Part II

Continuous-Time Control Theory

Chapter 6

Well-Posed Linear Systems (WPLS)

It must be remembered that there is nothing more difficult to plan, more doubtful of success, nor more dangerous to manage, than the creation of a new system. For the initiator has the enmity of all who would profit by the preservation of the old institutions and merely lukewarm defenders in those who would gain by the new ones.

— Niccolo Machiavelli (1469–1527)

In this chapter, we shall present a theory on most aspects WPLSs; only dynamic stabilization is left for the next chapter.

Section 6.1 treats the basic properties of WPLSs: stability, realization theory, generators and dual systems.

In Section 6.2, we list the basic facts on regularity, which means the existence of a feedthrough operator in a weak sense. This leads to generalizations of classical state-space and frequency-space formulae for the maps that comprise the system, e.g., of equations $x' = Ax + Bu,\ y = Cx + Du$ for the state and output and equation $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ for the transfer function are established in a weak sense using the Weiss extension of C.

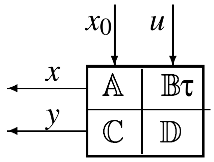
Not all WPLSs have a feedthrough operator D, but there are always ways to define a *compatible pair* (C_{ext}, D) s.t. the above formulae become valid; however this theory is less fruitful and hence less important than that of regular WPLSs. Compatible pairs are studied in Section 6.3, where we also present further results on regularity, [strong or weak] L^p impulse responses and H^p transfer functions, relations between a WPLS and its generators, and reachability and observability.

In Sections 6.4 and 6.5, we define and study coprime, spectral and lossless factorizations. The importance of these factorizations is due the fact that in many control problems, the existence of a (nonsingular) solution is equivalent to the equivalence of such a factorization; also dynamic feedback is intimately connected to coprime factorization. We also present two weak forms of coprimeness that are useful in infinite-dimensional settings, the weaker of them being invariant under (inverse) discretization and hence allowing us to reduce several results to the simpler discrete-time theory.

Sections 6.6 and 6.7 treat state feedback, output injection and static output feedback. In Section 6.8, we study systems whose semigroup is smoothing (e.g., $\mathbb{A}Bu_0 \in \mathrm{L}^p_{\text{loc}}(\mathbf{R}_+; H)$ for all $u_0 \in U$).

In Section 6.9, we show that a transfer function $\widehat{\mathbb{D}}$ has a realization with bounded B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane. We also establish analogous results for realizations with bounded C and for Pritchard–Salamon realizations.

To get a shorter introduction to WPLSs, just read the parts of Sections 6.1, 6.2, 6.4 and 6.6 that seem interesting. Throughout this chapter, the letters H, U, W, Y, Z, H_k, U_k and Y_k ($k \in \mathbf{N}$) denote arbitrary (complex) Hilbert spaces.



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$

Figure 6.1: Input/state/output diagram of the system Σ

6.1 WPLS theory

> *Foolproof systems don't take into account the ingenuity of fools.*
> — Gene Brown

We start by defining WPLSs; see p. 22 for a motivation. Our formulation follows that of Olof Staffans. The correspondence to Weiss' notation is explained on pp. 158 and 166. Recall also that $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2 = \{f \,|\, \mathrm{e}^{-\omega\cdot} f \in \mathrm{L}^2\}$.

Definition 6.1.1 (WPLS and stability) *Let* $\omega \in \mathbf{R}$. *An* ω-stable well-posed linear system on (U,H,Y) *is a quadruple*[1] $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$, *where* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, *and* $\mathbb{D}$ *are bounded linear operators of the following type:*

1. $\mathbb{A}(t)\colon H \to H$ *is a strongly continuous semigroup of bounded linear operators on H satisfying* $\sup_{t\in\mathbf{R}_+} \|\mathrm{e}^{-\omega t}\mathbb{A}(t)\| < \infty$;

2. $\mathbb{B}\colon \mathrm{L}^2_\omega(\mathbf{R};U) \to H$ *satisfies* $\mathbb{A}(t)\mathbb{B}u = \mathbb{B}\tau(t)\pi_- u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *and* $t \in \mathbf{R}_+$;

3. $\mathbb{C}\colon H \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ *satisfies* $\mathbb{C}\mathbb{A}(t)x = \pi_+\tau(t)\mathbb{C}x$ *for all* $x \in H$ *and* $t \in \mathbf{R}_+$;

4. $\mathbb{D}\colon \mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ *satisfies* $\tau(t)\mathbb{D}u = \mathbb{D}\tau(t)u$, $\pi_-\mathbb{D}\pi_+ u = 0$, *and* $\pi_+\mathbb{D}\pi_- u = \mathbb{C}\mathbb{B}u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *and* $t \in \mathbf{R}$.

We write $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$ *(or* $\Sigma \in \mathrm{WPLS}$ *if we do not wish to specify stability).*

The different components of $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *are named as follows: U is the* input space, *H the* state space, *Y the* output space, $\mathbb{A}$ *the* semigroup, $\mathbb{B}$ *the* reachability map, $\mathbb{C}$ *the* observability map, *and* $\mathbb{D}$ *the* I/O map *(input/output map) of* Σ.

We allow the right column (or the bottom row) to be empty, e.g., we call $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\end{smallmatrix}\right]$ a WPLS iff 1. and 2. are satisfied; this is equivalent for $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ 0 & 0\end{smallmatrix}\right]$ being a WPLS (for any Y). The same applies to $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{C}\end{smallmatrix}\right]$.

Intuitively, the reachability map $\mathbb{B}$ maps past controls into the present state, the observability map $\mathbb{C}$ maps the present state into future observations, and the I/O map $\mathbb{D}$ maps inputs into outputs in a causal way. The condition "4." imposed on $\mathbb{D}$ requires that $\mathbb{D} \in \mathrm{TIC}_\omega(U;Y)$ and that the Hankel operator induced by $\mathbb{D}$ is equal to $\mathbb{C}\mathbb{B}$. The definitions of this section are more extensively explained in [Sbook] (or in [S97b]–[S98c]).

[1]This is an ordinary matrix with operator-valued elements (cf. Appendix A). The rule lines are just for making the recognition of elements easier, especially in the case of multiblock elements.

Obviously, axioms 2. and 3. imply that $\mathbb{B}\pi_+ = 0 = \pi_-\mathbb{C}$. Since $\pi_+\mathrm{L}^2_\omega \subset \pi_+\mathrm{L}^2_{\omega'}$ and $\pi_-\mathrm{L}^2_{\omega'} \subset \pi_-\mathrm{L}^2_\omega$, continuously, for any $\omega' > \omega$, we can increase ω in Definition 6.1.1 (use also Lemma 2.1.11) when we identify $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ with their unique continuous (restricted) extensions:

Lemma 6.1.2 ($\mathbf{WPLS_\omega \subset WPLS_{\omega'}}$) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$ *for some* $\omega \in \mathbf{R}$. *Then* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_{\omega'}$ *for all* $\omega' > \omega$. □

(This is Lemma 2.4 of [S98a].)

We also note that if $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$ and $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{H}\\ \mathbb{C} & \mathbb{G}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$, then $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{H} & \mathbb{B}\\ \mathbb{C} & \mathbb{G} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$. However, having $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{C}\end{smallmatrix}\right] \in \mathrm{WPLS}$ does not imply that $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$ for any $\mathbb{D}$ (unless B or C is bounded; see Lemma 6.3.16), by Example 6.3.24.

Definition 6.1.3 (Stability) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\omega \in \mathbf{R}$.

If axiom 1. (resp. 2., 3., 4.) of Definition 6.1.1 holds, then we say that $\mathbb{A}$ *(resp.* $\mathbb{B}$*,* $\mathbb{C}$*,* $\mathbb{D}$*) is* ω*-stable and* Σ *is* internally *(resp.* input, output, I/O*)* ω-stable.

If Σ *is [internally]* ω*-stable and* $\mathrm{e}^{-\omega t}\mathbb{A}(t)x \to 0$ *strongly (resp. weakly) as* $t \to \infty$ *for all* $x \in H$*, then* Σ *is* [internally] strongly (resp. weakly) ω-stable*, and* $\mathbb{A}$ *is* strongly (resp. weakly) ω-stable. *We call* $\mathbb{B}$ strongly (resp. weakly) ω-stable *if* $\mathbb{B}$ *is* ω*-stable and* $\mathbb{B}\tau^t u \to 0$ *strongly (resp. weakly), as* $t \to +\infty$*, for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*. For* $\mathbb{C}$ *and* $\mathbb{D}$*, "strongly stable" or "weakly stable" means "stable".*

If $\Sigma \in \mathrm{WPLS}_\omega$ *for some* $\omega < 0$*, then* Σ *is* exponentially stable.*The prefix "0-" is usually omitted (e.g.,* Σ *is strongly stable iff it is stable and* $\mathbb{A}(t)x_0 \to 0$ *as* $t \to +\infty$*, for all* $x_0 \in H$*).*

An output stable and I/O-stable WPLS is called a Stable-Output (Well-Posed Linear) System *(or SOS-stable); we denote such systems by* SOS *(or* $\mathrm{SOS}(U,H,Y)$*). Thus,* $\mathrm{WPLS}_0 \subset \mathrm{SOS} \subset \mathrm{WPLS} := \cup_{\omega\in\mathbf{R}}\mathrm{WPLS}_\omega$.

Sometimes "strongly stable" is called "asymptotically stable" and "exponentially stable" is called "uniformly stable". Obviously, "exponentially strongly" is equivalent to "exponentially".

If Σ is minimal and finite-dimensional, then it is stable iff it is exponentially stable. Whenever $\dim H < \infty$, the system is strongly (or weakly) stable iff it is exponentially stable (iff the eigenvalues of the generator of $\mathbb{A}$ have negative real parts). Therefore, in finite-dimensional control theory, "stable" usually means "exponentially stable" (and the same applies to stabilizability and detectability); this practise is common also in earlier infinite-dimensional theory, but less common in the theory of WPLSs.

The WPLS Σ is exponentially stable iff $\mathbb{A}$ is exponentially stable, by Lemma 6.1.10(a1), but the ω-stability of $\mathbb{A}$ does not imply that of Σ. If Σ is strongly stable, then so is $\mathbb{B}$ (and $\mathbb{A}$), by Lemma 6.1.13 that

SOS-stability means that the system maps any initial state $x_0 \in H$ and input $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ continuously to the output $y = \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$. It is the weakest assumption that allows one to use the stable case methods for most control problems (this assumption was made in [WW] too).

We define dual systems as in Proposition 6.1 of [WW]:

Lemma 6.1.4 (Dual system Σ^{d}) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$, $\omega \in \mathbf{R}$. *Then its* dual system *(or (causal) adjoint system)*

$$\Sigma^{\mathrm{d}} := \left[\begin{array}{c|c} \mathbb{A}^{\mathrm{d}} & \mathbb{C}^{\mathrm{d}} \\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}^* & \mathbb{C}^* \boldsymbol{Я} \\ \hline \boldsymbol{Я}\mathbb{B}^* & \boldsymbol{Я}\mathbb{D}^* \boldsymbol{Я} \end{array}\right] \tag{6.1}$$

is in $\mathrm{WPLS}_\omega(Y,H,U)$. *Moreover,* $(\Sigma^{\mathrm{d}})^{\mathrm{d}} = \Sigma$. □

(We leave the simple verification of the lemma to the reader.) For the generators ($\left[\begin{smallmatrix}A^* & C^* \\ B^* & \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}A^* & C^* \\ B^* & D^*\end{smallmatrix}\right]$) of Σ^{d}, see Lemma 6.1.16 and Lemma 6.2.9(b). Note that our dual systems are causal unlike those in [S95]–[S98d].

