\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega} \geq \varepsilon^2 I$ iff $\|\mathbb{D}_{-\omega}u\|_2 \geq \varepsilon\|u\|_2$ for all $u \in \mathrm{L}^2$, which holds iff $\|\mathbb{D}v\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|v\|_{\mathrm{L}^2_\omega}$ for all $v \in \mathrm{L}^2_\omega = e^{\omega\cdot}\mathrm{L}^2$.

$\omega \geq 0$: By (2.5), we can allow $\omega = 0$ in (iv') and in (iv").

(b1) By (iv') this holds for π_+u in place of u. But $\pi_-u \in \mathrm{L}^2_\omega$, hence $\mathbb{D}\pi_-u \in \mathrm{L}^2_\omega$ too.

(b2) This follows from (b1) and causality, except that we have to show that $u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ assuming that $u \in \mathrm{L}^2_\omega$ and $\mathbb{D}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$.

For $T = 0$ we have $\|u\|_{\mathrm{L}^2_r} \leq \varepsilon^{-1}\|\mathbb{D}u\|_{\mathrm{L}^2_r} \to 0$, as $r \to +\infty$, hence $\pi_-u = 0$. Use time-invariance for $T \neq 0$.

(b3) Set $M := \|\mathbb{D}\|_{\mathrm{TIC}}$. Assume that at least one of the norms is finite. Then, by (b1), $u \in \mathrm{L}^2_\omega$, hence (b3) follows from (iv).

(c1) If $\mathbb{D}\mathbb{F} \in \mathrm{TIC}_\omega$ (i.e., $\|\mathbb{D}\mathbb{F}u\|_{\mathrm{L}^2_\omega} \leq M\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{C}_c^\infty(\mathbf{R}_+;U)$), then $\|\mathbb{F}\|_{\mathrm{TIC}_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}_\omega}$, by (c2) and (2.13) (with $X := \pi_+\mathcal{C}_c^\infty$ under the L^2_ω norm). Conversely, $\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}_\omega} \leq \|\mathbb{D}\|_{\mathrm{TIC}}\|\mathbb{F}\|_{\mathrm{TIC}_\omega}$.

(c2) "$\Leftarrow$": This is trivial. "$\Rightarrow$": If $\beta > \omega$, then

$$\|\mathbb{C}x\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}x\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|_{\mathcal{B}(X,\mathrm{L}^2_\omega)}\|x\|_X \quad (x \in X), \tag{4.14}$$

by (b3). Therefore, $\|\mathbb{C}\| \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|$.

(c3) Trivially, $\phi^{-1}f \in \mathrm{H}^\infty \Rightarrow \widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty$. Conversely, If $\widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty$, then $\phi^{-1}f = \widehat{\mathbb{V}}\widehat{\mathbb{D}}\phi^{-1}f$ is bounded, by (i), hence in H^∞, because $\phi^{-1}f$ is holomorphic, by Lemma D.1.2(h) (the zeros of ϕ are isolated, by Lemma D.1.2(e)). (In fact, the same equivalence holds whenever $\phi \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ is s.t. it is invertible outside a set having no limit points in $\mathbf{C}^+$.)

(d) This follows from Proposition 2.2.5. $\square$

We also need a “left invertibility over TIC” concept that is invariant under (inverse) discretization (see Theorem 13.4.5(g)) but still has at least the properties of Lemma 4.1.8(b1)–(d). Therefore, we define quasi–left invertibility as follows:

Lemma 4.1.8 (Quasi–left invertibility) *Assume that* $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *is s.t.*

$$\mathbb{D}u \notin \mathrm{L}^2 \quad \text{for all} \quad u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2. \tag{4.15}$$

Then $\mathbb{D}$ *is called* quasi-left-invertible (over TIC)*, and there is* $\varepsilon > 0$ *s.t. (a)–(e) hold for any* $\omega \geq 0$ *and any normed space* X.

(a) $\mathbb{D}^*\mathbb{D} \geq \varepsilon I$.

(b1) $\mathbb{D}u \in \mathrm{L}^2 \Leftrightarrow u \in \mathrm{L}^2$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b3) We have $\varepsilon\|u\|_{\mathrm{L}^2} \leq \|\mathbb{D}u\|_{\mathrm{L}^2} \leq \|\mathbb{D}\|_{\mathrm{TIC}}\|u\|_{\mathrm{L}^2}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(c1) We have $\mathbb{D}\mathbb{F} \in \mathrm{TIC} \Leftrightarrow \mathbb{F} \in \mathrm{TIC}$ *(and* $\|\mathbb{F}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}}$*) when* $\mathbb{F} \in \mathrm{TIC}_\infty(H,U)$.

