Chapter 5

Spectral Factorization ($\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$)

God created spectral factorizations; the rest is made by man.

— Frank Callier, in a discussion of the importance of spectral factorizations, indefinite inner spaces and Riccati equations, MTNS'98.

This chapter treats the spectral factorization (or canonical factorization) of MTI maps. Spectral factorization will be used in later chapters for the solution of several control problems.

In Section 5.1, we apply the early factorization theory of Israel Gohberg and Yuri Leiterer (not being a prophet, we cannot refer directly to [God]) to $\mathrm{MTI}^{\mathrm{L}^1}$ and $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ in continuous time and to ℓ^1 in discrete time. In Section 5.2, we adopt several $\mathrm{MTI}_{\mathrm{d}}$ factorization results to our setting and show that the factorization of MTI maps can be reduced to that of $\mathrm{MTI}^{\mathrm{L}^1}$ and $\mathrm{MTI}_{\mathrm{d}}$ maps. We thus obtain both positive and indefinite spectral factorization results for several MTI classes.

We also state a few other results concerning the spectral factorization of TI maps. By H, U and Y we again denote Hilbert spaces of arbitrary dimensions. (The results based on [GL73a] could be modified for arbitrary Banach spaces.)

Also Section 6.4 contains related results, but we have chosen its current place since that section is a prerequisite for Sections 6.6–6.7 and Chapter 7.

5.1 Auxiliary spectral factorization results

Grief can take care of itself; but to get the full value of a joy you must have somebody to divide it with.

— Mark Twain (1835–1910)

In this section we apply the spectral factorization theory of Gohberg and Leiterer to $\mathrm{MTI}^{\mathrm{L}^1}$ (Theorem 5.1.2), ℓ^1 (Theorem 5.1.3) and $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ (Corollary 5.1.4). In Section 5.2, we shall then refine these and other results to cover further classes and to provide more information on the factors.

First we define a spectral factorization:

Definition 5.1.1 (SpF) *A factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *is a* spectral factorization *of* $\mathbb{E} \in \mathrm{TI}(U)$ *if* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$.

For $\dim U < \infty$, this could be rephrased in the familiar form "if $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+; \mathbf{C}^{n\times n})$ and $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$, then $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ is a *spectral factorization* of $\widehat{\mathbb{E}}$" ($\in \mathrm{L}^\infty(\mathbf{C}^+; \mathbf{C}^{n\times n})$), by Theorems 2.1.2 and 3.3.1.

Even for a general U, the identity $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ can be written as "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ on $i\mathbf{R}$" when $\mathbb{E}, \mathbb{X}, \mathbb{Y} \in \mathrm{MTI}$, but for general $\mathbb{E} \in \mathrm{TI}$ we must be satisfied with the equality "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U))$", which need not imply pointwise equality anywhere (for separable U an equivalent formulation is that "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$"); see Theorem 3.1.3 for details. However, in this chapter we mainly study MTI maps, for which we have continuity and pointwise equality everywhere on $i\mathbf{R}$ regardless of U.

As the first spectral factorization result, we apply Theorem 5.1.6 to the Wiener class:

Theorem 5.1.2 ($\mathbf{MTIC}^{\mathbf{L}^1}$ spectral factorization) *Let* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(U)$*, i.e.,* $\widehat{\mathbb{E}} = E + \hat{f}$*, where* $E \in \mathcal{B}(U)$ *and* $f \in \mathrm{L}^1(\mathbf{R}; \mathcal{B}(U))$.

Then the Toeplitz operator $\pi_+\mathbb{E}\pi_+$ *is invertible iff* $\mathbb{E}$ *has a factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *with* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$.

If, in addition, $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ *(i.e.,* $f \in \mathrm{L}^1(\mathbf{R}; \mathcal{BC}(U))$*), then* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$.

Proof: By Lemma 5.1.7, we may apply Theorem 5.1.6 to obtain the above factorizations in the same way as in the proof of Theorem 5.1.3.

Note that $E + f \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ implies that $P_-^0((E + \widehat{f}) \circ \phi_{\mathrm{Cayley}}^{-1}) = \widehat{\pi_- f} \circ \phi_{\mathrm{Cayley}}^{-1} \in \mathfrak{C}_\infty$, and that $P_+((E + \widehat{f}) \circ \phi_{\mathrm{Cayley}}^{-1})$ can be seen to be a Fredholm operator as in the proof of Theorem 5.1.3 (alternatively, it follows easily from the fact that E must be invertible, by, e.g., Proposition 6.3.1(c)). See also Lemma 5.1.5. The parametrization of all factors is given in Lemma 6.4.5(i). □

Next we apply Theorem 5.1.6 to the *discrete Wiener class* $\widehat{\ell^1}$. We will use the following notation (as in Theorem 4.1.1 and in Section 13.1):

$$\begin{aligned}
\ell^1 &:= \ell^1(\mathbf{Z};\mathcal{B}(U)) := \{(a_j)_{j\in\mathbf{Z}} \mid a_j \in \mathcal{B}(U) \text{ and } \|(a_j)_{j\in\mathbf{Z}}\|_{\ell^1} := \sum_{j\in\mathbf{Z}} \|a_j\|_{\mathcal{B}(U)} < \infty\},\\
\ell^1_{\mathcal{BC}} &:= \{a \in \ell^1 \mid a_j \in \mathcal{BC}(U,Y) \text{ for all } j \neq 0\},\\
\ell^1_{\pm} &:= \{a \in \ell^1 \mid a_j = 0 \text{ for all } \pm j < 0\},\\
\ell^1_{\mathcal{BC},\pm} &:= \{a \in \ell^1_{\mathcal{BC}} \mid a_j = 0 \text{ for all } \pm j < 0\}.
\end{aligned} \tag{5.1}$$

We equip these spaces with *convolution multiplication*

$$(a_j)_{j\in\mathbf{Z}} * (b_k)_{k\in\mathbf{Z}} := \big(\sum_j a_j b_{n-j}\big)_{n\in\mathbf{Z}}. \tag{5.2}$$

As in Section 13.1, one can verifies that ℓ^1 with convolution multiplication is a Banach algebra, and the five other classes defined above are closed subalgebras. The *Z-transform* of $a = (a_j)_{j\in\mathbf{Z}} \in \ell^1$ is

$$\widehat{a} := \sum_{j\in\mathbf{Z}} a_j z^j \in \mathcal{C}(\overline{\mathbf{D}}) \cap \mathrm{H}^\infty(\mathbf{D}), \tag{5.3}$$

and $\widehat{a*b} = \widehat{a}\widehat{b}$. The class $\widehat{\ell^1}$ (obviously isomorphic to the Banach algebra ℓ^1) is sometimes called the discrete Wiener class. The canonical projection $\pi^+ : \ell^2(\mathbf{Z};U) \mapsto \ell^2(\mathbf{N};U)$ obviously satisfies $\widehat{\pi^+}\sum_{j\in\mathbf{Z}} z^j x_j := \sum_{j\in\mathbf{N}} z^j x_j$. Recall from Lemma D.1.15 that $\mathrm{L}^2(\partial\mathbf{D};U) = \{\sum_{j\in\mathbf{Z}} z^j x_j \mid \sum \|z_j\|_U^2 < \infty\}$ and that $\mathrm{H}^2(\mathbf{D};U) = \widehat{\pi^+}[\mathrm{L}^2(d\mathbf{D};U)]$.

Theorem 5.1.3 (Discrete $\boldsymbol{\ell^1}$ spectral factorization) *Let $\mathbb{E} \in \ell^1(\mathbf{Z};\mathcal{B}(U))$, i.e., $\widehat{\mathbb{E}} = \sum_{j=-\infty}^{\infty} z^j E_j$, where $E_j \in \mathcal{B}(U)$ for all j and $\sum_j \|E_j\| < \infty$.*

Then the Toeplitz operator $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+} \in \mathcal{B}(\mathrm{H}^2(\mathbf{D};U))$ is invertible iff $\widehat{\mathbb{E}}$ has a spectral factorization $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_-\,\widehat{\mathbb{E}}_+$ with $\mathbb{E}_+ \in \mathcal{G}\ell^1_+$ and $\mathbb{E}_- \in \mathcal{G}\ell^1_-$. If, in addition, $\mathbb{E} \in \ell^1_{\mathcal{BC}}$ (i.e., $E_j \in \mathcal{BC}(U)$ for $j \neq 0$), then $\widehat{\mathbb{E}}_+ \in \mathcal{G}\ell^1_{\mathcal{BC},+}$ and $\widehat{\mathbb{E}}_- \in \mathcal{G}\ell^1_{\mathcal{BC},-}$.