Here $(\boldsymbol{Я}u)(t) := u(-t)$ (hence $\boldsymbol{Я} : \mathrm{L}^2_\omega \to \mathrm{L}^2_{-\omega}$ is an isometric isomorphism). The adjoints are taken with respect to the L^2 inner product (i.e., without a weight function); e.g., for $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega(\mathbf{R};Y))$ we have that $\mathbb{C}^* \in \mathcal{B}(\mathrm{L}^2_{-\omega}(\mathbf{R};Y), H)$ and (cf. Lemma A.3.24)

$$\langle \mathbb{C}x, y\rangle_{\langle \mathrm{L}^2_\omega, \mathrm{L}^2_{-\omega}\rangle} := \int_{\mathbf{R}} \langle (\mathbb{C}x)(t), y(t)\rangle_Y \, dt = \langle x, \mathbb{C}^* y\rangle_H \quad (x \in H,\ y \in \mathrm{L}^2_{-\omega}(\mathbf{R};Y)). \tag{6.2}$$

(Equivalently, $\mathbb{C}^* = \mathbb{C}^{\mathrm{H}} e^{-2\omega\cdot}$, where $\mathbb{C}^{\mathrm{H}}$ is the adjoint of $\mathbb{C}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathrm{L}^2_\omega}$; note also that $\mathbb{C}^* = \mathbb{C}^*\pi_+$, $\mathbb{C}^{\mathrm{H}} = \mathbb{C}^{\mathrm{H}}\pi_+$.) This makes Σ^{d} independent of ω (cf. Lemma 6.1.2), and $\mathbb{C}^{\mathrm{d}}$ becomes α-stable iff $\mathbb{C}$ is α-stable for any $\alpha \in \mathbf{R}$; the same applies to $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$.

Note that $\mathrm{L}^2_{-\omega}$ is the dual of L^2_ω with respect to this (weightless) L^2 inner product. Note also that the standard involution rules apply, e.g., $(\mathbb{D}\mathbb{C})^* = \mathbb{C}^*\mathbb{D}^*$ and $(\mathbb{C}\mathbb{B})^* = \mathbb{B}^*\mathbb{C}^*$ regardless of ω.

In control theory, one usually assumes the system to have an initial value (at $t = 0$, w.l.o.g.) and a control (on $(0,\infty)$). Sometimes the system is assumed to be controlled from $-\infty$ to ∞ (usually the latter setting is only used in proofs of results for the former setting). We formulate this in detail:

Definition 6.1.5 (State and output — $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] : \left[\begin{smallmatrix}x_0 \\ u\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}x \\ y\end{smallmatrix}\right]$) *In the* initial value setting *with initial time zero,* initial value $x_0 \in H$, *and* control *(or* input*)* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, *the controlled* state $x(t) \in H$ *at time* $t \in \mathbf{R}_+$ *and the* output $y \in \mathrm{L}^2_\omega(\mathbf{R}_+,Y)$ *of* Σ *are given by (cf. Figure 6.1)*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{A}(t) & \mathbb{B}\tau(t) \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \\ \mathbb{C}x_0 + \mathbb{D}u \end{bmatrix}. \tag{6.3}$$

In the time-invariant setting, *the controlled state* $x(t) \in H$ *at time* $t \in \mathbf{R}$ *and the output* $y \in \mathrm{L}^2_\omega(\mathbf{R};Y)$ *of* Σ *with control (or input)* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *are given by*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{B}\tau(t)u \\ \mathbb{D}u \end{bmatrix}. \tag{6.4}$$

Sometimes we use the Salamon–Weiss notation ($\tau^t := \tau(t)$ and)

$$\begin{bmatrix} \mathbb{A}^t & \mathbb{B}^t \\ \mathbb{C}^t & \mathbb{D}^t \end{bmatrix} := \begin{bmatrix} \mathbb{A}(t) & \mathbb{B}\tau(t)\pi_{[0,t)} \\ \pi_{[0,t)}\mathbb{C} & \pi_{[0,t)}\mathbb{D}\pi_{[0,t)} \end{bmatrix} : \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x(t) \\ \pi_{[0,t)}y \end{bmatrix}. \tag{6.5}$$

Because of (6.3), we sometimes denote Σ by $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ (naturally, "$\Sigma : \left[\begin{smallmatrix}x_0 \\ u\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}x \\ y\end{smallmatrix}\right]$" refers to $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$).

To treat both settings at once, we sometimes allow for $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ in (6.3) (but we are interested only in cases where $x_0 = 0$ or $\pi_- u = 0$). By causality, the state and output are well defined for $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ too (with $y \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$); cf. Definition 2.1.1.

G. Weiss et al. use symbols $\begin{bmatrix}\mathbb{T}_t & \Phi_t \\ \Sigma_t & \mathbb{F}_t\end{bmatrix} := \begin{bmatrix}\mathbb{A}^t & \mathbb{B}^t \\ \mathbb{C}^t & \mathbb{D}^t\end{bmatrix}$, and define WPLSs by requiring this quadrable to be linear and continuous $H \times \mathrm{L}^2(\mathbf{R}_+;U) \to H \times \mathrm{L}^2(\mathbf{R}_+;Y)$ (i.e., by requiring Σ to be locally continuous) and to satisfy, instead of (2.)–(4.), the algebraic conditions

$$\mathbb{B}^{s+t}(u\Diamond_s v) = \mathbb{A}^t\mathbb{B}^s u + \mathbb{B}^t v, \tag{6.6}$$

$$\mathbb{C}^{s+t}x_0 = \mathbb{C}^s x_0 \Diamond_s \mathbb{C}^t\mathbb{A}^s x_0, \tag{6.7}$$

$$\mathbb{D}^{s+t}(u\Diamond_s v) = \mathbb{D}^s u \Diamond_s (\mathbb{C}^t\mathbb{B}^s u + \mathbb{D}^t v) \qquad (u,v \in \mathrm{L}^2(\mathbf{R}_+;U),\ x_0 \in H); \tag{6.8}$$

here $u\Diamond_s v := \pi_{[0,s)}u + \tau^{-t}v$ ($\mathbb{A}$ is still required to be a C_0-semigroup). By discretization (see Theorems 13.4.4 and 13.4.5, or the proof of Lemma 6.1.10), it then follows from Lemma 13.3.3(b) that Σ is ω-stable for any $\omega > \omega_A$ (where ω_A is the growth rate of $\mathbb{A}$), hence the two definitions of WPLSs (and that of D. Salamon) are equivalent (see [Sbook] for details).

Of these two notations, we try to use the one (often "$\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$") that leads to a simpler formula. Obviously, (6.3) implies that, for $s,t \geq 0$, we have

$$\begin{bmatrix}\tau^s x(t) \\ \pi_+\tau^s y\end{bmatrix} = \begin{bmatrix}x(t+s) \\ y(\cdot+s)\end{bmatrix} = \begin{bmatrix}\mathbb{A}^t & \mathbb{B}^t \\ \mathbb{C} & \mathbb{D}\end{bmatrix}\begin{bmatrix}x(s) \\ \pi_+\tau^s u\end{bmatrix}. \tag{6.9}$$

From axiom 4. of Definition 6.1.1 we conclude that

$$\pi_{[t,\infty)}\mathbb{D}\pi_{(-\infty,t)} = \tau^{-t}\mathbb{C}\mathbb{B}\tau^t \quad (t \in \mathbf{R}). \tag{6.10}$$

"Shift semigroup systems" are often used as realizations of given I/O maps; these systems are useful for WPLSs too; indeed, every well-posed I/O map has a realization as a WPLS:

Definition 6.1.6 (Realizations) *Let* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \in \mathbf{R}$. *If* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *for some Hilbert space* H*, then we call* Σ *(together with* H*) a* realization *of* $\mathbb{D}$.

We call the (strongly ω*-stable) system*

$$\Sigma_\omega := \left[\begin{array}{c|c}\pi_+\tau & \pi_+\mathbb{D}\pi_- \\ \hline I & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}_\omega(U,\mathrm{L}^2_\omega(\mathbf{R}_+;Y),Y) \tag{6.11}$$

the exactly ω-observable realization *of* $\mathbb{D}$.

The system $\Sigma_\omega \in \mathrm{WPLS}_\omega(U,H,Y)$*, where* $H := \{\pi_+\mathbb{D}\pi_- u \,|\, u \in \mathrm{L}^2(\mathbf{R};U)\}$ *and*

$$\|x_0\|_H^2 := \|x_0\|_{\mathrm{L}^2_\omega}^2 + \inf_{u\in\mathrm{L}^2_\omega,\ \pi_+\mathbb{D}\pi_-\, u = x_0}\|u\|_{\mathrm{L}^2_\omega}^2 \qquad (x_0 \in H), \tag{6.12}$$

is the minimal ω-stable exactly ω-observable realization *of* $\mathbb{D}$.

(Note that we have identified the two systems ("Σ_ω"), although the semigroup and the output map of the latter system are actually restrictions of those of the former (and the range spaces of the semigroup and the input map are restricted analogously). The situation with Lemma 6.1.2 was somewhat analogous. Despite the name, the latter realization is not the only minimal ω-stable exactly ω-observable realization of $\mathbb{D}$.)

Analogously, the ω-stable realization $\Sigma^\omega := \left[\begin{smallmatrix} \tau\pi_- & \pi_- \\ \hline \pi_+\mathbb{D}\pi_- & \mathbb{D} \end{smallmatrix}\right]$ is exactly ω-reachable (see Definition 6.3.25).

The minimal ω-stable realizations of $\mathbb{D}$ correspond naturally one-to-one to "admissible" ω-stable factorizations of the Hankel operator $\pi_+\mathbb{D}\pi_-$, as shown in [S99] (combined with Remark 6.1.9).

Obviously, Σ_ω is a WPLS over $(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y)$. It is also a WPLS over (U,H,Y):

Lemma 6.1.7 *Let* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \in \mathbf{R}$. *The* minimal ω-stable exactly ω-observable realization *of* $\mathbb{D}$ *is an* ω*-observable, exactly* ω*-reachable (hence minimal)* ω*-stable WPLS. This realization is exactly* ω*-observable iff* $\pi_+\mathbb{D}\pi_-[\mathrm{L}^2_\omega(\mathbf{R};U)]$ *is closed in* $\mathrm{L}^2_\omega(\mathbf{R};Y)$.

Proof: Set $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] := \Sigma_\omega$. Set $\mathcal{X} := \mathrm{Ker}(\mathbb{B})^\perp \subset \mathrm{L}^2_\omega$, so that $\|\mathbb{B}u\|_H^2 := \|\mathbb{B}u\|^2_{\mathrm{L}^2_\omega} + \|u\|^2_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{X}$. Thus, $T := \mathbb{B}|_{\mathcal{X}} : \mathcal{X} \to H$ becomes coercive, hence $T \in \mathcal{GB}(\mathcal{X}, H)$. Consequently, H is complete, hence a Hilbert space. Moreover, $H \underset{c}{\subset} \mathrm{L}^2_\omega$.

We have $\pi_+\tau\mathbb{B} = \mathbb{A}^t\mathbb{B} = \pi_+\tau^t\mathbb{D}\pi_- = \mathbb{B}\tau^t\pi_-$ $(t > 0)$, hence

$$\|\mathbb{A}^t\mathbb{B}u\|_H^2 = \|\tau^t\mathbb{B}u\|_H^2 + \|\tau^t u\|^2_{\mathrm{L}^2_\omega} \leq e^{2\omega t}\|\mathbb{B}u\|_H^2 \tag{6.13}$$

for $u \in \mathcal{X}$, $t \geq 0$, by (2.2), hence $\|\mathbb{A}^t\|_{\mathcal{B}(H)} \leq e^{\omega t}$, i.e., $\mathbb{A}|_H$ is an ω-stable semigroup on H (it need not be strongly ω-stable, but it is weakly ω-stable), because its semigroup properties are inherited from $\mathbb{A}$ and its strong (or C_0-)continuity on H follows from the fact that $\tau^t\mathbb{B}u \to \mathbb{B}u$ and $\tau^t u \to u$ in L^2_ω, as $t \to 0+$.