(c2) We have $\mathbb{D}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2)$ *(and* $\|\mathbb{C}\| \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|$*) when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega(\mathbf{R}_+;U))$.

(d) If also $\mathbb{F} \in \mathrm{TIC}(*,U)$ *is quasi-left-invertible, then so is* $\mathbb{D}\mathbb{F}$.

(e) For each $u_0 \in U \setminus \{0\}$*, we have* $\widehat{\mathbb{D}}u_0 \neq 0$ *on* $\mathbf{C}^+$ *and* $\|\widehat{\mathbb{D}}(ir)u_0\|_Y \geq \varepsilon\|u_0\|_U$ *for a.e.* $r \in \mathbf{R}$.

(f) If $\mathbb{E},\mathbb{F} \in \mathrm{TIC}$ *and* $\mathbb{D} = \mathbb{E}\mathbb{F}$*, then* $\mathbb{F}$ *is quasi-left-invertible.*

(g) Pseudo–left invertibility implies quasi–left invertibility, but the converse does not hold even for $U = \mathbf{C} = Y$.

(h) Quasi–left invertibility (over TIC*) implies left invertibility over* TI *(though not over* TIC*), but the converse does not hold even for* $U = \mathbf{C} = Y$.

From (e) we observe that a quasi-left-invertible $\widehat{\mathbb{D}}$ must have no zeros on $\overline{\mathbf{C}^+}$, but unlike for pseudo-left-invertible transfer functions, $|\widehat{\mathbb{D}}|$ may be arbitrarily small (on $\mathbf{C}^+$, not on $i\mathbf{R}$) and even zero at $+\infty$ (take $\widehat{\mathbb{D}}(s) = e^{-s}$). We do not know whether (e) is equivalent to quasi–left invertibility. A sufficient condition is obviously that for each $\omega \geq 0$, there is $\varepsilon_\omega > 0$ s.t. $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon_\omega$ when $0 < \operatorname{Re} s < \omega$.

Proof: We first show the existence of a number $\varepsilon > 0$ s.t. (a) is satisfied. Then we show that also (b1)–(g) are satisfied with the same ε.

(a) We assume that $\mathbb{D}^*\mathbb{D} \not\geq \varepsilon$ for any $\varepsilon > 0$, and construct an $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2$ s.t. $\mathbb{D}u \in \mathrm{L}^2$.

By assumption, $\mathbb{D}^*\mathbb{D} - 2^{-n} \not\geq 0$ $(n \in \mathbf{N})$. By Theorem 3.1.3(e1), for any $n \in \mathbf{N}$, there are $u_n \in U$ and $E_n \subset i\mathbf{R}$ s.t. $\|u_n\|_U = 1$, $m(E_n) > 0$ and $\langle([\widehat{\mathbb{D}}]^*[\widehat{\mathbb{D}}] - 2^{-n}I)u_n, u_n\rangle < 0$, i.e., $\|\widehat{\mathbb{D}}(s)u_n\| < 2^{-n}$, for $s \in E_n$.

Choose distinct points $ir_k \in E_k$ $(k \in \mathbf{N})$ as in Lemma D.1.24. Let $ir_\infty \in i\mathbf{R} \cup \{\infty\}$ be a limit point of $\{ir_k\}$. We assume that $r_\infty \in \mathbf{R}$ (case $r_\infty = \infty$ is analogous but easier (require, e.g., that $r_0 > 1$, $|r_{k+1}| > 3|r_k|$, and work as below), hence omitted).

W.l.o.g., we assume that $r_\infty = 0$ (replace $\widehat{\mathbb{D}}$ by $\widehat{\mathbb{D}}(\cdot - ir_\infty) \in \mathrm{H}^\infty$) and $|r_k| > 3|r_{k+1}|$ for all $k \in \mathbf{N}$ (choose a subsequence if necessary).