Proof: 1° *$\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible iff $\widehat{\mathbb{E}}$ has a spectral factorization:*
The first claim follows from (a) and (c) of Theorem 5.1.6, as soon as we have verified the assumptions of the Theorem.

One easily verifies that assumptions (1) and (2) of Theorem 5.1.6 hold (for both $\widehat{\ell^1}$ and $\widehat{\ell^1_{\mathcal{BC}}}$), where we have set $P_+\sum_{j=-\infty}^{\infty} z^j E_j := \sum_{j=0}^{\infty} z^j E_j$.

(3a) One easily deduces from [HP, p. 97], that the Laurent series of a holomorphic (around $\partial\mathbf{D}$) function converges absolutely on $\partial\mathbf{D}$. Conversely, the holomorphic function $\sum_{j\in\mathbf{Z}} r^{-|j|} E_j z^j$ converges to $\widehat{\mathbb{E}}$ in $\widehat{\ell^1}$, as $r \to 1-$. Finally, $\widehat{\ell^1}$ (equivalently, ℓ^1 as a convolution algebra) is inverse closed by Theorem 4.1.1(d).

(a) If $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible, then it is a Fredholm operator, hence the assumptions of (a) are satisfied in this case, and the converse follows from the implication (i)⇒(ii) of (c).

2° *The case* $\mathbb{E} \in \ell^1_{\mathcal{BC}}$*:* Below we show that the assumptions of (b) are satisfied, so that the form of $\widehat{\mathbb{E}}_\pm$ follows from (b).

Assumptions (1) and (2) were handled above.

Because $z^n \in \mathcal{R}(\partial\mathbf{D})$ for $n \in \mathbf{Z}$, the assumption (3b) is satisfied. Similarly, we see that $P^0_-\widehat{\mathbb{E}}$ is in $\mathfrak{C}_\infty$ when $\widehat{\mathbb{E}} \in \ell^1_{\mathcal{BC}}$, so it only remains to be shown that $P_+\widehat{\mathbb{E}}(z)$ is a Fredholm operator for all $z \in \overline{\mathbf{D}}$.

The invertibility of $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ implies the invertibility of $\widehat{\mathbb{E}}$ on $\partial\mathbf{D}$, by Lemma 5.1.5; in particular, $\widehat{\mathbb{E}}(1) \in \mathcal{GB}(U)$.

But $\widehat{\mathbb{E}}(z) = E_0 + \widehat{\mathbb{F}}(z)$, where $\widehat{\mathbb{F}}(z) := \sum_{j\neq 0} E_j z^j \in \mathcal{BC}(U)$ for all $z \in \overline{\mathbf{D}}$, hence $\widehat{\mathbb{E}}(z) = \widehat{\mathbb{E}}(1) + \widehat{\mathbb{F}}(z) - \widehat{\mathbb{F}}(1) \in \mathcal{GB}(U) + \mathcal{BC}(U)$, and $\mathcal{GB} + \mathcal{BC}$ operators are Fredholm operators, by Lemma A.3.4(B4).

Finally, $\widehat{\mathbb{E}}, \widehat{\mathbb{E}}_-, \widehat{\mathbb{E}}_-^{-1} \in \ell^1_{\mathcal{BC}}$ implies that $\widehat{\mathbb{E}}_+, \widehat{\mathbb{E}}_+^{-1} \in \ell^1_{\mathcal{BC}}$. □

Due to isomorphism, the above is equivalent to the following:

Corollary 5.1.4 ($\mathbf{MTI}_{\mathbf{d},T\mathbf{Z}}$ spectral factorization) *Let* $T \in \mathbf{R}$*, and let* $\mathbb{E} \in \mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$*, i.e.,* $\mathbb{E} = \sum_{k\in\mathbf{Z}} E_j \delta_{jT}$*, where* $E_j \in \mathcal{B}(U)$ *for all* j *and* $\|\mathbb{E}\|_{\mathrm{MTI}} := \sum_j \|E_j\| < \infty$*.*

Then the Toeplitz operator $\pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ *is invertible iff* $\mathbb{E}$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *with* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(U)$*. If, in addition,* $\mathbb{E} \in \mathrm{MTI}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}$ *(i.e.,* $E_j \in \mathcal{BC}(U)$ *for* $j \neq 0$*), then* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}(U)$*.*

Proof: This is Theorem 5.1.3 rephrased according to the isomorphism stated in Theorem 13.4.5 (note that $\widetilde{I}\widehat{\pi^+}(E_j)_{j\in\mathbf{Z}}\widehat{\pi^+} = \pi_+\mathbb{E}\pi_+$, $\widetilde{I}[\ell^1_+] = \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, and $\widetilde{I}[\ell^1_-] = \{\mathbb{Y}^* \mid \mathbb{Y} \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}\}$). □

The rest of this sections consists only of results that are needed for the proofs of the above results.

We start the proofs with an auxiliary lemma:

Lemma 5.1.5 *Let* $\widehat{\mathbb{E}} \in \mathcal{C}(\partial\mathbf{D};\mathcal{B}(H))$ *and set* $\widehat{\mathbb{F}} := \widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(H))$*. Then* $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is invertible iff* $\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{\pi_+}$ *is invertible. Moreover, if* $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is invertible, then* $\widehat{\mathbb{E}} \in \mathcal{GC}(\partial\mathbf{D};\mathcal{B}(H))$*.*

Proof: The equivalence follows from Theorem 13.2.3(a1)&(b1)&(c). If $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible on $\mathrm{H}^2(\mathbf{D};H)$, then $\widehat{\mathbb{E}}$ is invertible on $\mathrm{L}^2(\partial\mathbf{D};H)$, by discretized Lemma 6.4.6, hence then $\widehat{\mathbb{E}}$ is invertible in $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};H)$, by Theorem 3.1.3(a1), hence in $\mathcal{C}(\partial\mathbf{D};\mathcal{B}(H))$, by Theorem F.1.9(s4) (applied to $Q := [0,2\pi)$). □

The following “raw result” from [GL-Crit] and [GL73a] is the basis for the above factorization results:

Theorem 5.1.6 *Let* H *be a Hilbert space. Let* $\mathcal{R}(\partial\mathbf{D})$ *be the set of rational scalar functions with poles outside* $\partial\mathbf{D}$*. Let* $\mathfrak{C} \subset \mathcal{C}(\partial\mathbf{D};\mathcal{B}(H))$ *be a Banach algebra with a norm* $\|\cdot\|_{\mathfrak{C}}$ *s.t.*

(1) $\sup_{\partial\mathbf{D}} \|\widehat{\mathbb{E}}(\cdot)\|_{\mathcal{B}(H)} \le c\|\widehat{\mathbb{E}}\|_{\mathfrak{C}}$ *for all* $\widehat{\mathbb{E}} \in \mathfrak{C}$ *for some* $c > 0$*, and*

(2) $\mathfrak{C}$ *is the direct sum* $\mathfrak{C}^+ \oplus \mathfrak{C}_0^-$*, where* $\mathfrak{C}^+ = \mathfrak{C} \cap \mathrm{H}^\infty(\mathbf{D};\mathcal{B})$*,* $\mathfrak{C}^- = \mathfrak{C} \cap \mathrm{H}^\infty(\overline{\mathbf{D}}^c;\mathcal{B})$*, and* $\mathfrak{C}_0^- = \{f \in \mathfrak{C}^- \mid f(\infty) = 0\}$.