Because $\mathcal{X} \subset \mathrm{L}^2_\omega$ is closed, the orthogonal projection $P : \mathrm{L}^2_\omega \to \mathcal{X}$ is continuous, hence so is $\mathbb{B} = TP \in \mathcal{B}(\mathrm{L}^2_\omega, H)$. Obviously, $\mathbb{C}$ remains continuous with this stronger topology of $H \subset \mathrm{L}^2_\omega$ and the other properties of the exactly ω-observable realization of $\mathbb{D}$ are preserved (except that Σ is exactly ω-reachable iff $\mathbb{B}$ is coercive on $\mathcal{X}$, equivalently, iff $\mathbb{B}$ has a closed-range, i.e., iff the Hankel operator $\pi_+\mathbb{D}\pi_-$ (on L^2_ω) has a closed range). It follows that $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$. $\square$

Example 6.1.8 The exactly observable realization of $\mathbb{D} := \tau(-1) \in \mathrm{TIC}(\mathbf{C})$ is given by

$$\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] := \left[\begin{array}{c|c} \pi_+\tau & \tau(-1)\pi_{[-1,0)} \\ \hline I & \tau(-1) \end{array}\right] \in \mathrm{WPLS}_0(U,H,Y), \tag{6.14}$$

where $U = \mathbf{C} = Y$ and $H := \mathrm{L}^2(\mathbf{R}_+;Y)$. Thus,

$$\left[\begin{array}{c|c} \mathbb{A}^t & \mathbb{B}^t \\ \hline \mathbb{C}^t & \mathbb{D}^t \end{array}\right] = \left[\begin{array}{c|c} \pi_+\tau^t & \tau^{t-1}\pi_{[\max(0,t-1),t)} \\ \hline \pi_{[0,t)} & \tau^{-1}\pi_{[0,t-1)} \end{array}\right] : \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x(t) \\ \pi_{[0,t)}y \end{bmatrix} \tag{6.15}$$

and

$$\Sigma^{\mathrm{d}} := \left[\begin{array}{c|c} \mathbb{A}^{\mathrm{d}} & \mathbb{C}^{\mathrm{d}} \\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}} \end{array}\right] := \left[\begin{array}{c|c} \tau(-\cdot)\pi_+ & \text{Я} \\ \hline \tau^{-1}\pi_{[-1,0)}\text{Я} & \tau(-1) \end{array}\right]. \tag{6.16}$$

See Examples 6.2.14, 6.3.7, 8.3.12 and 9.8.15 for more on Σ. ◁

We often assume that $\omega = 0$; the corresponding results for general ω can be obtained by shifting stability (we extend here Remark 2.1.6):

Remark 6.1.9 (Shifting stability) *Let* $\alpha, \omega \in \mathbf{R}$. *Let* $\mathcal{T}_\alpha$ *be the* stability shift *(or scaling operator)* $\mathbb{E} \mapsto \mathrm{e}^{\alpha\cdot}\mathbb{E}\mathrm{e}^{-\alpha\cdot}$. *Then* $\mathcal{T}_\alpha$ *is an isometric isomorphism of* TI_ω *onto* $\mathrm{TI}_{\omega+\alpha}$ *and of* TIC_ω *onto* $\mathrm{TIC}_{\omega+\alpha}$ *(because* $[\pi_+]\mathrm{L}^2_{\omega+\alpha} = \mathrm{e}^{\alpha\cdot}[\pi_+]\mathrm{L}^2_\omega$*, isometrically).*

Obviously, $\mathcal{T}_\alpha\pi_\pm = \pi_\pm\mathcal{T}_\alpha$, $\mathcal{T}_\alpha\tau(t) = \tau(t)\mathcal{T}_\alpha$, $\tau(t)\mathrm{e}^{\omega\cdot} = \mathrm{e}^{\omega t}\mathrm{e}^{\omega\cdot}\tau(t)$ *(*$t \in \mathbf{R}$*), and we have*

$$\mathcal{T}_\alpha(\mathbb{E}\mathbb{F}) = (\mathcal{T}_\alpha\mathbb{E})(\mathcal{T}_\alpha\mathbb{F}), \qquad \mathcal{T}_\alpha(\beta\mathbb{E}+\gamma\mathbb{F}) = \beta\mathcal{T}_\alpha\mathbb{E}+\gamma\mathcal{T}_\alpha\mathbb{F}, \tag{6.17}$$

$$(\mathcal{T}_\alpha\mathbb{E})^{-1} = \mathcal{T}_\alpha\mathbb{E}^{-1}, \qquad (\mathcal{T}_\alpha\mathbb{E})^* = \mathcal{T}_{-\alpha}\mathbb{E}^*, \tag{6.18}$$

$$(\mathcal{T}_\alpha\mathbb{E})^d = \mathcal{T}_\alpha\mathbb{E}^d, \qquad \widehat{\mathcal{T}_\alpha\mathbb{E}} = \tau(-\alpha)\widehat{\mathbb{E}}. \tag{6.19}$$

The operator $\mathcal{T}_\alpha$ *can be extended to a bijection of* $\mathrm{WPLS}_\omega(U,H,Y)$ *onto* $\mathrm{WPLS}_{\omega+\alpha}(U,H,Y)$*, by the rule*

$$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \mapsto \Sigma_\alpha := \left[\begin{array}{c|c} \mathbb{A}_\alpha & \mathbb{B}_\alpha \\ \hline \mathbb{C}_\alpha & \mathbb{D}_\alpha \end{array}\right] := \left[\begin{array}{c|c} \mathrm{e}^{\alpha\cdot}\mathbb{A} & \mathbb{B}\mathrm{e}^{-\alpha\cdot} \\ \hline \mathrm{e}^{\alpha\cdot}\mathbb{C} & \mathrm{e}^{\alpha\cdot}\mathbb{D}\mathrm{e}^{-\alpha\cdot} \end{array}\right]. \tag{6.20}$$

This extended bijection preserves all properties of Σ *modulo the change in stability; e.g, the bijection does not affect the norms of* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ *and* $\mathbb{D}$ *(remember that their domains are changed by the amount* α*), nor the regularity of* $\mathbb{D}$ *(because* $\widehat{\mathcal{T}_\alpha\mathbb{D}}(s) = \widehat{\mathbb{D}}(s-\alpha)$*), as one easily verifies (see [Sbook] for details).*

Moreover, this shift commutes with the multiplication by static operators and with the valid sums and compositions of operators, hence the shift of a system (and its admissible state or output feedback operators) corresponds to the same shift of the closed-loop system (an analogous remark applies to all closed loop systems corresponding any definition given in Summary 6.7.1). □

(See Section 6.2 for regularity and Summary 6.7.1 for feedback. The formula $\widehat{\mathcal{T}_\alpha\mathbb{E}} = \widehat{\mathbb{E}}(\cdot-\alpha)$ refers to Theorem 3.1.3(a1); for $\mathbb{E} \in \mathrm{TIC}_\infty$ it also covers Theorem 6.2.1.)

In Lemma 6.2.9(c) we will show that the generators of $\mathcal{T}_\alpha \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ are $\left[\begin{smallmatrix} A+\alpha I & C \\ B & * \end{smallmatrix}\right]$ (or $\left[\begin{smallmatrix} A+\alpha I & C \\ B & D \end{smallmatrix}\right]$ if Σ is WR). Note that $\alpha > 0$ decreases stability, i.e., shifts the transfer function to the right. If $\mathbb{D}u = \mu * u$ for all $u \in \mathrm{L}^2_\omega$ for a measure μ then $\mathbb{D}_\alpha u = (\mathrm{e}^{\omega\cdot}\mu) * u$ for all $u \in \mathrm{L}^2_{\omega+\alpha}$ (see Definition 2.6.3 and Lemma D.1.12(d) for details).

A system is almost as stable as its semigroup:

Lemma 6.1.10 (Exp. stability) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $-\infty < \omega < \omega' < \infty$*. Then*

(a1) Σ *is exponentially stable iff* $\mathbb{A}$ *is exponentially stable.*

(a2) $\Sigma \in \mathrm{WPLS}_\omega$ *and* $\mathbb{B}\tau \in \mathrm{TIC}_\omega(U,H)$ *whenever* $\omega > \omega_A$*.*

(b1) If $\mathbb{B}$ *is* ω*-stable, then* $\mathbb{B}\tau$ *and* $\mathbb{D}$ *are* ω'*-stable.*

(b2) If $\mathbb{B}\tau$ *is* ω*-stable, then* $\mathbb{B}$ *and* $\mathbb{D}$ *are* ω*-stable.*

(b3) If $\mathbb{C}$ *is* ω*-stable, then* $\mathbb{D}$ *is* ω'*-stable,* $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2_\omega$*, and hence Lemma 2.1.13 applies.*

(c1) If $\mathbb{A}$ *is exponentially stable, then* $\mathbb{A},\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2)$ *and* $\mathbb{B}\tau,\mathbb{D} \in \mathrm{TIC}_{\mathrm{exp}}(U,*)$*.*

(c2) If $\mathbb{B}$ *is exponentially stable, then so are* $\mathbb{D}$ *and* $\mathbb{B}\tau$*.*

(c3) If $\mathbb{C}$ *is exponentially stable, then so is* $\mathbb{D}$*.*

See Lemma A.4.5 and Theorem 6.7.10(d) for further equivalent conditions.

When A is bounded (or $\mathbb{A}$ is compact or differentiable), exponential stabilizability is equivalent to the condition $\sigma(A) \subset \mathbf{C}^-$; for general infinite-dimensional systems the latter condition is strictly weaker, as illustrated in Example 5.1.4 of [CZ].

(The "spectrum determined growth condition" $\sup \mathrm{Re}\, \sigma(A) = \omega_A$ holds for any bounded A and any compact or differentiable semigroup; see [CZ] or [Sbook] for details.)

Proof: (Part (a1) was independently proved by D. Salamon and G. Weiss.) All this follows from Lemma 13.3.8 through discretization, see Theorems 13.4.4 and 13.4.5. □

If $\mathbb{C}$ is stable, then $\mathbb{D}$ is "almost stable", by (b3) above. We often use this fact combined with Lemma 2.1.13, hence we give the conclusions here explicitly:

Lemma 6.1.11 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$ *and let* $\mathbb{C}$ *be stable. Then* $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2$*; in fact,* $\mathbb{D}\pi_{[-T,T)} \in \mathcal{B}(\mathrm{L}^2)$ *for all* $T > 0$*. Moreover,* $\mathbb{D} \in \mathrm{TIC}_\omega$ *for all* $\omega > 0$*.*

See Lemma 2.1.13 for further implications.