For each $k \in \mathbf{N}$, we set

$$\varepsilon_k := \min\{|r_k|/2, 2^{-k}\} \tag{4.16}$$

and find $f_k := f_{t_k,r_k} \in \mathrm{L}^2$ for $\varepsilon = \varepsilon_k$ and $E = E_k$ as in Lemma D.1.24 (with $p = 2$). Set $v_n := \sum_{k=1}^n f_k u_k \in \mathrm{L}^2$ ($n \in \mathbf{N}$).

Given $\omega > 0$, there is $N \in \mathbf{N}+1$ s.t. $2^{-N} < \omega$, and hence $\|f_k\|_{\mathrm{L}^2_\omega} < \varepsilon_k \leq 2^{-k}$ for all $k > N$; in particular, $v_n \to u$ in L^2_ω for some $u \in \mathrm{L}^2_\omega$, as $n \to \infty$, by Lemma A.3.4(L1) (the limit (equivalence class) u is independent of ω, by Lemma D.1.4(b3)).

Analogously, we see that $\widehat{u}(s)$ converges absolutely on $\{s \in \mathbf{C} \,|\, |s| \geq \varepsilon\}$, for any $\varepsilon > 0$; in particular, $\widehat{u} \in \mathrm{H}(\mathbf{C}^+; U)$ has a unique continuous extension $\widehat{u} \in \mathcal{C}(\overline{\mathbf{C}^+} \setminus \{0\}; U)$. If we had $u \in \mathrm{L}^2$, then we would have $\widehat{u} \in \mathrm{L}^2(i\mathbf{R}; U)$, by Theorem 3.3.1(b)&(a1)(1.).

However, intervals $I_n := i(r_n - \varepsilon_n, r_n + \varepsilon_n) \subset i(r_n/2, 3r_n/2)$ ($n \in \mathbf{N}$) are disjoint, and hence $\|\widehat{f}_k\|^2_{\mathrm{L}^2(I_n)} \leq 2\varepsilon_n \varepsilon_k^2$, so that

$$\|\widehat{u}\|_{\mathrm{L}^2(i\mathbf{R};U)} \geq \sum_{n=1}^\infty \|\widehat{u}\|_{\mathrm{L}^2(I_n;U)} \geq \sum_{n=1}^\infty \left(\|\widehat{f}_n\|_{\mathrm{L}^2(I_n)} - \sum_{n \neq k=1}^\infty \|\widehat{f}_k\|_{\mathrm{L}^2(I_n)} \right) \tag{4.17}$$

$$\geq \sum_{n=1}^\infty (1 - \varepsilon_n - (2\varepsilon_n)^{1/2} \sum_{k=1}^\infty \varepsilon_k \geq \sum_{n=1}^\infty (1 - \varepsilon_n - (2\varepsilon_n)^{1/2}) = \infty. \tag{4.18}$$

Therefore, $u \notin \mathrm{L}^2$. It only remains to show that $\mathbb{D}u \in \mathrm{L}^2$. But

$$\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(i\mathbf{R};U)} \leq \|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(E_n;U)} + \|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(i\mathbf{R}\setminus E_n;U)} \tag{4.19}$$

$$< \sqrt{2\pi}\varepsilon_n + \varepsilon_n \|\mathbb{D}\| \leq 2^{-n}(\sqrt{2\pi} + \|\mathbb{D}\|) = M2^{-n}, \tag{4.20}$$

where $M := \sqrt{2\pi} + \|\mathbb{D}\|$, because $\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_Y \leq |\widehat{f}_n|\varepsilon_n$ on E_n, $\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_Y \leq |\widehat{f}_n|\|\mathbb{D}\|$ a.e. on $i\mathbf{R}$, and $\|\widehat{f}\|_2 = \sqrt{2\pi}$, $\|\widehat{f}\|_{\mathrm{L}^2(i\mathbf{R}\setminus E_n)} < \varepsilon_n$.

Consequently, $\mathbb{D}v_n \to y$ in L^2, for some $y \in \mathrm{L}^2$, by Lemma A.3.4(L1). Choose $\omega > 0$. Because $v_n \to u$ in L^2_ω, we have $\mathbb{D}v_n \to \mathbb{D}u$ in L^2_ω, hence $\mathbb{D}u = y$ a.e., in particular, $\mathbb{D}u \in \mathrm{L}^2$.