Let $P_+ : \mathfrak{C} \to \mathfrak{C}^+$ *and* $P_-^o := I - P_+ : \mathfrak{C} \to \mathfrak{C}_0^-$ *be the corresponding projections. Let* $\widehat{\mathbb{E}} \in \mathcal{G}\mathfrak{C}$*. Then we have the following:*

(a) Let functions holomorphic on a neighborhood of $\partial\mathbf{D}$ *be a dense subset of* $\mathfrak{C}$*, and let* $\mathfrak{C}$ *be inverse closed in* C *(i.e., if* $\widehat{\mathbb{E}} \in \mathfrak{C} \cap \mathcal{G}C(\partial\mathbf{D};\mathcal{B}(H))$*, then* $\widehat{\mathbb{E}}^{-1} \in \mathfrak{C}$*).*

Then $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is a Fredholm operator on* $\mathrm{H}^2(\mathbf{D};H)$ *iff* $\widehat{\mathbb{E}}$ *has a factorization of the form*

$$\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_- G \widehat{\mathbb{E}}_+, \quad \widehat{\mathbb{E}}_+ \in \mathcal{G}\mathfrak{C}^+,\ \widehat{\mathbb{E}}_- \in \mathcal{G}\mathfrak{C}^-,\ G(z) = P_0 + \sum_{j=1}^n z^{\kappa_j} P_j, \tag{5.4}$$

where $n \in \mathbf{N}$*,* P_j $(j = 1,\dots,n)$ *are disjoint one-dimensional projections,* $P_0 = I - \sum_j P_j$*, and* $\kappa_j \in \mathbf{Z}\setminus\{0\}$*,*

(b) Let the rational functions $\sum_{j=1}^n r_j T_j$ $(r_j \in \mathcal{R}(\partial\mathbf{D}),\ T_j \in \mathcal{B}(H)$ *for all* $j)$ *be a dense subset of* $\mathfrak{C}$*.*

Let $P_+\widehat{\mathbb{E}}(z)$ *be a Fredholm operator for all* $z \in \overline{\mathbf{D}}$*, and let* $P_-^o\widehat{\mathbb{E}} \in \mathfrak{C}_\infty$*. Then* $\widehat{\mathbb{E}}$ *has the factorization (5.4) with* $\widehat{\mathbb{E}}_- - I, \widehat{\mathbb{E}}_-^{-1} - I \in \mathfrak{C}_\infty^- := \mathfrak{C}_\infty \cap \mathrm{H}^\infty(\overline{\mathbf{D}}^c;\mathcal{B})$.

Here the set $\mathfrak{C}_\infty$ *is the closure (in* $\mathfrak{C}$*) of rational* $\mathcal{BC}$*-valued operators*

$$\sum_{j=1}^n r_j T_j, \quad r_j \in \mathcal{R}(\partial\mathbf{D}),\ T_j \in \mathcal{BC} \text{ for all } j. \tag{5.5}$$

(c) Let all the assumptions of (a) or those of (b) be satisfied, and let $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_- G \widehat{\mathbb{E}}_+$ *be the resulting factorization. Then the following are equivalent:*

(i) $G = I$*,*

(ii) $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is invertible on* $\mathrm{H}^2(\mathbf{D};H)$*,*

(iii) $\widehat{\pi_+}(\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}})\widehat{\pi_+}$ *is invertible on* $\mathcal{L}\pi_+\mathrm{L}^2(\mathbf{R};H)$.

Moreover, if (i) holds and we set $\widehat{\mathbb{X}} := \widehat{\mathbb{E}}_+ \circ \phi_{\text{Cayley}}$*,* $\widehat{\mathbb{Y}}(s) := (\widehat{\mathbb{E}}_- \circ \phi_{\text{Cayley}})(-\bar{s})^*$*, then* $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$*,* $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}} = \widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}$ *on* $i\mathbf{R}$*, and all spectral factorizations of* $\widehat{\mathbb{E}}$ *are given by* $\widehat{\mathbb{E}} = (\widehat{\mathbb{E}}_- T)(T\widehat{\mathbb{E}}_+)$*,* $T \in \mathcal{GB}(U)$ *(i.e.,* $\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}} = (T\widehat{\mathbb{Y}})^*(T\widehat{\mathbb{X}})$*).*

We remark that the original results in [GL73a] and [GL-Crit] are given in a more abstract and general form.

Do not mix $\widehat{\pi^+}\widehat{\mathbb{E}}$ with $P_+\widehat{\mathbb{E}}$ ((the restriction of) P_+ is an operator on $\widehat{\text{MTI}}$, i.e., it operatres $\widehat{\mathbb{E}}$, whereas $\widehat{\pi^+}$ is an operator on H^2 (so are $\widehat{\mathbb{E}}$ and $P_+\widehat{\mathbb{E}}$ too)).

If H is finite-dimensional, then it is possible to formulate the theorem without a reference to Fredholm operators; see Theorems II.3.1 and II.4.1 of [CG81].

Proof: (a) This is Theorem 2 of [GL-Crit] (use (13.29) and note that "PA" in [GL-Crit] refers to composition, i.e., $PAu := P(Au)$, as stated on p. 102 of [GL72]; P does not operate directly on A).

(b) This is Theorem 2.1, p. 40, of [GL73a] (cf. pp. 38–39 of [GL73a]).

(c) 1° By Lemma 5.1.5, (ii) and (iii) are equivalent. Because $\widehat{\mathbb{E}}_- \circ \phi_{\text{Cayley}} \in \mathrm{H}^\infty(\mathbf{C}^-;\mathcal{B}(H))$, we have $\widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$; clearly $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$ too. If (i) holds, then $(\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}})(ir) = \widehat{\mathbb{Y}}(-\overline{ir})^*\widehat{\mathbb{X}}(ir) = \widehat{\mathbb{Y}}(ir)^*\widehat{\mathbb{X}}(ir)$ for $r \in \mathbf{R}$, and the uniqueness claim follows from Lemma 6.4.5(a).

Thus, only (i)$\Leftrightarrow$(iii) is left to be proved.

2° "(i)$\Leftrightarrow$(iii)": Set $\widehat{\mathbb{G}} := D \circ \phi_{\text{Cayley}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(H))$. Then $\mathbb{G} \in \mathrm{TI}(H)$, $\mathbb{X},\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(H)$, and $\mathbb{F} = \mathbb{Y}^*\mathbb{G}\mathbb{X}$.

Because $\pi_+\mathbb{X}\pi_+ = \mathbb{X}\pi_+$ and $\pi_+\mathbb{Y}^*\pi_+ = \pi_+\mathbb{Y}^*$ are invertible on $\pi_+\mathrm{L}^2$ (the inverses are $\pi_+\mathbb{X}^{-1}\pi_+$ and $\pi_+\mathbb{Y}^{-*}\pi_+$), the equation

$$\pi_+\mathbb{F}\pi_+ = \pi_+\mathbb{Y}^*\mathbb{G}\mathbb{X}\pi_+ = \pi_+\mathbb{Y}^*\pi_+\mathbb{G}\pi_+\mathbb{X}\pi_+ \tag{5.6}$$

implies that $\pi_+\mathbb{G}\pi_+$ is invertible iff $\pi_+\mathbb{F}\pi_+$ is. By Lemma 5.1.5, this can be paraphrased as "$\widehat{\pi^+}G\widehat{\pi^+}$ is invertible on $\mathrm{H}^2(\mathbf{D};H)$ iff $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is". Of course, $G = I$ is sufficient, so we study the necessity:

Let $\widehat{\pi^+}G\widehat{\pi^+}$ be invertible. If $u_j = P_ju_j \in H$ and $\kappa_j > 0$, then $u := 1u_j \notin \widehat{\pi^+}G\widehat{\pi^+}\mathrm{H}^2$, because $P_j\widehat{\pi^+}G\widehat{\pi^+} = \widehat{\pi^+}s_j^{\kappa}\widehat{\pi^+}P_j$; similarly no κ_j can be negative, hence $n = 0$ and $G = P_0 = I$. □

(The proofs in [GL-Crit] and [GL73a] go as follows: first it is assumed that $\mathbb{E}$ is holomorphic around the unit circle, then this is applied to "rational finite-dimensional $\mathbb{E}$'s", then the density of such operators in $\mathrm{MTI}^{\mathrm{L}^1}$ is combined with the fact that any element near the identity has a spectral factorization.)

The above theorem can be applied to the Wiener class:

Lemma 5.1.7 *The class* $\mathfrak{C} := \{\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1} \,|\, \mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(U)\}$ *with* $\|\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1}\|_{\mathfrak{C}} := \|\mathbb{E}\|_{\mathrm{MTI}^{\mathrm{L}^1}}$ *satisfies the assumptions of (1), (2), (a) and (b) (on* $\mathfrak{C}$*) of Theorem 5.1.6 when we define the projection* $P_+ \in \mathcal{B}(\mathfrak{C},\mathfrak{C}^+)$ *by*

$$P_+ : (E + \widehat{f}) \circ \phi_{\text{Cayley}}^{-1} \mapsto E + \widehat{\pi_+ f} \circ \phi_{\text{Cayley}}^{-1} \tag{5.7}$$

for $E \in \mathcal{B}(U)$, $f \in \mathrm{L}^1(\mathbf{R};U)$ *(i.e., for* $E + \widehat{f} \in \widehat{\mathrm{MTI}^{\mathrm{L}^1}}(U)$*). Moreover, in this case,* $\mathfrak{C}_\infty$ *corresponds to maps* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ *whose feedthrough operator is compact.*

Proof: We remark that now $\mathfrak{C}$ denotes the Cayley transforms of functions in $\widehat{\mathrm{MTI}^{\mathrm{L}^1}}$, with their original norm, exactly as in [GL73a, Theorem 4.3].