Proof: We have $\mathbb{D}\pi_{[0,1)} = \mathbb{D}^1 + \tau^{-1}\mathbb{C}\mathbb{B}^1 \in \mathcal{B}(\mathrm{L}^2([0,1);U),\mathrm{L}^2)$, hence the claims follow from Lemma 2.1.13. □

A map is stable iff it maps stable inputs to stable outputs:

Lemma 6.1.12 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$ *and* $\omega \in \mathbf{R}$*. Then* $\mathbb{C}$ *is* ω*-stable iff* $\mathbb{C}[H] \subset \mathrm{L}^2_\omega$*, and* $\mathbb{D}$ *is* ω*-stable iff* $\mathbb{D}[\pi_+\mathrm{L}^2_\omega] \subset \mathrm{L}^2_\omega$*.*

Proof: Let $\mathbb{C}H \subset \mathrm{L}^2_\omega$. Then $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$, by Lemma A.3.6, because $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\alpha)$ for $\alpha := \max\{\omega_A+1,\omega\}$. The converse is trivial. The claim on $\mathbb{D}$ is Lemma 2.1.10(e). □

If(f) $\mathbb{B}$ is stable, then the strong stability of $\mathbb{A}$ implies that of $\mathbb{B}$:

Lemma 6.1.13 (Strongly stable $\mathbb{B}$) *Let* $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$. *Then the following are equivalent:*

(i) $\mathbb{B}$ *is strongly stable;*

(ii) $\mathbb{B}$ *is stable and* $\mathbb{B}\tau^t u \to 0$*, as* $t \to +\infty$*, for all* $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$*;*

If $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ *is strongly stable or* $\mathbb{B}$ *is exponentially stable, then* $\mathbb{B}$ *is strongly stable.*

Proof: 1° *(i)⇔(ii):* Obviously, (i) implies (ii). Assume (ii). Then we may allow for any $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)$ (ii) (replace t by $t+T$ for suitable $T \in \mathbf{R}$)), hence for any $u \in \mathrm{L}^2(\mathbf{R};U)$, by continuity; thus, (i) holds.

2° If $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ is strongly stable, and $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$, then $\mathbb{B}\tau^{T+t}u = \mathbb{A}^t\mathbb{B}\tau^T u \to 0$, as $t \to +\infty$, thus, then $\mathbb{B}$ is strongly stable, by 1°.

3° If $\mathbb{B}$ is ω-stable, $\omega < 0$, then $\|\mathbb{B}\tau^t u\|_H \le M\|\tau^t u\|_{\mathrm{L}^2_\omega} \le M\mathrm{e}^{\omega t}\|u\|_{\mathrm{L}^2_\omega} \to 0$, as $t \to +\infty$, for all $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$. Thus, then $\mathbb{B}$ is strongly stable, by 1°. □

Assume for a while that $\dim H < \infty$. Then $\mathbb{A}$ is stable iff $\sigma(A) \subset \overline{\mathbf{C}^-}$ and $\mathbb{A}$ is strongly (or exponentially) stable iff $\sigma(A) \subset \mathbf{C}^-$. However, for a non-strongly stable $\mathbb{A}$ (say $H = \mathbf{C}$ and $A \in i\mathbf{R}$), maps $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are unstable unless the non-strongly stable poles of $\mathbb{A}$ are unreachable or unobservable.

The strong stability of $\mathbb{A}$ does not imply that of $\mathbb{B}$, $\mathbb{C}$ or $\mathbb{D}$, not even for bounded A, B, C and D (cf. Lemma 6.1.16):

Example 6.1.14 ($\mathbb{A}$ strongly stable $\not\Rightarrow$ $\mathbb{B}/\mathbb{C}/\mathbb{D}$ stable) (It follows from Lemma 6.3.26(f) (or (d)), that all systems below are minimal.)

(a) Let $Y := H := U := \ell^2(\mathbf{N}+1)$ (with natural base $\{e_k := \chi_{\{k\}}\}_{k=1}^\infty$). Define $A \in \mathcal{B}(H)$ by setting $Ae_k := -k^{-1}e_k$ $(k \in \mathbf{N}+1)$.

Then $\|A\| \le 1$ and $\mathbb{A}^t e_k := \mathrm{e}^{At}e_k = \mathrm{e}^{-t/k}e_k$ $(k \in \mathbf{N}+1,\ t \ge 0)$, hence $\|\mathbb{A}^t\| \le 1$ $(t \ge 0)$. Because $\mathbb{A}$ is stable and $\mathbb{A}e_k \to 0$ for all k, the semigroup $\mathbb{A}$ is strongly stable, by Lemma A.3.4(H1). Since $A = A^*$, we have $\mathbb{A} = \mathbb{A}^*$.

By Lemma 6.3.16(a), the operators $\left[\begin{smallmatrix}A & I\\ I & 0\end{smallmatrix}\right] \in \mathcal{B}(U \times U)$ generate a wpls $\Sigma := \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(U,U,U)$. By Theorem 6.2.11,

$$\widehat{\mathbb{A}} = \widehat{\mathbb{B}\tau} = \widehat{\mathbb{C}} = \widehat{\mathbb{D}} = (s-A)^{-1} = \mathrm{diag}((s+k^{-1})^{-1})_{k\in\mathbf{N}+1}. \tag{6.21}$$

Because $\mathbb{A}$ is obviously not exponentially stable, we have $\widehat{\mathbb{A}} \notin \mathrm{H}^\infty$ and $\widehat{\mathbb{A}} \notin \mathrm{H}^2_{\mathrm{strong}}$, by Lemma A.4.5, hence $\mathbb{D}$, $\mathbb{C}$ and $\mathbb{B}\tau$ are unstable. Since $\mathbb{B} = \mathbb{C}^{\mathrm{d}}$, also $\mathbb{B}$ is unstable.

(b) If, instead, we take $Ce_k := k^{-1/2}e_k$, then $\widehat{\mathbb{C}}(s)e_k = \widehat{\mathbb{D}}(s)e_k = \frac{k^{1/2}}{sk+1}e_k$ $(k \in \mathbf{N}+1)$, hence $\|\widehat{\mathbb{C}}x\|^2_{H^2} = \sum_k \int_{i\mathbf{R}} |x_k \frac{k^{1/2}}{sk+1}|^2 = \pi\sum_k |x_k|^2 = \pi\|x\|_2^2$ for each $x \in H$, so that then $\left[\begin{smallmatrix}\mathbb{A}\\ \hline \mathbb{C}\end{smallmatrix}\right]$ is strongly stable but still $\mathbb{D}$ is unstable (since $\|\widehat{\mathbb{D}}e_k\|_\infty = k^{1/2}$ $(k \in \mathbf{N}+1)$).

One could show that this system is exponentially stabilizable (take $K = 2$) but not detectable.

(c) Exchange C and $B = I$ in (b) to have $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ strongly stable but $\mathbb{C}$ and $\mathbb{D}$ unstable.

(d) By Example 9.13.14 (see $\mathbb{A}_{\circlearrowleft}$ and $\mathbb{A}^{\mathrm{d}}_{\circlearrowleft}$), we can have $\mathbb{A}$, $\mathbb{C}$ and $\mathbb{D}$ strongly stable but $\mathbb{B}$ and $\mathbb{B}\tau$ unstable, and $\mathbb{A}$, $\mathbb{B}$, $\mathbb{B}\tau$ and $\mathbb{D}$ strongly stable but $\mathbb{C}$ unstable.

(e) By (d), we can have Σ minimal and $\mathbb{B}\tau$ stable without Σ being exponentially stable. $\triangleleft$

(This still leaves the open question whether $\mathbb{D}$ can be unstable when both $\mathbb{C}$ and $\mathbb{B}$ are stable (and Σ is well-posed; cf. Example 6.3.24). By using realization (6.11) we obtain that this is the case iff some unstable $\mathbb{D} \in \mathrm{TIC}_\infty$ has a stable Hankel operator $\pi_+\mathbb{D}\pi_-$; thus, it might be that the answer to our question is known.)

As one can easily verify, in [S97b], [S98b] and [S98c] the stability assumptions on $\mathbb{A}$ and $\mathbb{B}$ were not important:

Remark 6.1.15 (SOS and [Staffans]) *In [S97b], [S98b] and [S98c], except in [S97b, Lemma 21] and [S98a, Lemma 3.5(ii)], we may drop the assumptions on the stability of semigroups and input maps, if we do the same on conclusions.* $\square$

In the sequel, we will refer to these articles with these weaker assumptions without any further mention.

(We could, in addition, replace joint stabilizability and detectability by r.c.-SOS-stabilizability In particular, Sections 5–7 of [S98b] are true with these two replacements and for indefinite S too, cf. [S98c, Remark 7.7]; however, we do not need this.)

Next we will present the generating operators A, B and C of a WPLS; the existence of a feedthrough operator D depends on the regularity of the system and is therefore studied in Section 6.2. As mentioned above, these lead to classical formulae $x' = Ax + Bu$, $y = Cx + Du$ and $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ and others in a weak sense.

Following the customary practice, we shall set

$$H_1 := \mathrm{Dom}(A),\ \ H_1^* := \mathrm{Dom}(A^*),\ \ H_{-1} := (H_1^*)^*,\ \ H_{-1}^* := (H_1)^*, \tag{6.22}$$

where A is the generator of $\mathbb{A}$. We shall take adjoints w.r.t. the pivot space H; e.g., $H_{-1} \times H_1^* \to \mathbf{C}$ denotes the unique continuous extension of the restriction of the inner product $H \times H \to \mathbf{C}$ to $H \times H_1^*$, see Definition A.3.23 for details.

A detailed description of this process is given in the lemma and definition below.

Lemma 6.1.16 ($\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$, H_1, H_{-1}) *Let* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$, $\omega \in \mathbf{R}$. *Let A be the generator of $\mathbb{A}$ and let $\alpha \in \sigma(A)^c$.*

We set $H_1 := \mathrm{Dom}(A)$ with $\|x\|_{H_1} := \|(\alpha - A)x\|_H$ (this is equivalent to the graph norm), and define H_{-1} to be the completion of H under the norm $\|(\alpha - A)^{-1} \cdot \|_H$ (thus $H_1 \subset H \subset H_{-1}$; H_1 and H_{-1} are independent of α modulo an equivalent norm).

The following hold:

(a) The unique extension $\mathbb{A}_{H_{-1}}$ of $\mathbb{A}$ onto H_{-1} is a semigroup isomorphic to the original $\mathbb{A}$ and the generator of $\mathbb{A}_{H_{-1}}$ an extension of A; we identify the two. The situation with $\mathbb{A}$ and $\mathbb{A}|_{H_1}$ is the same.

Thus, $\mathbb{A}(t)$ is in $\mathcal{B}(H)$, $\mathcal{B}(H_1)$ and $\mathcal{B}(H_{-1})$ for $t \geq 0$, $A \in \mathcal{B}(H_1, H)$, and $A \in \mathcal{B}(H, H_{-1})$. However, by $\mathrm{Dom}(A)$ *we always denote $H_1 = \{x_0 \in H \,|\, Ax_0 \in H\}$. The map $\alpha - A$ is an isometric isomorphism of H_n onto H_{n-1} ($n = 0, 1$).*

(b) There is a unique input operator $B \in \mathcal{B}(U, H_{-1})$ *s.t.*

$$\mathbb{B}\tau(t)u = (\mathbb{A}B * u)(t) = \int_0^t \mathbb{A}(t-s)Bu(s)\,ds \in H \quad (u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U),\ t \geq 0) \tag{6.23}$$

(above and below, the integration is carried out in H_{-1}). Consequently,

$$\mathbb{B}\tau(t)u = (\mathbb{A}B * u)(t) = \lim_{T\to\infty} \int_{-T}^t \mathbb{A}(t-s)Bu(s)\,ds \in H \quad (t \in \mathbf{R},\ u \in \mathrm{L}^2_\omega(\mathbf{R};U)), \tag{6.24}$$

where the limit can be taken in H. Moreover, $x = \mathbb{A}x_0 + \mathbb{B}\tau u$ satisfies $x' = Ax + Bu$ in H_{-1} a.e. on $\mathbf{R}_+$ and $x(t) - x_0 = \int_0^t (Ax + Bu)\,dm$ for all $t \geq 0$, $x_0 \in H$, $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$.

(c) There is a unique output operator $C \in \mathcal{B}(H_1, Y)$ *s.t.*

$$(\mathbb{C}x_0)(t) = C\mathbb{A}(t)x_0 \quad (x_0 \in H_1,\ t \geq 0). \tag{6.25}$$

We say that Σ is generated by $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$, *and we call* $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ *the* generators *of* Σ*; they are independent of* α *and* ω *(as long as* $\Sigma \in \mathrm{WPLS}_\omega$*). Also the following hold:*

(d) $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ determine $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \end{smallmatrix}\right]$ uniquely and $\mathbb{D}$ modulo an additive constant from $\mathcal{B}(U, Y)$ (cf. Lemma 6.2.9(a) and Lemma 6.3.10(d)).