(b3) We have $|\mathbb{D}u\|_2 \geq \varepsilon\|u\|_2$ for all $u \in \mathrm{L}^2([T, +\infty); U)$ and all $T \in \mathbf{R}$, by (a) and time-invariance, hence for all $u \in \mathrm{L}^2(\mathbf{R}; U)$, because both sides of the inequality are continuous on L^2. Because $\mathrm{L}^2_\omega(\mathbf{R}_-; U) \subset \mathrm{L}^2$, we have $\mathrm{L}^2_\omega(\mathbf{R}; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U) \subset \mathrm{L}^2(\mathbf{R}_-; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U)$, hence $\|\mathbb{D}u\|_2 = \infty$ whenever $u \in \mathrm{L}^2_\omega(\mathbf{R}; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U)$ and $\|u\|_2 = \infty$. Take $M := \|\mathbb{D}\|_{\mathrm{TIC}}$ to obtain (b3).

(b1) This follows from (b3).

(c1)&(c2) See the proof of Proposition 4.1.7(c1)&(c2).

(d) This is obvious (note also that now $\|\mathbb{D}\mathbb{F}u\|_2 \geq \varepsilon\varepsilon_{\mathbb{F}}\|v\|_2$ for all $v \in \mathrm{L}^2$).

(e) Inequality $\|\widehat{\mathbb{D}}(ir)u_0\|_Y \geq \varepsilon\|u_0\|_U$ follows from Theorem 3.1.3(e1) (cf. the beginning of the proof).

Assume then that $\widehat{\mathbb{D}}(s_0)u_0 = 0$ for some $u_0 \in U \setminus \{0\}$ and $s_0 \in \mathbf{C}^+$. Then

$\widehat{\mathbb{D}}(\cdot)(s-s_0)^{-1}u_0 \in \mathrm{H}(\mathbf{C}^+;U)$, by Lemma D.1.2(j) and a simple computation, although $u := (\cdot - s_0)^{-1}u_0 \in \mathrm{H}^2(\mathbf{C}^+_{\mathrm{Re}\,s_0+1};U) \setminus \mathrm{H}^2(\mathbf{C}^+;U)$.

(f) If $u \in \mathrm{L}^2$, then $\|\mathbb{F}u\|_2 \geq \varepsilon\|\mathbb{E}\|^{-1}\|u\|_2$. If $u \notin \mathrm{L}^2$, then $\mathbb{D}u \notin \mathrm{L}^2$, hence then $\mathbb{F}u \notin \mathrm{L}^2$.

(g) Implication follows from Proposition 4.1.7(b3); note that $\mathbb{D} := \tau^{-1} \in \mathrm{TIC}(U)$ is quasi-left-invertible but not p.r.c. (since $\widehat{\mathbb{D}}(s) = e^{-s}$).

(h) By (a), quasi-left-invertible maps are left invertible over TI (but not necessarily over TIC, by (g)).

Let $\widehat{\mathbb{D}}(s) := (s-1)/(s+1) \in \mathrm{H}^\infty(\mathbf{C}^+)$. Then $\mathbb{D}^*\mathbb{D} = I$, hence $\mathbb{D}^*$ is the inverse of $\mathbb{D}$ in $\mathrm{TI}(\mathbf{C})$, but $\mathbb{D}$ is not quasi-left-invertible, by (e) (alternatively, because $\widehat{u} := (s-1)^{-1} \in \mathrm{H}^2(\mathbf{C}^+_2) \setminus \mathrm{H}^2(\mathbf{C}^+)$ and $\widehat{\mathbb{D}}\widehat{u} = (s+1)^{-1} \in \mathrm{H}^2$). □

The noncausal case is simple:

Lemma 4.1.9 (Noncausal corona theorem) *If $\mathcal{A}$ is one of the classes* TI, CTI, MTI, $\mathrm{MTI_d}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{\mathbf{S}}$, $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$, *then $\mathcal{A}(U,Y)$ is left inverse closed in* TI, *and $\mathbb{E} \in \mathcal{A}(U,Y)$ is left invertible iff $\mathbb{E}^*\mathbb{E} \geq \varepsilon I$ on L^2 (iff $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$, provided that U is separable or $\mathcal{A} \neq$ TI).*

If $\mathcal{A}$ is one of the classes TI, CTI, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{T\mathbf{Z}}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, *then any left invertible element $\mathbb{E} \in \mathcal{A}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented to an invertible operator.*

Proof: 1° *Left invertibility:* By Lemma A.3.1(c1)(ii)&(v), "$\mathbb{E}^*\mathbb{E} \geq \varepsilon I$" is necessary. (It is equivalent to "$\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$", by Theorem 3.1.3(d), provided that U is separable of $\widehat{\mathbb{E}}$ is continuous.)