Conditions (1) and (2) can be proved as in [CG81, pp. 62–63] (which treats the case $B = \mathbf{C}$).

(a) The functions holomorphic on a neighborhood of $\partial\mathbf{D}$ are contained in $\mathfrak{C}$, by Lemma D.1.23. The density follows from that the rational functions (case (b) below). Inverse-closedness was shown in Theorem 4.1.1(a) (use the Cayley transform).

(N.B. Although [GL73a, p. 44] suggests that the strongly measurable (*sil'no izmerimyj*) Wiener class would do, this strong measurability must mean Bochner measurability (with respect to the uniform operator norm), not measurability with respect to the strong topology, because (in (b)) the closure of the Fourier inverse transforms of rational functions (with poles off $\mathbf{R}\cup\{\infty\}$) is $\mathrm{L}^1(\mathbf{R};B)$, not $\mathrm{L}^1_{\mathrm{strong}}(\mathbf{R};B)$; a similar remark applies to (a).)

(b) 1° To show that the rational functions are dense in $\widehat{\mathrm{MTI}^{\mathrm{L}^1}}(B)$, we work as follows:

Now for $\widehat{\mathbb{E}} = E + \widehat{f} \in \widehat{\mathrm{MTI}^{\mathrm{L}^1}}(B)$ we may replace $\widehat{f}$ by $\sum_{k=1}^n T_k\hat{\chi}_{E_k}$, with $T_k \in \mathcal{B}(B)$ for all k, by the density of simple functions in L^1 (see Theorem B.3.11(a1)), and then replace each $\hat{\chi}_{E_k}$ by some rational function, by [CG81, pp. 62–63], to end up with a rational function close to $\widehat{\mathbb{E}}$, as required.

2° $\mathfrak{C}_\infty$: If $E \in \mathcal{BC}(U)$ and $f \in \mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))$, then the above approximation provides (use Theorem B.3.11(a1) with $B := \mathcal{BC}(U)$) a rational function of form (5.5); conversely, if $\widehat{\mathbb{E}}\circ\phi_{\mathrm{Cayley}}$ is of the form (5.5), then $\widehat{\mathbb{E}} - E \in \mathcal{L}\mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))$, by Lemma D.1.23. □

Notes

During the third quarter of the last century, Budjanu, Gohberg and several others developed an extensive theory on the factorization of $\mathrm{MTI}^{\mathrm{L}^1}$ maps and of maps in certain other TI classes for the purposes of singular integral equation theory. Some of the articles (mainly in Russian) also treat a more general factorization where $\pi_+\mathbb{E}\pi_+$ and $\mathbb{Y}$ need not be invertible (and the word "spectral" or "canonical" is dropped).

Soon this theory became popular also amidst control theorists, and today many articles in the infinite-dimensional control theory are based principally on spectral factorization. Also the factorization theory is still being rapidly developed.

In case $\dim U < \infty$, the most important results can be found in English in the book [CG81]; a somewhat more up-to-date book on the subject is [LS]. Both books also contain the case ("generalized spectral factorization") where the discrete-time ($\mathcal{Z}$-transformed) factors are allowed to be in $\mathcal{G}\mathrm{H}^2$ over the unit disc instead of $\mathcal{G}\mathrm{H}^\infty$ (cf. Example 8.4.13). These books also have extensive reference lists.

The applications of this theory given in this section are rather straightforward, and most of them have finite-dimensional analogies in the literature.

5.2 MTI spectral factorization ($\mathbb{E}, \mathbb{Y}, \mathbb{X} \in$ MTI)

Each Man is in his Spectre's power
Until the arrival of that hour,
When his Humanity awake,
And cast his Spectre into the Lake.
— William Blake (1757–1827)

The purpose of this section is to establish part "(II)⇔(III)" of the equivalence on page 21 for several systems and problems. We build up a series of lemmas on MTI ending up with two existence theorems on spectral factorization, both of which cover several MTI subclasses (and TI in the positive case).

Our strategy is the following: We first show that if a map $\mathbb{E} \in \mathrm{MTI}(U)$ has an invertible *Toeplitz operator* $\pi_+\mathbb{E}\pi_+$ (on $\mathrm{L}^2(\mathbf{R}_+;U)$), then also the discrete (atomic) part of the map has an invertible Toeplitz operator (Lemma 5.2.3(b)). Then we adopt several $\mathrm{MTI_d}$ factorization results to our setting and show that by factorizing first the discrete part of a MTI map using these results and then the "remainder $\mathrm{MTI}^{\mathrm{L}^1}$ part" by using the results of Section 5.1, one obtains a spectral factorization of the original map (Theorems 5.2.7 and 5.2.8).

We start by listing some basic facts on spectral factorization:

Lemma 5.2.1 (Spectral Factorization) *Let* $\mathbb{E} \in \mathrm{TI}(U)$.

(a) Then $\mathbb{E} \gg 0$ *iff* $\mathbb{E}$ *has the spectral factorization* $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$.

If this is the case, then all spectral factorizations of form $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$ *are given by* $\mathbb{E} = (L\mathbb{X})^*(L\mathbb{X})$, *where* $L \in \mathcal{GB}(U)$ *is unitary.*

Assume now that $\mathbb{E} \in \mathrm{TI}(U)$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$. *Then we have the following:*

(b) The Toeplitz operator $\pi_+\mathbb{E}\pi_+$ *is invertible on* $\pi_+\mathrm{L}^2$, *and* $\pi_+\mathbb{X}^{-1}\pi_+\mathbb{Y}^{-*}\pi_+$ *is its inverse.*

(c) If $\mathbb{E} \in \mathrm{TI}_{-\omega} \cap \mathrm{TI}_\omega$ *for some* $\omega > 0$, *then* $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U) \cap \mathrm{TIC}_{-\omega}(U)$.

(d) If $\mathbb{E} = \mathbb{E}^*$, *then* $\mathbb{Y} = \mathbb{X}^* S$ *for some* $S = S^* \in \mathcal{GB}(U)$; *thus, then* $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$.

If, in addition, $\mathbb{E} \in \mathrm{TI}_\omega(U)$ *for some* $\omega \neq 0$, *then* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U)$.

(e) The map $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}\boldsymbol{Я} \in \mathrm{TI}(U)$ *has the* co-spectral factorization $\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$
(obviously, $\mathbb{X}^{\mathrm{d}}, \mathbb{Y}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(U)$*).*

(f) All spectral factorizations of $\mathbb{E}$ *are given by* $\mathbb{E} = (L^{-*}\mathbb{Y})^*(L\mathbb{X})$, *where* $L \in \mathcal{GB}(U)$.

Theorems 5.2.7 and 5.2.8 below list classes that are closed w.r.t. spectral factorization and for which the converse of (b) holds (i.e., the Toeplitz operator $\pi_+\mathbb{E}\pi_+$ is invertible iff $\mathbb{E}$ has a spectral factorization).

The uniqueness result in (f) says that $X := \widehat{\mathbb{X}}(+\infty) \in \mathcal{GB}(U)$ can be chosen arbitrarily, and it determines $\mathbb{X}$, $\mathbb{Y}$ and $Y = EX^{-1} \in \mathcal{GB}(U)$ (we have $E,X,Y \in \mathcal{GB}(U)$, by Proposition 6.3.1(a3)).

Proof: (Part of this is given in Lemma 4.3 of [S98c].)

(a) This is Lemma 4.3(iv) of [S98c], but its proof needs clarification for the unseparable case: use Theorem 3.7, p. 54 and Theorem 3.4, p. 50 of [RR] to obtain $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$, where $\operatorname{Ran}(\mathbb{Z}) = \mathrm{L}^2(\mathbf{R};U_2)$ for some closed $U_2 \subset U$ (because $\mathbb{Z}$ is coercive and "outer"), find $E \in \mathcal{GB}(U_2,U)$ (such an E exists, by Lemma 2.2.1(c4)) and set $\mathbb{X} := E\mathbb{Z}$.

(We would obtain an alternative, control-theoretic proof of (a) (with $\mathbb{D}^*J\mathbb{D}$ in place of $\mathbb{E}$, $\mathbb{D}\in$ TIC) as in Theorem 14.3.2.)