(e) The generators of Σ^{d} are given by $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$.

(See Definition 6.2.3 and Lemma 6.2.9(a) for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and Definition 6.1.17 for $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$.) The specific number α chosen above is irrelevant; only the topology of H_{-1} matters in application, not the particular norm of H_{-1} (the norms corresponding to different α's are equivalent).

Note from (c) that C is determined by

$$Cx_0 := (\mathbb{C}x_0)(0) \quad (x_0 \in H_1), \tag{6.26}$$

and B is determined by $B^*x_0 := (\mathbb{B}^{\mathrm{d}}x_0)(0) = (\mathbb{B}^*x_0)(0)$ ($x_0 \in H_1^* := \mathrm{Dom}(A^*)$).

We note for (b) that the (convolution) integrals converge in H_{-1}, i.e., $\mathbb{A}(t - \cdot)Bu(\cdot) \in \mathrm{L}^1((-\infty, t); H_{-1})$, for each $t \in \mathbf{R}_+$, by the Hölder Inequality (we have $\mathbb{A}B \in \mathcal{C}(\mathbf{R}_+; \mathcal{B}(U, H_{-1})) \subset \mathrm{L}^2_{\mathrm{loc}}$). However, the values of the integrals belong to H too. Recall from (B.18) that if any integral converges in H, then it converges in H_{-1}, with the same value.

In Theorem 6.2.13(a1), we shall show that x is the unique strong solution of "$x' = Ax + Bu$, $x(0) = x_0$" in the sense that $Ax + Bu$ is the distributional derivative (and derivative a.e.) of x For further details and formula $x' = Ax + Bu$ (and "$y = Cx + Du$"), see Theorem 6.2.13 and Chapter 4 of [Sbook]; for the generators of closed-loop systems, see Proposition 6.6.18.

As above, we will denote the generators with same letters as corresponding operators (as in [S95]–[S01] and [Sbook]).

Proof: (a) See Lemma A.4.6.

(b) Formula (6.23) and the existence claim for B and follow easily from Theorem 3.9 of [W89a] (note that $\pi_-\tau(t)u \in \mathrm{L}^2_\omega$ for any $\omega \in \mathbf{R}$ and that $\mathbb{B} = \mathbb{B}\pi_-$, by Definition 6.1.1). Formula (6.24) (cf. Remark 30 of [S97b]) follows from (6.23), because $\pi_{[-T,\infty)}u \to u$ in L^2_ω, by Corollary B.3.8. The rest is from Theorem 3.9 of [W89a] (and Lemma B.7.6).

(c) This is from Theorem 3.3 of [W89b].

(d) This is obvious (use density and continuity).

(e) By Lemma A.4.2(f), the generator of $\mathbb{A}^*$ is A^*. It is easy to verify that $\langle u, \mathbb{B}^* x_0\rangle_H = \langle u(-\cdot), B^*\mathbb{A}(\cdot)^* x_0\rangle_{\mathrm{L}^2}$ for $u \in \mathcal{C}_c(\mathbf{R}_-;U)$, $x_0 \in H_1^*$ (cf. 6.1.17). By density (note that $B^*\mathbb{A}(\cdot)^*x_0 \in \mathcal{C}(\mathbf{R}_+;U)$), $\mathbb{B}^*x_0 = B^*\mathbb{A}(\cdot)^*x_0$ for $x_0 \in H_1^*$, hence B^* corresponds to $\text{Я}\mathbb{B}^*$. By exchanging the roles of Σ and Σ^{d}, we obtain that C^* corresponds to $\mathbb{C}^*\text{Я}$. □

As explained above, the spaces H_1^* and H_{-1}^* are defined as H_1 and H_{-1}, respectively, but with A^* in place of A, and $H_{-1}^* \times H_1 \to \mathbf{C}$ is the unique continuous extension of $H \times H_1 \to \mathbf{C}$. The adjoints C^* and B^* are defined w.r.t. these two sesquilinear forms (see Lemma A.3.24), so that

$$\langle Bu_0, x_0\rangle_{\langle H_{-1},H_1^*\rangle} = \langle u_0, B^*x_0\rangle_U \qquad \text{for all } u_0 \in U,\ x_0 \in H_1^*, \tag{6.27}$$

$$\langle y_0, Cx_0\rangle_Y = \langle C^*y_0, x_0\rangle_{\langle H_{-1}^*,H_1\rangle} \qquad \text{for all } y_0 \in Y,\ x_0 \in H_1. \tag{6.28}$$

Let us state this and a bit more formally:

Definition 6.1.17 ($B^*, C^*, H_1^*, H_{-1}^*, H_B, H_C^*$) *Let the assumptions of Lemma 6.1.16 hold. We set $H_1^* := \mathrm{Dom}(A^*) \subset H \subset H_{-1}^* :=$ the completion of H under the norm $\|(\bar\alpha - A^*)^{-1}\cdot\|_H$.*

We extend $\langle h, x\rangle_{\langle H_1,H_{-1}^*\rangle} := \langle h, x\rangle_H$ for all $h \in H_1$, $x \in H$ continuously to $H_1 \times H_{-1}^*$ to get an interpretation of H_{-1}^* as the dual of H_1, and we do the same for $\langle h^*, x\rangle_{\langle H_1^*,H_{-1}\rangle} := \langle h^*, x\rangle_H$ for all $h^* \in H_1^*$, $x \in H$ (all these pairings are sesquilinear).

The adjoints $C^* \in \mathcal{B}(Y, H_{-1}^*)$ and $B^* \in \mathcal{B}(H_1^*, U)$ are taken with respect to these pairings. As above, A^* means the adjoint $A^* : \mathrm{Dom}(A^*) \to H$ of the unbounded operator A as well as its unique continuous extension $A^* \in \mathcal{B}(H, H_{-1}^*)$ (for which $\langle Az, x\rangle_H = \langle z, A^*x\rangle_{\langle H_1,H_{-1}^*\rangle}$ for all $z \in H_1$, $x \in H$). Finally, we define the Hilbert spaces (see Lemma A.3.16)

$$H_B := (\alpha - A)^{-1}[H + BU] = \{x_0 \in H \,|\, Ax_0 + Bu_0 \in H \text{ for some } u_0 \in U\} \subset H \tag{6.29}$$

with $\|z\|_{H_B} := \inf\{(\|x\|_H^2 + \|u\|_U^2)^{1/2} \,|\, (\alpha - A)^{-1}(x + Bu) = z\}$, and $H_C^* := (\bar\alpha - A^*)^{-1}[H + C^*Y] \subset H$ with $\|z\|_{H_C^*} := \inf\{\|(x,y)\|_{H\times Y} \,|\, (\bar\alpha - A^*)^{-1}(x + C^*y) = z\}$. Thus, $H_1 \subset H_B \subset H \subset H_{-1}$ and $H_1^* \subset H_C^* \subset H \subset H_{-1}^*$, continuously (see Corollary A.3.7).

We set $H_{C,K}^* := H^*_{\left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right]}$ when $\left[\begin{array}{c|c} A & B \\ \hline C & \\ K & \end{array}\right]$ generate a WPLS.

All spaces in this definition are independent of $\alpha \in \sigma(A)^c$ modulo an equivalent norm, by the Resolvent Equation and Corollary A.3.7.

A mnemonic: $H_1^* := (H^*)_1 \neq (H_1)^* = (H^*)_{-1} =: H_{-1}^*$. See also Lemma 6.1.16(e). Example 6.2.14 contains examples of spaces and (extended) operators defined above, but usually suffices to remember (6.27)–(6.28). See Lemma 6.3.18 and [Sbook, Lemma 4.2.18] or [S97b, Lemma 32] for further details.

We follow the standard convention to call B *bounded* and write "$B \in \mathcal{B}(U,H)$", when $B \in \mathcal{B}(U,H_{-1})$ is such that $Bu_0 = B_0u_0$ for all $u_0 \in U$ for some $B_0 \in \mathcal{B}(U,H)$ (equivalently, $(\mathbb{A}x_0 + \mathbb{B}u)' = Ax + B_0u$ a.e. for all $x_0 \in H$ and all $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$). Cf. also part 1° of the proof of Theorem 9.9.6(a). Similarly, we call C *bounded* if "$C \in \mathcal{B}(H,Y)$". Obviously, B is bounded iff B^* is bounded.

We call B or C *unbounded* if it is not bounded. For B, this does not agree with the meaning of unboundedness in functional analysis (but it does agree for C and for B^*). In physical examples, unbounded input and output operators appear typically in connection with boundary control or boundary observation, respectively.

We have followed the WPLS formulation of O. Staffans ([S97a]–[S01]), hence we share most of his notation, but we use the part of the notation of G. Weiss that we feel more elegant or practical (cf. Definition 6.1.5).

Readers familiar with existing WPLS literature might wish to consult the following "translation table" between the notation of this book, Staffans and Weiss. Here "$H_1 \overset{\mathrm{S}}{=} W \overset{\mathrm{W}}{=} X_1$" means that we use H_1 for the W of [S95]–[S98d], which in turn equals the X_1 of [W94a], [WW] etc. We exclude the sign "$\overset{\mathrm{S}}{=}$" when the symbol of [S95]–[S98d] coincides with that of this book, e.g., by "$H \overset{\mathrm{W}}{=} X$" we mean that Staffans and we use H (for the state space) where Weiss uses X.

System theory (Chapter 6):
Our notation $\overset{\mathrm{S}}{=}$ Staffans' notations $\overset{\mathrm{W}}{=}$ Weiss' notation:
$H \overset{\mathrm{W}}{=} X$ (the state space), $U \overset{\mathrm{W}}{=} U$ (the input space), $Y \overset{\mathrm{W}}{=} Y$ (the output space) (these are complex (possibly unseparable) Hilbert spaces).
$\mathbb{A}^t := \mathbb{A}(t) \overset{\mathrm{W}}{=} \mathbb{T}_t \in \mathcal{B}(H)$ for all $t \geq 0$ (the semigroup), $A \overset{\mathrm{W}}{=} A$ (the infinitesimal generator of $\mathbb{A}$).
$\mathbb{B} \in \mathcal{B}([\pi_-]\mathrm{L}^2_\omega(\mathbf{R};U);H))$ (the reachability map) is the (unique) operator for which we have $\mathbb{B}^t := \mathbb{B}\tau(t)\pi_{[0,t)} \overset{\mathrm{W}}{=} \Phi_t$,
$\mathbb{C} \overset{\mathrm{W}}{=} \Psi_\infty \in \mathcal{B}(H,[\pi_+]\mathrm{L}^2_\omega(\mathbf{R};Y))$ (the observability map) ($\mathbb{C}^t := \pi_{[0,t)}\mathbb{C} \overset{\mathrm{W}}{=} \Psi_t$),
$\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ (the I/O map) is the (unique) causal, time-invariant operator for which $\pi_+\mathbb{D}\pi_+ \overset{\mathrm{W}}{=} \mathbb{F}_\infty$ (and hence $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)} \overset{\mathrm{W}}{=} \mathbb{F}_t$); see Lemma 2.1.3.
$H_1 \overset{\mathrm{S}}{=} W \overset{\mathrm{W}}{=} X_1 := \mathrm{Dom}(A)$, $H_{-1} \overset{\mathrm{S}}{=} V \overset{\mathrm{W}}{=} X_{-1} := \mathrm{cl}_{\|(\alpha-A)^{-1}\cdot\|}(H)$, hence $H_1 \underset{c}{\subset} H \underset{c}{\subset} H_{-1}$;
$H_1^* \overset{\mathrm{S}}{=} V^* \overset{\mathrm{W}}{=} Z_1 := \mathrm{Dom}(A^*)$, $H_{-1}^* \overset{\mathrm{S}}{=} W^* \overset{\mathrm{W}}{=} Z_{-1} := \mathrm{cl}_{\|(\bar{\alpha}-A^*)^{-1}\cdot\|}(H)$, analogously.
$B_{\mathrm{s}}^* \overset{\mathrm{S}}{=} \overline{B^*} \overset{\mathrm{W}}{=} (B^*)_\Lambda = B_\Lambda^* = \text{s-lim}_{s\to+\infty} B^*s(s-A^*)^{-1}$, hence $H_1^* \underset{c}{\subset} \mathrm{Dom}(B_{\mathrm{s}}^*) \underset{c}{\subset} H$;
$C_{\mathrm{s}} \overset{\mathrm{S}}{=} \overline{C} \overset{\mathrm{W}}{=} C_\Lambda := \text{s-lim}_{s\to+\infty} Cs(s-A)^{-1}$, $K_{\mathrm{s}} \overset{\mathrm{S}}{=} \overline{K} \overset{\mathrm{W}}{=} F_\Lambda$, (see Proposition 6.2.8).
$\widehat{\mathbb{D}} \overset{\mathrm{W}}{=} \mathbb{H} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ (the transfer function; see Theorem 6.2.1).
Optimization theory (Chapters 8–12):
$\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) \overset{\mathrm{W}}{=} \Pi \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ (the Popov function for stable $\mathbb{D}$),