Conversely, if $\mathbb{E}^*\mathbb{E} \gg 0$, then the formula $\mathbb{V} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}^* \in \mathcal{A}$ (by Theorem 4.1.1 and Lemma 4.1.3(a)) provides a left-inverse for $\mathbb{E}$.

2° *Complementation:* (Clearly left invertibility is necessary.) Classes CTI, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ and $\mathrm{MTI}_{T\mathbf{Z}}$ can be handled by using the methods of the proof of Theorem 4.1.6(c) (based on the fact that they have contractible maximal ideals); a left-invertible $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented as follows:

By Lemma 6.4.7, there is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$ s.t. $\mathbb{X}^*\mathbb{X} = \mathbb{E}^*\mathbb{E}$. By [CG97, Lemma 2.2] (and Theorem 3.1.3(a)&(c)), we can complement the isometric $\mathbb{N} := \mathbb{E}\mathbb{X}^{-1} \in \mathrm{TI}$ to a unitary $\begin{bmatrix}\mathbb{N} & \mathbb{F}\end{bmatrix} \in \mathcal{G}\mathrm{TI}$, hence $\begin{bmatrix}\mathbb{E} & \mathbb{F}\end{bmatrix} = \begin{bmatrix}\mathbb{N} & \mathbb{F}\end{bmatrix}\begin{bmatrix}\mathbb{X} & 0\\ 0 & I\end{bmatrix} \in \mathcal{G}\mathrm{TI}$. □

Lemma 4.1.10 (No ∞-dim. Corona Theorem) *Let U be infinite-dimensional and $\dim Y \geq \dim U$. Then there is $\mathbb{D} \in \mathrm{TIC}(U,Y)$ s.t. $\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq I$ on $\mathbf{C}^+$, but $\mathbb{V}\mathbb{D} \neq I$ for all $\mathbb{V} \in$ TIC.*

Consequently, $\mathbb{D}$ cannot be complemented to an invertible element of TIC (because the proof of implication "(vii)⇒(i)" of Theorem 4.1.6 applies to this case too). Cf. also Proposition 4.1.7.

Proof: A counter-example is constructed in [Treil89], below its Theorem 1, assuming U and Y to be separable (we may need to multiply the counter-example by a positive constant). In the general case, write U as $U_1 \times U_2$

and Y as $U \times Y_2$ (modulo an isometric isomorphism), where U_1 is separable (this is possible because $\dim Y \geq \dim U$, by Lemma 2.2.1(c3)). Let $\widehat{\mathbb{F}}$ be as in the counter-example, and set $\mathbb{D} := \left[\begin{smallmatrix} \mathbb{F} & 0 \\ 0 & I \end{smallmatrix}\right]$. Then $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq I$ on $\mathbf{C}^+$, but if $\mathbb{V}\mathbb{D} = I = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{B}(U_1 \times U_2)$, then $\mathbb{V}_{11}\mathbb{F} = I$, hence then $\mathbb{V} \notin \mathrm{TIC}$. $\square$

Notes

Several sufficient conditions for the invertibility of a measure with values in a Banach algebra are given in [Gri]. Much of Theorem 4.1.1(f) is based on those results. The article [Gri] also provides further results and treats very general measures.

The monograph [Vid] is an excellent classical reference for the connection between dynamic stabilization, coprime factorization and the Corona Theorem. It contains the principal ideas of Lemmas 4.1.4 and 4.1.5 and Theorem 4.1.6 and applications for several classes.

The complementation result Theorem 4.1.6(c) for TIC is due to V.A. Tolokonnikov [Tolokonnikov]; see pp. 288–298 of [Nikolsky] for a presentation in English, further results, norm estimates and historical remarks. These results, being more recent than those of [Vid], do not seem to be widely known.

The original Corona Theorem is due to L. Carleson [Carleson]. A matrix-valued Corona Theorem ($\mathcal{B}(\mathbf{C}^n, \mathbf{C}^{nm})$ only) was given in [Fuhrmann68]. An extension of the matrix-valued Corona Theorem with an arbitrary $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ in place of $1_{\mathcal{A}}$ in (4.7) is given in [Anderson].