(b) Using the fact that $\pi_+\mathbb{Y}^{-*}\pi_+ = \pi_+\mathbb{Y}^{-*}$ etc. (by causality), one easily verifies that $\pi_+\mathbb{X}^{-1}\pi_+\mathbb{Y}^{-*}\pi_+$ is the inverse of $\pi_+\mathbb{Y}^*\mathbb{X}\pi_+$ on $\pi_+\mathrm{L}^2$ (i.e., that their product is π_+).

(c) Because $\mathbb{X} = \mathbb{Y}^{-*}\mathbb{E} \in \mathrm{TI}_{-\omega}$, we have $\mathbb{X}^{-1} \in \mathrm{TI}_{-\varepsilon'>0}$ for some $\varepsilon' > 0$, by Lemma 2.2.7, hence $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{-\varepsilon}\cap\mathrm{TI}_{-\omega}\cap\mathrm{TIC} = \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U,Y)\cap\mathrm{TIC}_{-\omega}(U,Y)$ for some $\varepsilon > 0$. The same holds for $\mathbb{Y}$, because $\mathbb{Y} = \mathbb{X}^{-*}\mathbb{E}^*$.

(d) Now $\mathbb{E} = \mathbb{Y}^*\mathbb{X} = \mathbb{E}^* = \mathbb{X}^*\mathbb{Y}$, hence $\mathbb{Y} = S\mathbb{X}$ for some $S \in \mathcal{GB}(U)$, by (f). The latter claim is obtained from Proposition 5.2.2 (because $\mathbb{E} \in \mathrm{TI}_\omega \Rightarrow \mathbb{E}^* \in \mathrm{TI}_{-\omega}$).

(e) Obviously $\mathbb{E}^{\mathrm{d}} = \mathbb{Y}^*\mathbb{X} \Rightarrow \mathbb{E} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$, and, by Lemma 2.2.3, $\mathbb{X} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$.

(f) Let $\mathbb{E} = \mathbb{Y}_0^*\mathbb{X}_0$ also be a spectral factorization. Then $L := \mathbb{Y}_0^{-*}\mathbb{Y}^* = \mathbb{X}_0\mathbb{X}^{-1} \in$ TIC and $L^* = \mathbb{Y}\mathbb{Y}_0^{-1} \in$ TIC, hence $L \in \mathcal{B}(U)$, by Lemma 2.1.7. Obviously, $L = \mathbb{X}_0\mathbb{X}^{-1}$ is invertible. □

In part (c) above, we stated that the spectral factorization of an "exponentially stable" map is exponentially stable. Below we shall prove this claim and the fact that the same holds with MTI or something analogous in place of TI:

Proposition 5.2.2 (Exponentially stable SpF) *Let $\mathcal{A} \subset$ TI be inverse-closed and adjoint-closed (cf. Theorem 4.1.1 and Lemma 4.1.3), and set $\widetilde{\mathcal{A}} := \mathcal{A}\cap$ TIC and*

$$\mathcal{A}_r := \{\mathrm{e}^{r\cdot}\mathbb{E}\mathrm{e}^{-r\cdot} \bigm| \mathbb{E}\in\mathcal{A}\}, \quad \widetilde{\mathcal{A}}_r := \{\mathrm{e}^{r\cdot}\mathbb{D}\mathrm{e}^{-r\cdot} \bigm| \mathbb{D}\in\widetilde{\mathcal{A}}\}, \quad r\in\mathbf{R}. \tag{5.8}$$

Assume that $\omega > 0$, $\mathbb{E} \in \mathcal{A}_{-\omega}(U)\cap\mathcal{A}_\omega(U)$ (e.g., $\mathbb{E} = \mathbb{E}^* \in \mathcal{A}_{-\omega}(U)$), and $\mathbb{E}$ has the spectral factorization $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, where $\mathbb{Y},\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}(U)$.

Then $\mathbb{Y},\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y)\cap\widetilde{\mathcal{A}}_{-\omega}(U,Y)$ for some $\varepsilon > 0$; in particular, $\mathbb{X}^{\pm1}$ and $\mathbb{Y}^{\pm1}$ are exponentially stable.

In particular, if $\mathbb{E} = \mathbb{E}^* \in$ MTI is "exponentially MTI", i.e., $\mathbb{E} \in \mathrm{MTI}_{-\varepsilon}$ for some $\varepsilon > 0$, then its (possible) spectral factors are "exponentially MTI".

Proof: Because $\mathbb{X} = \mathbb{Y}^{-*}\mathbb{E} \in \mathcal{A}_{-\omega}$ (recall that $\{\mathbb{F}^* \bigm| \mathbb{F}\in\mathcal{A}_\omega\} = \mathcal{A}_{-\omega}$, by Lemma 4.1.3(b)), we have $\mathbb{X}^{-1} \in \mathcal{A}_{-\varepsilon}$ for some $\varepsilon > 0$, by Lemma 2.2.7. But $\mathcal{A}_r\cap\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{A}}_r$ for $r\in\mathbf{R}$, hence $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y)\cap\widetilde{\mathcal{A}}_{-\omega}(U,Y)$. Similarly, $\mathbb{Y} = \mathbb{X}^{-*}\mathbb{E}^* \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y)\cap\widetilde{\mathcal{A}}_{-\omega}(U,Y)$. □

Now we turn our attention to MTI maps. Our first task is to show that the invertibility of the Toeplitz operator of $\mathbb{E} \in \mathrm{MTI}$ implies that of its discrete part:

Lemma 5.2.3 (MTI Toeplitz) *For* $\mathbb{E} \in \mathrm{MTI}$ *(*$= \mathrm{MTI_d} + \mathrm{L}^1*$*) we write* $\mathbb{E}_\mathrm{d} := \Pi(\mathbb{E})$ *for the discrete part (of the form* $\sum_{k=1}^\infty L_k\delta(\cdot - t_k)*$*)*

Let $\mathbb{E} \in \mathrm{MTI}(U,Y)$.

(a) If $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ *then* $\mathbb{E}_\mathrm{d} \in \mathcal{G}\mathrm{MTI_d}$.

(b) If $\mathbb{E} \in \mathrm{MTI}(U,Y)$ *and* $\pi_+\mathbb{E}\pi_+ \in \mathcal{GB}(\mathrm{L}^2)$*, then* $\pi_+\mathbb{E}_\mathrm{d}\pi_+ \in \mathcal{GB}(\mathrm{L}^2)$ *(and* $\mathbb{E}, \mathbb{E}_\mathrm{d} \in \mathcal{G}\mathrm{TI}$*)*

(c) If $\mathbb{E} \in \mathrm{MTI}(U)$ *and* $\mathbb{E} \geq 0$*, then* $\mathbb{E}_\mathrm{d} \geq 0$.

Proof: (a) This is contained in Theorem 4.1.1.

(b) We will prove that if the Toeplitz operator $\mathbf{T}_\mathbb{E} := \pi_+\mathbb{E}\pi_+$ of $\mathbb{E} \in \mathrm{MTI}$ is coercive, i.e., $\|\mathbf{T}_\mathbb{E} u\| \geq \varepsilon u$ for all $u \in \pi_+\mathrm{L}^2$, then so is $\mathbf{T}_{\mathbb{E}_\mathrm{d}}$ (with the same $\varepsilon > 0$).

Claim (b) follows from this, because T is invertible iff T and T^* are coercive, by Lemma A.3.1(c3). (The last two claims follow from (a) and Lemma 2.2.2(a1).)

Define $\mathbb{F} \in \mathrm{MTI_d}$ and $f \in \mathrm{L}^1$ by $\mathbb{F} := \mathbb{E}_\mathrm{d}$ and $f* := \mathbb{E} - \mathbb{F}$ (so that $\mathbb{E} = \mathbb{F}u + f * u$ for all $u \in \mathrm{L}^2$).