$\mathcal{P} \overset{\text{S}}{=} \Pi \overset{\text{W}}{=} X \in \mathcal{B}(H)$ (the Riccati operator);
$\mathbb{X} \overset{\text{W}}{=} \Xi \in \mathcal{G}\mathrm{TIC}(U)$ (the spectral factor, see Definition 6.4.4)
$X \overset{\text{W}}{=} D \in \mathcal{B}(U)$ (its feedthrough operator, see Definition 6.2.3; in particular, $X^*SX \overset{\text{W}}{=} D^*D$ for $S \gg 0$; in applications we often take $X = I$, hence then $S \overset{\text{W}}{=} D^*D$ and $X \overset{\text{W}}{=} D^{-1}\Xi$);
$K \overset{\text{W}}{=} F \in \mathcal{B}(H_1, U)$ (the state feedback operator) $D^*JD \overset{\text{W}}{=} R \in \mathcal{B}(U)$ (see Remark 9.1.14).

See Remark 9.1.14 for further differences. The notation in recent works of Staffans (e.g., [Sbook]) is closer to that of Weiss and this book than that of [S95]–[S98d]. Most other existing WPLS theory seems to use notation close to that of Weiss with some exceptions in the direction of this book.

Notes

The history of WPLSs is explained in the notes to Chapter 2 of [Sbook]; see also p. 23. The early history is explained in [Helton76a]; the final rise of the theory is due to [Sal87], [Sal89], [W89a], [W89b] and [W89c], whose contributions include Lemma 6.1.16, hence the alternative name *Salamon–Weiss systems*. Ruth Curtain [Curtain89] gives a detailed account of this process and its relations to the rest of control theory.

This abstract formulation has for long been widely used also in the special case of bounded input and output operators, where also the definition of a system through generators would be possible. Naturally, in that case the theory becomes much simpler and more elegant, although very restrictive, hence it serves as a nice introduction to the general case; see [CZ] for a mature and rather extensive presentation.

This book is system theory oriented, but the literature is full of practical applications of WPLSs and its special cases, dating back to [Sal87] (see also [CZ]).

The name WPLS, part of Lemma 6.1.13, Remark 6.1.15 and most of each definition in this section are due to Olof Staffans. Lemmas 6.1.2 and 6.1.12 and Lemma 6.1.10(a) are well-known. Realization (6.11) is from Section 4 of [Sal89]. Variants of realization (6.12) and Lemma 6.1.7 were presented in [KMR] in a Pritchard–Salamon setting. The main new contribution of this section are the relations between the stabilities of different parts of a system (most of 6.1.10–6.1.14).

The reader interested in a more detailed study on WPLSs than that of this chapter must read [Sbook]. That monograph also treats the case where L^2 signals are replaced by L^p signals $(1 \leq p \leq \infty)$ and U, H and Y are allowed to be Banach spaces. Although most results and proofs on system theory and feedback theory are the same in this more general context as in our L^2 Hilbert space setting, the latter seems necessary for fruitful optimization theory and hence it is the one usually treated in the literature; this has also motivated our choice.

6.2 Regularity ($\exists\widehat{\mathbb{D}}(+\infty)$)

That is the usual method, but not mine–
My way is to begin with the beginning;
The regularity of my design
Forbids all wandering as the worst of sinning.
— Lord Byron (1788–1824), "The Bride of Abydos"

learning or genius, reader,

In this section we define certain regularity concepts for an I/O map and study the basic implications of regularity, including the classical formulae $y = Cx + Du$ and $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$. This section does not contain essentially new results. Further results on regularity and the relations between a system and its generators are given in Section 6.3.

If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a finite-dimensional system (cf. (1.7)), then its transfer function $D + C(\cdot - A)^{-1}B$ has the limit D at infinity. In general, well-posed transfer functions having such a limit are called regular. For a mnemonic, we write formally

$$\mathbb{D} \in \mathrm{TIC}_\infty \text{ is regular } \Leftrightarrow \exists\widehat{\mathbb{D}}(+\infty); \tag{6.30}$$

see Definition 6.2.3 for exact definitions.

Most system theory can be written in terms of "integral maps" $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$, but part of the theory requires feedthrough operators. In particular, the Riccati equations used to solve optimal control problems are written in terms of the generators, including the feedthrough operator, of the system involved. Such problems can be solved in terms of factorization of the I/O map, but this approach is less useful in practical applications and therefore usually only serves as a path to Riccati equations. Fortunately, all transfer functions of practical interest seem to be regular.

The Laplace transform, defined by $\widehat{u} : s \mapsto \int_{\mathbf{R}} e^{-st} u(t)\, dt \in U$, maps function $u \in \mathrm{L}^2_\omega$ onto H^2_ω, and is an isomorphism onto, isometric times the factor $\sqrt{2\pi}$; see Appendix D for details. Well-posed I/O maps correspond to transfer functions that are bounded on some right half-plane $\mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}$, i.e., that are proper; we recall this fact from Theorem 2.1.2:

Theorem 6.2.1 (Transfer functions) *Let* $\omega \in \mathbf{R}$. *For each* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ *there is a unique function* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$*, called the* transfer function *(or symbol or Laplace transform) of* $\mathbb{D}$*, s.t.* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ *on* $\mathbf{C}^+_\omega$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$*. The mapping* $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ *is an isometric isomorphism.* □

We often identify functions and corresponding multiplication operators, i.e., we consider $\widehat{\mathbb{D}}$ both as a function and as an operator on H^2_ω. Note that $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\omega$ iff $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\omega$, in particular $\widehat{\mathbb{D}^{-1}} = (\widehat{\mathbb{D}})^{-1}$ if $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\omega$.

Next we recall the last claim of Lemma 3.3.8:

Lemma 6.2.2 ($\mathbb{D}^{\mathrm{d}}$) *Let* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$*. Then* $\mathbb{D}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{D}^*\boldsymbol{Я} \in \mathrm{TIC}_\omega(U,Y)$ *and* $(\mathbb{D}^{\mathrm{d}})^{\mathrm{d}} = \mathbb{D}$*. Moreover,* $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ *for* $s \in \mathbf{C}^+_\omega$. □

George Weiss has given eight equivalent characterizations of strong regularity in [W94a, Theorem 5.8] and for weak regularity in [SW01a]; a more thorough study on these concepts is given in [Sbook]. We have chosen the simplest characterizations as the definitions: the transfer function should have a limit at infinity. However, some applications require that this limit converges in a very strong sense whereas others allow for a weaker convergence, therefore we define twelve different (combinations of) attributes of regularity:

Definition 6.2.3 (Regularity) *Let $\omega \in \mathbf{R}$, $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, and $\mathbb{D} \in \mathcal{B}(U,Y)$.*

The map $\mathbb{D}$ is called regular (R) *with* feedthrough operator $\widehat{\mathbb{D}}(+\infty) := D \in \mathcal{B}(U,Y)$ *if $\widehat{\mathbb{D}}(s) \to D$ as $s \to +\infty$ on $(\omega, +\infty)$.*

If $\widehat{\mathbb{D}}(s) \to D$ as $\operatorname{Re} s \to +\infty$ *on $\mathbf{C}^+_\omega$, then we call $\mathbb{D}$* line-regular (LR).

If $\mathbb{D}$ is regular and there is $\alpha > \omega$ s.t. $\widehat{\mathbb{D}}(\beta + iy) \to D$ as $y \to +\infty$, for all $\beta > \alpha$, then we call $\mathbb{D}$ vertically regular (VR).

If $\mathbb{D}$ is stable (or $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$) and $\widehat{\mathbb{D}}(s) \to D$ as $s \in \mathbf{C}^+$ and $|s| \to \infty$, then we call $\mathbb{D}$ half-plane regular (HPR).

In the above definitions, we modify the word "regular" with the word weakly (W), strongly (S) *or* uniformly (U) *according to the sense of convergence (of $\widehat{\mathbb{D}}(s) \to D$).*

Thus, WR *means weakly regular,* SHPR *means strongly half-plane-regular,* ULR *means uniformly line-regular etc.*

We call ($\widehat{\mathbb{D}}$ and) $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS *strongly regular and write "$\mathbb{D} \in \mathrm{SR}$" if the I/O map $\mathbb{D}$ is strongly regular, and we do analogously also for the other regularity concepts defined above.*

When Σ is WR, the generators *of Σ refer to the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we say that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$* generate Σ *and we follow the classical convention to denote a system by its generators as in "$\Sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$" (cf. Lemma 6.1.16 and Definition 6.3.8).*

(Sometimes in the literature, the term "regular" means "strongly regular", and no other forms of regularity are defined. The operator D in "VR" refers to the one in "R", hence D is always the same for different forms of regularity.)

As above, we shall denote feedthrough operators by same letters as the corresponding WR operators ($D := \widehat{\mathbb{D}}(+\infty)$). The same also applies to the generators of the other components of WPLSs, as in Lemma 6.1.16. See Lemma 2.1.13 for more on the condition $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$.

We shall now make a few remarks on these different forms of regularity. Obviously, weak regularity is weaker than any other form of regularity. The definition of weak regularity means the existence of $D \in \mathcal{B}(U,Y)$ s.t.

$$\langle y_0, \widehat{\mathbb{D}}(s) u_0 \rangle \to \langle y_0, D u_0 \rangle \text{ as } s \in (\omega, +\infty) \text{ and } s \to +\infty, \text{ for all } u_0 \in U,\ y_0 \in Y. \tag{6.31}$$

It is apparent that whenever $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\widehat{\mathbb{D}}(s)u_0$ converges weakly as $s \to +\infty$, for each $u_0 \in U$, then the limit is necessarily Du_0 for some (unique) $D \in \mathcal{B}(U,Y)$ satisfying $\|D\|_{\mathcal{B}(U,Y)} \le \|\mathbb{D}\|_{\mathrm{TIC}_\omega(U,Y)}$, hence then (and only then) $\mathbb{D}$ is WR.

Similarly, $\mathbb{D} \in \mathrm{ULR}(U,Y)$ with feedthrough operator $D \in \mathcal{B}(U,Y)$ iff $\|\widehat{\mathbb{D}}(s) - D\| \to 0$ as $\operatorname{Re} s \to +\infty$. Most systems treated in the literature are ULR; this includes the systems of [FLT], [LT00a] and [LT00b] (see also Lemma 6.3.16). We note that $\mathbb{D}$ is HPR iff $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}^+;\mathcal{B})$ and $\widehat{\mathbb{D}} \circ \phi_{\text{Cayley}}$ has a limit at -1.