Let $\delta > 0$ be arbitrary. Let $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ be otherwise arbitrary but $\|u\|_2 = 1$. By Lemma D.1.11(b), there is $T_\delta > 0$ s.t.

$$T > T_\delta \Longrightarrow \|\widehat{f}(\cdot)\widehat{u}(\cdot - iT)\|_2 < \delta. \tag{5.9}$$

Because $\widehat{\mathbb{F}}(i\cdot)$ is almost periodic, by Lemma C.1.2(h2), there is $T > T_\delta$ s.t. $\|\widehat{\mathbb{F}}(it) - \widehat{\mathbb{F}}(i(t-T))\| < \delta$ for all $t \in \mathbf{R}$. Therefore (recall that $\widehat{\pi_+}\widehat{u} := \widehat{\pi_+ u}$, hence $\|\widehat{\pi_+}\| = \|\pi_+\| = 1$; note also that $\|e^{iT\cdot}u\|_2 = \|u\|_2 = 1$ and $\mathcal{L}(e^{iT\cdot}u) = \widehat{u}(\cdot - iT)$)

$$\begin{aligned}
\|\pi_+\mathbb{F}u\|_2 = {} & \|\pi_+ e^{iT\cdot}\mathbb{F}u\|_2 && = \|\widehat{\pi_+}\mathcal{L}(e^{iT\cdot}\mathbb{F}u)\|_2 \\
= {} & \|\widehat{\pi_+}\widehat{(\mathbb{F}u)}(\cdot - iT)\|_2 && = \|\widehat{\pi_+}\widehat{\mathbb{F}}(\cdot - iT)\widehat{u}(\cdot - iT)\|_2 \\
\geq {} & \|\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{u}(\cdot - iT)\|_2 - \|\widehat{\pi_+}[\widehat{\mathbb{F}}(\cdot - iT) - \widehat{\mathbb{F}}]\widehat{u}(\cdot - iT)\|_2 \\
\geq {} & \|\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{u}(\cdot - iT)\|_2 - \|\delta\widehat{u}\|_2 && \geq \|\widehat{\pi_+}\widehat{\mathbb{E}}\widehat{u}(\cdot - iT)\|_2 - \|\widehat{\pi_+}\widehat{f}\widehat{u}(\cdot - iT)\|_2 - \delta \\
\geq {} & \|\widehat{\pi_+}\widehat{\mathbb{E}}\widehat{u}(\cdot - iT)\|_2 - \delta - \delta && = \|\pi_+\mathbb{E}e^{iT\cdot}u\|_2 - 2\delta \\
= {} & \|\pi_+\mathbb{E}\pi_+ e^{iT\cdot}u\|_2 - 2\delta && = \varepsilon - 2\delta.
\end{aligned}$$

Because $\delta > 0$ was arbitrary, $\|\pi_+\mathbb{F}u\|_2 \geq \varepsilon = \varepsilon\|u\|_2$.

(c) To obtain a contradiction, assume that $\mathbb{E} \in \mathrm{MTI}(U)$, $\mathbb{E} \geq 0$ and $\mathbb{E}_\mathrm{d} \not\geq 0$. Then $\widehat{\mathbb{E}} \geq 0$ on $i\mathbf{R}$ but there is $u_0 \in U$ s.t. $g := \langle u_0, \widehat{\mathbb{E}}_\mathrm{d}(\cdot)u_0\rangle_U$ satisfies $g(ir_0) \not\geq 0$ for some $r_0 \in \mathbf{R}$.

Set $\delta := d(g(ir_0), [0, +\infty))0$. Let $f \in \mathrm{L}^1(\mathbf{R}; \mathcal{B}(U))$ be the one for which $\mathbb{E} = \mathbb{E}_\mathrm{d} + f*$. By Lemma D.1.11(b), there is $T_\delta > |r_0|$ s.t. $\|\widehat{f}(ir)\|_{\mathcal{B}(U)} < \delta$ for $|r| > T_\delta$.

Because $\widehat{\mathbb{E}}_{\rm d}(i\cdot)$ is almost periodic, by Lemma C.1.2(h2), there is $T > 2T_\delta$ s.t. $\|\widehat{\mathbb{E}}_{\rm d}(it)-\widehat{\mathbb{E}}_{\rm d}(i(t-T))\| < \delta/2$ for all $t\in\mathbf{R}$. Then the distance from

$$\langle u_0,\widehat{\mathbb{E}}(ir_0+T)u_0\rangle = g(ir_0+T)+\langle u_0,\widehat{f}(ir_0+T)u_0\rangle \tag{5.10}$$

to $[0,+\infty)$ is greater than $\delta-\delta/2>0$, a contradiction, as required. $\square$

Next we state the spectral factorization results of Yuri Karlovich [Karlovich91] and others for matrix-valued atomic measures:

Lemma 5.2.4 *Let* $\mathbb{E}=\mathrm{MTI}_{\rm d}(\mathbf{C}^n)$*. Then* $\mathbb{E}$ *has a spectral factorization* $\mathbb{E}=\mathbb{Y}^*\mathbb{X}$ *with* $\mathbb{Y},\mathbb{X}\in\mathcal{G}\mathrm{MTIC}_{\rm d}(\mathbf{C}^n)$ *iff* $\pi_+\mathbb{E}\pi_+\in\mathcal{G}(\pi_+\mathrm{L}^2)$.

Moreover, if $\mathrm{supp}_{\rm d}(\mathbb{E})\subset\mathbf{S}$*, where* $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$*, then* $\mathrm{supp}_{\rm d}(\mathbb{X}),\mathrm{supp}_{\rm d}(\mathbb{Y})\subset\mathbf{S}$.

In particular, if the atoms of $\widehat{\mathbb{E}}$ are at points nT, $n\in\mathbf{Z}$ for some $T>0$, then so are those of $\mathbb{X}$.

(Note from Lemma 5.2.1(f) that the factorization is unique modulo a multiplicative constant.)

Proof: The first claim is a rephrasement of the equivalence "4)$\Leftrightarrow$6)" of Theorem 7 of [Karlovich93] (use the fact that $\mathbb{E}\pi_+ + \pi_-$ is invertible on $\mathrm{L}^2(\mathbf{R};\mathbf{C}^n)$ iff $\pi_+\mathbb{E}\pi_+$ is invertible on $\mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^n)$, by Lemma A.1.1(b1)&(b2)).

The $\mathbf{S}$-claim follows from [RSW, Theorem 6.1]. $\square$

We shall use the following lemma in the positive case:

Lemma 5.2.5 (MTI$_{\rm d}$ SpF when $\|I-\mathbb{E}\|<1$) *Assume that* $\mathbb{E}\in\mathrm{MTI}_{\rm d}(U)$ *and* $\|I-\mathbb{E}\|_{\mathrm{TI}(U)}<1$*. Then* $\mathbb{E}=\mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X},\mathbb{Y}\in\mathcal{G}\mathrm{MTIC}_{\rm d}(U)$.

Moreover, if $\mathrm{supp}_{\rm d}(\mathbb{E})\subset\mathbf{S}$*, where* $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$*, then* $\mathrm{supp}_{\rm d}(\mathbb{X}),\mathrm{supp}_{\rm d}(\mathbb{Y})\subset\mathbf{S}$.

Proof: 1° $\mathbb{E}=\mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X},\mathbb{Y}\in\mathcal{G}\mathrm{MTIC}_{\rm d}(U)$*:* Condition $\|I-\mathbb{E}\|_{\mathrm{TI}(U)}<1$ is equivalent to condition $\sup_{t\in\mathbf{R}}\|I-\widehat{\mathbb{E}}(it)\|_{\mathcal{B}(U)}<1$, by Theorem 3.1.3(d) and Theorem 2.6.4(e1).

By Theorem I of [BR], it follows that $\mathbb{E}=\mathbb{Y}^*\mathbb{X}$ for some $\mathbb{X},\mathbb{Y}\in\mathrm{MTIC}_{\rm d}(U)$.

2° $\mathrm{supp}_{\rm d}(\mathbb{X}),\mathrm{supp}_{\rm d}(\mathbb{Y})\subset\mathbf{S}$*:* Assume that $\mathrm{supp}_{\rm d}(\mathbb{E})\subset\mathbf{S}$ and $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$. We shall examine the proof of Theorem I so as to show that $\mathrm{supp}_{\rm d}(\mathbb{X}),\mathrm{supp}_{\rm d}(\mathbb{Y})\subset\mathbf{S}$. (Note that the Fourier transform of [BR] has an extra reflection; this is just a matter of notation.)