Although weak regularity suffices for most results on regularity, only uniform line-regularity makes the invertibility of a map equivalent to the invertibility of its feedthrough operator, as noted in Proposition 6.3.1(c), and hence allows for several results similar to the finite-dimensional ones. Therefore, "WR" and "ULR" are the most important ones of the twelve concepts above.

If $n := \dim Y < \infty$, then weak, strong and uniform convergence coincide (with the componentwise convergence of $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ \vdots \\ \mathbb{D}_n \end{bmatrix}$).

A scalar example of an irregular transfer function is $\widehat{\mathbb{D}}(s) := \cos(\log s)$ (due to Kirsten Morris), but we do not know any physically motivated examples (crossing our fingers in hope that there were none). On the other hand, a bounded control operator B or observation operator C makes a system ULR, by Lemma 6.3.16; in particular, any Pritchard–Salamon system is ULR, by Lemma 6.9.4. Further examples of assumptions that guarantee a certain amount of regularity are given in Sections 2.6, 6.3 and 6.8.

We shall often write inclusions such as $\mathcal{B} \subset \mathrm{ULR} \subset \mathrm{TIC}_\infty$ or $\mathcal{B} \subset \mathrm{H}^\infty_{-1} \subset \mathrm{H}^\infty$ without specifying the input and output spaces (recall that $T \in \mathcal{B}$ is identified with $\mathbb{D}_T \in \mathrm{TIC}$ defined by $\mathbb{D}_T u := Tu$ for all $u \in \mathrm{L}^2$); the definition below makes this notation rigorous. (We often apply it with substitutions $\mathcal{X} := \{$all Hilbert spaces$\}$, $\mathcal{A}' := \mathrm{TIC}_\infty$ (or some smaller class) and $\mathcal{A}$ equal to some subclass of TIC.)

Definition 6.2.4 ($\mathcal{A} \underset{a}{\subset} \mathcal{A}'$) *Let both $(\mathcal{A},\mathcal{X})$ and $(\mathcal{A}',\mathcal{X})$ be as in Lemma A.1.1 (or as in Remark A.1.3).*

If $\mathcal{A}(U,Y) \subset \mathcal{A}'(U,Y)$ for all $U,Y \in \mathcal{X}$, then we call $\mathcal{A}$ an algebraic subclass *of $\mathcal{A}'$ and write $\mathcal{A} \underset{a}{\subset} \mathcal{A}'$.*

Thus, $\mathrm{UHPR} \underset{a}{\subset} \mathrm{ULR} \underset{a}{\subset} \mathrm{SLR} \underset{a}{\subset} \mathrm{SR} \underset{a}{\subset} \mathrm{TIC}_\infty$ etc. Note that it follows that $\mathcal{A}(U,Y)$ is a subgroup of $\mathcal{A}'(U,Y)$ and $\mathcal{A}(U)$ is a subring of $\mathcal{A}'(U)$, for each $U,Y \in \mathcal{X}$ (where $\mathcal{X}$ is, e.g., the collection of all Hilbert spaces).

Because limits commute with most basic operations, such operations preserve regularity:

Lemma 6.2.5 (Regularity preserved) *Any form of regularity is preserved under sums and scalar multiplication, under (left or right) multiplication by static operators, and under convergence in* TIC_ω *($\omega \in \mathbf{R}$).*

All strong and uniform properties are preserved under composition of maps. All weak and uniform properties are preserved under taking causal adjoints ($\mathbb{D} \mapsto \mathbb{D}^{\mathrm{d}} := \text{Я}\mathbb{D}^*\text{Я}$).

In general, uniformly $*$ $\Longrightarrow$ strongly $*$ $\Longrightarrow$ weakly $*$, when $*$ is anything suitable from Definition 6.2.3; similarly, $*$ half-plane-regular $\Longrightarrow$ $*$ line-regular $\Longrightarrow$ $*$ regular, $*$ half-plane-regular $\Longrightarrow$ $*$ vertically regular $\Longrightarrow$ $*$ regular.

Furthermore, $\mathbb{D}\mathbb{E}$ is WR with feedthrough DE (but $\mathbb{E}\mathbb{D}$ need not be WR) if $\mathbb{D}$ is WR and $\mathbb{E}$ is SR (or $\mathbb{D}^{\mathrm{d}}$ is SR and $\mathbb{E}$ is WR).

Thus, $\mathrm{TIC}_\omega(U,Y)\cap \mathrm{WR}$ is a closed subspace of TIC_ω ($\omega\in\mathbf{R}$), the same holds for SR, UR, ULR or any other regularity property in place of WR, and $\mathbb{D}\mapsto D$ is a bounded linear operation on any such subspace.

Proof: Obviously, the limits commute with sums, scalar multiplication and multiplication by static operators (e.g., for $L\in\mathcal{B}$, the map $L\mathbb{D}$ or $\mathbb{D}L$ has (at least) the same regularity properties as $\mathbb{D}$ has).

Clearly $\|D\|\le\|\widehat{\mathbb{D}}\|_{\mathrm{H}^\infty_\omega}=\|\mathbb{D}\|_{\mathrm{TIC}_\omega}$. Thus, if $\mathbb{D}_n\to\mathbb{D}$ in $\mathrm{TIC}_\omega(U,Y)$, and $\mathbb{D}_n\in\mathrm{WR}$ for all n, then $\{D_n\}$ converges in $\mathcal{B}(U,Y)$, and one easily verifies that $\mathbb{D}$ is WR and $D_n\to D$ weakly. An analogous claim holds for any other weak or uniform regularity property.

Similarly strong and uniform limits commute with composition; e.g., if $\mathbb{D},\mathbb{E}\in\mathrm{SR}$, then $\mathbb{D}\mathbb{E}\in\mathrm{SR}$ with feedthrough DE (see Lemma A.3.1(j2)).

From Lemma 6.2.2 we observe that weak and uniform limits commute with causal adjointing (e.g., if $\mathbb{D}$ is WR, then $\mathbb{D}^{\mathrm{d}}$ is WR with feedthrough D^*).

The claims on $\mathbb{D}\mathbb{E}$ and $\mathbb{E}\mathbb{D}$ follow from Lemma A.3.1(j2) and Example 6.2.6 (with $\mathbb{D}\mapsto\mathbb{D}^{\mathrm{d}}$, $\mathbb{E}\mapsto\mathbb{D}$). □

As the formulation of the above lemma hints, strong regularity is not inherited by adjoints, and weak regularity is not preserved under composition:

Example 6.2.6 ($\mathbb{D}\mathbb{D}^{\mathbf{d}}$ is not WR) Let $\mathbf{N}:=\{0,1,2,3,\ldots\}$, $U=\ell^2(\mathbf{N};\mathbf{C})$, $Y=\mathbf{C}$, $f(s):=s/(1+s)^2$, and

$$\widehat{\mathbb{D}}(s)u:=\sum_{n\in\mathbf{N}}f(10^{-n}s)u_n. \tag{6.32}$$

Then $\mathbb{D}\in\mathrm{TIC}$, $D=0$, $\mathbb{D}$ is strongly half-plane-regular, $\mathbb{D}^{\mathrm{d}}\in\mathrm{WR}\setminus\mathrm{SR}$ and $\mathbb{D}\mathbb{D}^{\mathrm{d}}\notin\mathrm{WR}$. ◁

(The above claims are explicitly or implicitly contained in the computations of Example 8.1 of [SW01b].)

It follows that weak regularity is not preserved under feedback or cascade connection (see the remarks below Proposition 6.6.18).

We have called $D:=\widehat{\mathbb{D}}(+\infty)$ the feedthrough operator of $\mathbb{D}$. This is justified, since the step response $\mathbb{D}\pi_+u_0$ is close to Du_0 near $t=0$ (in the average):

Proposition 6.2.7 ("$\mathbb{D}^{\!\!\!\not\,} u_0\to Du_0$") *A map $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ is WR iff*

$$\operatorname*{w-lim}_{t\to0+}\frac{1}{t}\int_0^t\mathbb{D}\chi_{\mathbf{R}_+}u_0\,dm=:Du_0 \tag{6.33}$$

exists for all $u_0\in U$. If this is the case, then $D=\widehat{\mathbb{D}}(+\infty)$. Analogously, $\mathbb{D}$ is SR iff (6.33) converges strongly for each $u_0\in U$. □

(This is given in Theorem 4.6 of [SW01a] and Theorem 5.8 of [W94a]; it is the original definition of regularity.)

To obtain the convergence $(\mathbb{D}\chi_{\mathbf{R}_+}u_0)(t)\to Du_0$ (instead of the above convergence in the average), we seem to need a stronger assumption; e.g., for $\mathbb{D}\in\mathrm{SMTIC}_\infty$, $\mathbb{D}\chi_{\mathbf{R}_+}u_0$ becomes continuous with value Du_0 at zero, by Theorem 2.6.4(i3).

Recall from Lemma 6.1.16 and Definition 6.1.17 that $H_1 \underset{c}{\subset} H_B \underset{c}{\subset} H \underset{c}{\subset} H_{-1}$ for any $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(*,H,*)$, and that H_1 is the domain of the output operator C.

When $\mathbb{D} \in \mathrm{TIC}_\infty$ is the I/O map of a WPLS, the weak regularity of $\mathbb{D}$ is equivalent to the existence of the *(weak) Weiss extension* C_w of the output operator of the system, as shown below. This extremely important operator allows one to write the output of the system in the form $y(t) = C_\mathrm{w}x(t) + Du(t)$ (a.e.), and the transfer function as $\widehat{\mathbb{D}}(s) = D + C_\mathrm{w}(s-A)^{-1}B$, as shown later in this section.

Proposition 6.2.8 ($C_\mathbf{w}, C_\mathbf{s}, C_{\mathbf{L,s}}, C_{\mathbf{L,w}}, B^*_\mathbf{w}, \ldots$) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *have generating operators* $\left[\begin{smallmatrix} A & B \\ C & * \end{smallmatrix}\right]$ *(cf. Definition 6.1.17), and let* $\omega \in \mathbf{R}$.

(a1) Σ *is WR iff* $H_B \subset \mathrm{Dom}(C_\mathrm{w}) := \{x_0 \in H \,|\, \exists C_\mathrm{w}x_0 := \text{w-lim}_{s\to+\infty} Cs(s-A)^{-1}x_0\}$.
If Σ *is WR, then* $C_\mathrm{w} \in \mathcal{B}(H_B,Y)$.

(a2) Σ *is SR iff* $H_B \subset \mathrm{Dom}(C_\mathrm{s}) := \{x_0 \in H \,|\, \exists C_\mathrm{s}x_0 := \lim_{s\to+\infty} Cs(s-A)^{-1}x_0\}$.
If Σ *is SR, then* $C_\mathrm{s} \in \mathcal{B}(H_B,Y)$.

(a3) Σ *is regular in a certain sense iff* $C_\mathrm{w}x_0 := \lim_{s\to+\infty} Cs(s-A)^{-1}x_0$ *converges in the corresponding sense, for each* $x_0 \in (\alpha - A)^{-1}BU$ *(here* $\alpha \in \mathbf{C}^+_{\omega_A}$ *is irrelevant).*

(b1) We set $\|x_0\|_{\mathrm{Dom}(C_\mathrm{w})} := \|x_0\|_H + \sup_{s>\omega_A+1} \|Cs(s-A)^{-1}x_0\|_Y$.