By Lemma C.1.2(f5), we have $\|A-I\|_{\mathcal{B}(\overline{\mathrm{AP}})}\le\|A-I\|_\infty<1$. From Lemma C.1.2(f4) we observe that $\|\Pi_{[0,+\infty)}\|_{\mathcal{B}(\overline{\mathrm{AP}})}\le 1$. Consequently,

$$\mathbb{X}_+\phi := \sum_{k\in\mathbf{N}}(\Pi_{[0,+\infty)}(I-A)\Pi_{[0,+\infty)})^k\phi \tag{5.11}$$

converges in $\overline{\mathrm{AP}}(\mathbf{R};U)$, for all $\phi\in U$, where $\overline{\mathrm{AP}}$ denotes the Besicovich space (see Lemma C.1.2(f4)). Since the atoms of $(\Pi_{[0,+\infty)}(I-A)\Pi_{[0,+\infty)})^k$ belong to $\mathbf{S}\cup\{0\}=\mathbf{S}$, for each $k\in\mathbf{N}$, also the atoms of $\mathbb{X}_+\phi$ belong to $\mathbf{S}$. Because $\phi\in U$

was arbitrary, the atoms of $\mathbb{X}_+$ belong to $\mathbf{S}$. By p. 18 of [BR], $\mathbb{X}_+ \in \widehat{\mathrm{MTIC}_\mathrm{d}}$, hence $\mathbb{X}_+ \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}$,

By analogous proofs, one shows that $\mathbb{Y}_+ \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}$ and $\mathbb{X}_- \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}^*$. From the formula $A = (I + \mathbb{X}_-)\mathbb{Y}_+$ of p. 51 of [GL-Identity] (which is the basis of the proof of [BR]), we observe that $\mathbb{Y}_+ = \mathbb{X}$, $I + \mathbb{X}_- = \mathbb{Y}^*$, where $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ is a spectral factorization of $\mathbb{E}$. This completes the proof (recall Lemma 5.2.1(f)). $\square$

Indeed, the positive case is obtained as a corollary:

Corollary 5.2.6 *Let* $\mathbb{E} \in \mathrm{MTI}_\mathrm{d}(U)$. *Then* $\mathbb{E} \gg 0 \Leftrightarrow \mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d}(U)$.

Moreover, if $\mathrm{supp}_\mathrm{d}(\mathbb{E}) \subset \mathbf{S}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *then* $\mathrm{supp}_\mathrm{d}(\mathbb{X}), \mathrm{supp}_\mathrm{d}(\mathbb{Y}) \subset \mathbf{S}$.

Proof: Set $\mathbb{F} := \mathbb{E}/\|\mathbb{E}\|_{\mathrm{TI}} \in \mathrm{MTI}_\mathrm{d}(U)$. Then $I \ge \mathbb{F} \gg 0$, hence $\|I - \mathbb{F}\| < 1$, by Lemma A.3.1(b9). By Lemma 5.2.5, it follows that $\mathbb{F} = \mathbb{Y}^*\mathbb{Z}$ for some $\mathbb{Y}, \mathbb{Z} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. Use Lemma 5.2.1(a)&(f) to observe that $\mathbb{F} = \widetilde{\mathbb{X}}^*\widetilde{\mathbb{X}}$ for some $\widetilde{\mathbb{X}} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. Set $\mathbb{X} := \|\mathbb{E}\|_{\mathrm{TI}}^{1/2}\widetilde{\mathbb{X}} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. $\square$

Now we can combine the above results to two theorems:

Theorem 5.2.7 (MTI spectral factorization) *Let* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *and let either (1.) or (2.) hold, where*

(1.) $\mathcal{A}$ *is one of the classes* $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_{T\mathbf{Z}}$, $\mathrm{MTI}_{T\mathbf{Z}}^{\mathcal{BC}}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, *and* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}^{\mathcal{BC}}$;

(2.) $\dim U < \infty$ *and* $\mathcal{A}$ *is one of the classes* MTI, MTI_d, $\mathrm{MTI}_\mathbf{S}$, *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$.

Let $\mathbb{E} \in \mathcal{A}(U)$, *and set* $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$. *Then the Toeplitz operator* $\mathbf{T}_\mathbb{E} := \pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ *is invertible iff* $\mathbb{E}$ *has a spectral factorization*

$$\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \text{ where } \mathbb{X}, \mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{5.12}$$

Let $\mathbb{E}$, $\mathbb{X}$ *and* $\mathbb{Y}$ *be as above. Then* $\mathbb{E}_\mathrm{d} = \mathbb{Y}_\mathrm{d}^*\mathbb{X}_\mathrm{d}$ *is also a spectral factorization (in* MTI_d*), where* $()_\mathrm{d}$ *denotes the discrete part. Moreover, if* $\mathrm{supp}_\mathrm{d}(\mathbb{E}) \subset \mathbf{S}$, *then* $\mathrm{supp}_\mathrm{d}(\mathbb{X}) \subset \mathbf{S}$ *and* $\mathrm{supp}_\mathrm{d}(\mathbb{Y}) \subset \mathbf{S}$.

If, in addition, $\omega > 0$ *and* $\mathbb{E} \in \mathcal{A}_{-\omega} \cap \mathcal{A}_\omega$, *then* $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$ *for some* $\varepsilon > 0$; *in particular,* $\mathbb{X}^{\pm 1}$ *and* $\mathbb{Y}^{\pm 1}$ *are exponentially stable.*

If we merely know that $\mathbb{E} \in \mathrm{TI}$, then it is no longer guaranteed that the "canonical factors" $\mathbb{X}$ and $\mathbb{Y}$ are stable, see the notes on p. 148 for details.

However, positive results can be given also for certain other cases, see Theorems 5.2.8 and 9.2.14.

Proof: 1° *The discrete part* $\mathbb{E}_\mathrm{d} = \mathbb{W}^*\mathbb{Z}$:
If (5.12) holds, then $\mathbf{T}_\mathbb{E}^{-1} = \mathbf{T}_{\mathbb{X}^{-1}}\mathbf{T}_{\mathbb{Y}^{-1}}$, so assume that $\mathbf{T}_\mathbb{E}$ is invertible. Then so is $\mathbf{T}_{\mathbb{E}_\mathrm{d}}$, by Lemma 5.2.3(b), so we can factor $\mathbb{E}_\mathrm{d}$ as $\mathbb{W}^*\mathbb{Z}$, $\mathbb{W}, \mathbb{Z} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d}$ ($\mathbb{Z}, \mathbb{W} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d}^{\mathcal{BC}}$, if $\mathcal{A} = \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathcal{A} = \mathrm{MTI}_{T\mathbf{Z}}^{\mathcal{BC}}$, or $\mathcal{A} = \mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}^{\mathcal{BC}}$, because

in those cases $\mathbb{E}_{\rm d}\in \mathrm{MTI}_{\rm d}^{\mathcal{BC}}$) by Corollary 5.1.4 in case (1.) and by Lemma 5.2.4 in case (2.).

Moreover, if $\mathrm{supp_d}(\mathbb{E})\subset \mathbf{S}$, then $\mathrm{supp_d}(\mathbb{Z}),\mathrm{supp_d}(\mathbb{W})\subset \mathbf{S}$, by Lemma 5.2.4 in case (2.); in case (1.) either $\mathbf{S}=\{0\}$, in which case $\mathbb{E},\mathbb{Z},\mathbb{W}\in \mathcal{B}(U)$, or $\mathbf{S}$ contains a set of the form $T'\mathbf{Z}$ for some $T'>0$, in which case $\mathrm{supp_d}(\mathbb{Z}),\mathrm{supp_d}(\mathbb{W})\subset T'\mathbf{Z}\subset\mathbf{S}$, by Corollary 5.1.4.

If $\mathbb{E}-\mathbb{E}_{\rm d}=0$, then we can take $\mathbb{X}:=\mathbb{Z}$, $\mathbb{Y}:=\mathbb{W}$ to obtain the required result, but in the general case we proceed as follows.

2° *The absolutely continuous part* $\mathbb{E}_{\rm ac}:=\mathbb{E}-\mathbb{E}_{\rm d}$*:*
Because L^1* is an ideal of MTI, and $\mathbb{E}_{\rm ac}\in\mathrm{L}^1*$, we have $\mathbb{W}^{-*}\mathbb{E}_{\rm ac}\mathbb{Z}^{-1}=g*$ for some $g\in\mathrm{L}^1$, hence

$$\mathbb{W}^{-*}(\mathbb{E}_{\rm d}+\mathbb{E}_{\rm ac})\mathbb{Z}^{-1}=I+g \tag{5.13}$$

can be factorized as $\mathbb{T}^*\mathbb{S}$ with $\mathbb{T},\mathbb{S}\in\mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$ (with $\mathbb{T},\mathbb{S}\in\mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$, if ($\mathbb{E}_{\rm ac}$ and hence) $g*\in\mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))*$), by Theorem 5.1.2 and the invertibility of $\mathbf{T}_{\mathbb{W}^{-*}\mathbb{E}\mathbb{Z}^{-1}}$, which follows from Lemma 2.2.2(b).

Thus, we have $\mathbb{E}=\mathbb{Y}^*\mathbb{X}$, where $\mathbb{Y}:=\mathbb{T}\mathbb{W}\in\mathcal{G}\mathrm{MTIC}$ and $\mathbb{X}:=\mathbb{S}\mathbb{Z}\in\mathcal{G}\mathrm{MTIC}$. Moreover, $\mathbb{Y}_{\rm d}=\mathbb{T}_{\rm d}\mathbb{W}_{\rm d}=\mathbb{W}_{\rm d}=\mathbb{W}$ and $\mathbb{X}_{\rm d}=\mathbb{S}_{\rm d}\mathbb{Z}_{\rm d}=\mathbb{Z}_{\rm d}=\mathbb{Z}$, hence $\mathbb{E}_{\rm d}=\mathbb{Y}_{\rm d}^*\mathbb{X}_{\rm d}$.

The last paragraph of the theorem follows from Proposition 5.2.2. □

The assumption that the input space U must be finite-dimensional, is probably superfluous even in case (2.); there may be expected results in this direction in the near future. We have written this work based more on hypotheses (see Hypothesis 8.4.7) than on classes, in order for the reader to easily incorporate any new factorization results to this work.

In the uniformly positive case (which can be used for minimization problems, positive and bounded real lemmas and analogous), we do not need any dimensionality restrictions:

Theorem 5.2.8 (Positive MTI SpF) *Let U be a Hilbert space, let $\mathbf{S}=\mathbf{S}-\mathbf{S}\subset\mathbf{R}$, and let $\mathcal{A}$ be one of the classes* TI, MTI, $\mathrm{MTI_d}$, $\mathrm{MTI}_{\mathbf{S}}$, $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_{T\mathbf{Z}}$, $\mathrm{MTI}_{T\mathbf{Z}}^{\mathcal{BC}}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, *and* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}^{\mathcal{BC}}$.

Let $\mathbb{E}\in\mathcal{A}(U)$, and set $\widetilde{\mathcal{A}}:=\mathcal{A}\cap$ TIC. *Then $\mathbb{E}\gg 0$ iff $\mathbb{E}$ has a spectral factorization*

$$\mathbb{E}=\mathbb{X}^*\mathbb{X},\ \textit{where}\ \mathbb{X}\in\mathcal{G}\widetilde{\mathcal{A}}(U). \tag{5.14}$$

Let $\mathbb{E}$ and $\mathbb{X}$ be as above. Then $\mathbb{E}_{\rm d}=\mathbb{X}_{\rm d}^*\mathbb{X}_{\rm d}$ is also a spectral factorization (in $\mathrm{MTI_d}$*), where $\cdot_{\rm d}$ denotes the discrete part. Moreover, if* $\mathrm{supp_d}(\mathbb{E})\subset\mathbf{S}$*, then* $\mathrm{supp_d}(\mathbb{X})\subset\mathbf{S}$.

If, in addition, $\omega>0$ and $\mathbb{E}\in\mathcal{A}_{-\omega}$, then $\mathbb{X}\in\mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y)\cap\widetilde{\mathcal{A}}_{-\omega}(U,Y)$ for some $\varepsilon>0$.

If $\mathbb{E}\geq 0$, then $\mathbb{E}\gg 0$ iff $\mathbb{E}$ is invertible; equivalently, iff $\pi_+\mathbb{E}\pi_+$ is invertible (by Lemma A.3.1(b1) and Lemma 2.2.2(d)).

Proof: 1° If (5.14) holds, then, $\mathbb{E}\gg 0$, by, e.g., Lemma 5.2.1(a).

2° Let $\mathbb{E} \gg 0$. Then $\pi_+\mathbb{E}\pi_+ \gg 0$, by Lemma 2.2.2(d), hence the claims follow from Theorem 5.2.7 (because $\mathbb{E} = \mathbb{X}^*S\mathbb{X}$, by Lemma 5.2.1(d), and clearly $S \gg 0$, so $\mathbb{E} = (S^{1/2}\mathbb{X})^*(S^{1/2}\mathbb{X})$, $\mathbb{X} \in \widetilde{\mathcal{A}}$) when we note that the assumption $\dim U < \infty$ in the proof of Theorem 5.2.7 can be removed in this positive case, by using, in case (2.), Corollary 5.2.6 instead of Lemma 5.2.4 and Lemma 5.2.3(c) instead of (b).

3° The exponential (measure) stability is obtained as in Theorem 5.2.7. □

We have presented our spectral factorization results for several subclasses of MTI in order to get more specific information on the smoothness of the spectral factors. For example, the uniform half-plane-regularity of $\mathrm{MTIC}^{\mathrm{L}^1}$ maps allows us to even simplify the Riccati equations, provided that the Popov operator belongs to $\mathrm{MTI}^{\mathrm{L}^1}$.

Notes

Except for (e) and the $\mathrm{TIC}_{\mathrm{exp}}$ claims, Lemma 5.2.1 is contained in Lemma 4.3 of [S98c]. Proposition 5.2.2 was established in the finite-dimensional positive $\mathrm{MTI}_{T\mathbf{Z}}$ case in Lemma 3.3 of [Winkin], by using analytic extensions.

For finite-dimensional U, most of Theorem 5.2.8 (for $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{\mathrm{d}}$ and $\mathrm{MTI}_{T\mathbf{Z}}$) is contained in Theorem 3.1M of [Winkin] (and in [CW99]). Also Lemma 5.2.3(c) and our strategy to start with a factorization of the discrete part are from [Winkin]. (The space U is assumed to be finite-dimensional and $\mathbb{E}$ is assumed to be uniformly positive everywhere in [Winkin].)

As obvious from the proof, Lemma 5.2.4 is essentially contained in [RSW] (originating in [Karlovich91] and the joint articles of Yuri Karlovich and Ilya Spitkovsky et al.).

Lemma 5.2.5 and Corollary 5.2.6 are essentially contained in [BR] except for the claims on $\mathbf{S}$. Our proofs use the ideas of the proof of Theorem 6.1 of [RSW]. The exact assumption in [BR] is "$|\langle\widehat{\mathbb{E}}u_0, u_0\rangle| \geq \varepsilon\|u_0\|_U^2$", hence slightly more general than "$\mathbb{E} \gg 0$" (but does not allow for, e.g., $\mathbb{E} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$). See [Karlovich93] (particularly Theorems 14 and 15) for similar (not analogous) factorization results for [semi-]almost periodic functions (with values in $\mathbf{C}^{n\times n}$), for further equivalent conditions and for factorizations of functions with non-invertible Toeplitz operators.

For any $\mathbb{E} \in \mathrm{TI}$ s.t. $\mathbb{E} \geq 0$, the invertibility of the Toeplitz operator is equivalent to the existence of a spectral factorization, by Theorem 5.2.8. For a general $\mathbb{E} \in \mathrm{TI}$ s.t. $\pi_+\mathbb{E}\pi_+$ is invertible, we need an extra assumption, as in Theorems 5.2.7, Theorem 9.2.14 or 9.14.4.

Indeed, for a general $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^n)$ s.t. $\pi_+\mathbb{E}\pi_+$ is invertible, we only know the existence of a "Generalized canonical factorization" (see Section 9.15, [CG81] or [LS]) where $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$ and the Cayley transforms of $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathrm{H}(\mathbf{C}^+; \mathbf{C}^{n\times n})$ are invertible in H^2 over the unit disc. By Example 8.4.13, these (unique modulo a multiplicative constant) generalized factors need not be well-posed (i.e., $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{Y}}$ may be unbounded at infinity) unless $n = 1$.

In and below Theorem 9.14.6, we extend the above results on generalized canonical factorization for infinite-dimensional U (assuming that $\mathbb{E}$ is the Toeplitz operator corresponding to some cost function; this is the only case for which such factorizations are needed in control theory). In this weaker result, $\widehat{\mathbb{X}}(\bar{\cdot})^*$ and $\widehat{\mathbb{X}}^{-1}$ are only known to be $\mathrm{H}^2_{\text{strong}}$ over the unit disc ($\mathrm{H}^2_{\text{strong}}(\mathbf{D}; \mathcal{B}(U))$).

Obviously, the Cayley transform makes spectral factorization of $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U))$ maps equivalent to the spectral factorization of $\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$ maps. Unlike in continuous time, the generalized discrete-time canonical factors are always well-posed, by Theorem 9.14.6, but still not necessarily stable, as noted in Example 8.4.13. See also the notes on pp. 141 and 543.