Consequently, $\mathrm{Dom}(C_\mathrm{w})$ *becomes a Banach space and* $\mathrm{Dom}(C_\mathrm{s})$ *becomes its closed subspace. The operators* $C_\mathrm{s} : \mathrm{Dom}(C_\mathrm{s}) \to Y$ *and* $C_\mathrm{w} : \mathrm{Dom}(C_\mathrm{w}) \to Y$ *are continuous and* $H_1 \subset \mathrm{Dom}(C_\mathrm{s}) \subset \mathrm{Dom}(C_\mathrm{w}) \subset H$ *continuously.*

(b2) Part (b1) holds for any operator $C \in \mathcal{B}(H_1,Y)$ *and any* C_0*-semigroup generator* A.

(c1) Σ *is WR iff* $H_B \subset \mathrm{Dom}(C_{\mathrm{L,w}}) := \{x_0 \in H \,|\, \exists C_{\mathrm{L,w}}x_0 := \text{w-lim}_{t\to 0+} \frac{1}{t} C \int_0^t \mathbb{A}^r x_0\, dr\}$.

(c2) We have $x_0 \in \mathrm{Dom}(C_{\mathrm{L,w}})$ *iff* $\lim_{t\to 0+} \frac{1}{t}\langle \mathbb{C}^* \chi_{[0,t)} y_0, x_0\rangle_H$ *exists for all* $y_0 \in H$*; if this is the case, then this limit is equal to* $\langle y_0, C_{\mathrm{L,w}}x_0\rangle_Y$.

(c3) We have $x_0 \in \mathrm{Dom}(C_{\mathrm{L,w}})$ *iff* $\text{w-lim}_{t\to 0+} \frac{1}{t}\int_0^t (\mathbb{C}x_0)(r)\, dr$ *exists; if this is the case, then this limit is equal to* $C_{\mathrm{L,w}}x_0$.

(c4) The strong forms of (c1)–(c3) also hold; the corresponding (strong Lebesgue) extension of C *is denoted by* $C_{\mathrm{L,s}}$.

(c5) We set $\|x_0\|_{\mathrm{Dom}(C_{\mathrm{L,w}})} := \|x_0\|_H + \sup_{t\in(0,1]} \|C\frac{1}{t}\int_0^t \mathbb{A}^r x_0\, dr\|_Y$.

Consequently, $\mathrm{Dom}(C_{\mathrm{L,w}})$ *becomes a Banach space, and* $\mathrm{Dom}(C_{\mathrm{L,s}})$ *becomes its closed subspace. We have* $C_{\mathrm{L,w}} \in \mathcal{B}(\mathrm{Dom}(C_{\mathrm{L,w}}),Y)$*, and* $C_{\mathrm{L,s}} \in \mathcal{B}(\mathrm{Dom}(C_{\mathrm{L,s}}),Y)$. *Moreover,* $H_1 \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,w}}) \underset{c}{\subset} \mathrm{Dom}(C_\mathrm{w}) \underset{c}{\subset} H$.

(c6) $\mathrm{Dom}(A) \ni \frac{1}{t}\int_0^t \mathbb{A}^r x_0\, dr \to x_0$ *in* $\mathrm{Dom}(C_{\mathrm{L,s}})$*, as* $t \to 0+$*, for each* $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$. *In particular,* $\mathrm{Dom}(A)$ *is dense in* $\mathrm{Dom}(C_{\mathrm{L,s}})$.

(d1) $C \subset C_{\mathrm{L,s}} \subset C_\mathrm{s} \subset C_\mathrm{w}$, $C \subset C_{\mathrm{L,s}} \subset C_{\mathrm{L,w}} \subset C_\mathrm{w}$, *hence* $B^* \subset B^*_{\mathrm{L,s}} \subset B^*_\mathrm{s} \subset B^*_\mathrm{w}$.

(d2) The domains of all operators of (d1) are dense in H.

(e) We have $B \in \mathcal{B}(U, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$, and its adjoint is $B^*_{\mathrm{L,s}}$ (when $\mathrm{Dom}(B^*_{\mathrm{L,s}})^*$ (not H_{-1}) is considered as the range space of B and H (as always) the pivot space).

In particular, for any $u_0 \in U$, the functional $Bu_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})^* \underset{c}{\subset} H_{-1} := \mathrm{Dom}(A^*)^*$ is the restriction of any of $(B^*_{\mathrm{L,w}})^*u_0$, $(B^*_{\mathrm{s}})^*u_0$ and $(B^*_{\mathrm{w}})^*u_0$.

(f) $A \in \mathcal{B}(H_B, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$.

By $C \subset C_{\mathrm{s}}$ we mean that C_{s} is an extension of C (i.e., $H_1 = \mathrm{Dom}(C) \subset \mathrm{Dom}(C_{\mathrm{s}})$ and $C_{\mathrm{s}}x_0 = Cx_0$ for all $x_0 \in \mathrm{Dom}(C)$). By B^*_{w} (resp. B^*_{s}, $B^*_{\mathrm{L,s}}$, $B^*_{\mathrm{L,w}}$) we always refer to $(B^*)_{\mathrm{w}}$ (resp. $(B^*)_{\mathrm{s}}$, $(B^*)_{\mathrm{L,s}}$, $(B^*)_{\mathrm{L,w}}$).

Note that, by (a1), $H^*_C \subset \mathrm{Dom}(B^*_{\mathrm{w}}) := \{x \in H \,\big|\, \exists B^*_{\mathrm{w}}x := \text{w-lim}_{s\to+\infty} B^*s(s-A^*)^{-1}x\}$ iff Σ^{d} is WR, and in that case, $B^*_{\mathrm{w}} \in \mathcal{B}(H^*_C, U)$.

Obviously, $C_{\mathrm{w}} = C_{\mathrm{s}} = C_{\mathrm{L,s}} = C_{\mathrm{L,w}} = C$ if C is bounded ($C \in \mathcal{B}(H,Y)$), and $H_B = H_1 \subset C_{\mathrm{s}}$ if B is bounded; hence Σ is SR if B or C is bounded. See Example 6.2.14 for a nontrivial example of C_{w} and B^*_{w}.

Note that we have equipped the domains of these four extensions of C with special norms instead of graph norms in order to make the domains Banach spaces (i.e., complete; it was noted in [W89b] that if $\dim Y < \infty$, then $C_{\mathrm{L,s}}$ cannot be closable unless C is bounded).

For more information on the *strong Weiss extension* C_{s} (or on $C_{\mathrm{L,s}}$), see [W89b] and [W94b, Section 5] or [Sbook]; for more information on the *weak Weiss extension* C_{w} (or on $C_{\mathrm{L,w}}$), see [SW01a], [WW, Sections 2 & 4] or [Sbook].

Proof: For the definitions of $H_B := (\alpha - A)^{-1}[H + BU]$ and H^*_C, see Definition 6.1.17.

(b) The SR counterpart of this is given on pp. 42–43 of [W94b]; the same proof applies for the WR claim mutatis mutandis, by [SW01a], and the requirements in (b2) are enough for this. (The details are in Proposition 4.3 of [W89b], also [W94a, p. 848] is relevant.) One can replace $\omega_A + 1$ by any other value $> \omega_A$ to obtain an equivalent norm, by the resolvent equation (Lemma A.4.4(a)).

(a1)&(a2) This follows from [S97b, Proposition 36], Lemma A.3.6 and (b).

(a3) Let $\mathbb{D}$ be WR. By (a1), Lemma A.4.4(a), and the linearity of C_{w}, we have

$$\begin{aligned} Cs(s-A)^{-1}(\alpha-A)^{-1}Bu_0 &= \frac{s}{s-\alpha}\big(C_{\mathrm{w}}(\alpha-A)^{-1}Bu_0 - C_{\mathrm{w}}(s-A)^{-1}Bu_0\big) \\ &\to 1\cdot C_{\mathrm{w}}(\alpha-A)^{-1}Bu_0, \end{aligned} \tag{6.34}$$

as $s \to +\infty$, in the sense that $C_{\mathrm{w}}(s-A)^{-1}Bu_0 = \widehat{\mathbb{D}}(s) - D$ converges to zero (e.g., weakly, strongly, vertically, ..., but not independently on α), hence the claim.

(c4) The strong version of (c1) is Theorem 5.8 of [W94a]. The proofs of (c2)–(c3) apply in the strong case mutatis mutandis.

(c1) See Theorem 4.6 of [SW01a].

(c3) We have $\int_0^r \mathbb{C}x_0\,dm = C\int_0^r \mathbb{A}x_0\,dm$ for all $x_0 \in H$, because this holds on H_1 and both sides are continuous $H \to Y$. Thus, the extensions in (c1) and (c3) are equal.

(c2) We have $\langle y_0, \frac{1}{t}\int_{\mathbf{R}} \pi_{[0,t)}\mathbb{C}x_0\rangle_Y = \frac{1}{t}\int_{\mathbf{R}}\langle y_0, \pi_{[0,t)}\mathbb{C}x_0\rangle_Y = \frac{1}{t}\langle \pi_{[0,t)}y_0, \mathbb{C}x_0\rangle_{\mathrm{L}^2}$, hence (c2) follows from (c3) (use Lemma A.3.4(i3)).

(c5) It is shown in Proposition 4.3 of [W89b] that $\mathrm{Dom}(C_{\mathrm{L,s}})$ is a Banach space; the weak case is analogous (see [WW] or [Sbook]). The continuity of $C_{\mathrm{L,w}}$ and $C_{\mathrm{L,s}}$ is obvious, and so are the inclusions (use Lemma A.3.6 for continuity), except for $\mathrm{Dom}(C_{\mathrm{L,w}}) \subset \mathrm{Dom}(C_{\mathrm{w}})$, which is given in (d1).

(c6) This is given in the proof of Theorem 5.2 of [W94b]. It is an open problem whether H_1 is always dense in $\mathrm{Dom}(C_{\mathrm{s}})$.

(d1) See Proposition 4.2 of [SW01a].

(N.B. V. Katsnelson [KW] has constructed a system where $\mathrm{Dom}(C_{\mathrm{L,w}}) = \mathrm{Dom}(C_{\mathrm{L,s}}) \neq \mathrm{Dom}(C_{\mathrm{s}}) = \mathrm{Dom}(C_{\mathrm{w}})$. Example 6.2.6 shows that we may have $\mathrm{Dom}(C_{\mathrm{L,s}}) \neq \mathrm{Dom}(C_{\mathrm{L,w}})$ and $\mathrm{Dom}(C_{\mathrm{s}}) \neq \mathrm{Dom}(C_{\mathrm{w}})$. Thus, all inclusions in (d1) may be strict, as noted in Proposition 4.2 of [SW01a].)

(d2) Sets $\mathrm{Dom}(A)$ and $\mathrm{Dom}(A^*)$ are dense in H, hence so are any of their supersets.

(e) Note first that $\mathrm{Dom}(A^*) \underset{c}{\subset} \mathrm{Dom}(B^*_{\mathrm{L,s}}) \underset{c}{\subset} H \underset{c}{\subset} \mathrm{Dom}(B^*_{\mathrm{L,s}})^* \underset{c}{\subset} H_{-1}$, and that the first two (and the last) inclusions are dense, by (c6) and Lemma A.3.24 (we needed the density of H_1 in $\mathrm{Dom}(B^*_{\mathrm{L,s}})$ to obtain $\mathrm{Dom}(B^*_{\mathrm{L,s}})^* \subset H_{-1}$ (one-to-one)). For all $x_0 \in \mathrm{Dom}(A^*)$, $u_0 \in U$, we have

$$\langle x_0, Bu_0\rangle_{\langle H_1^*, H_{-1}\rangle} = \langle B^*x_0, u_0\rangle_U = \langle B^*_{\mathrm{L,s}}x_0, u_0\rangle_U = \langle x_0, (B^*_{\mathrm{L,s}})^*u_0\rangle_{\langle Z, Z^*\rangle}, \tag{6.35}$$

where $Z := \mathrm{Dom}(B_{\mathrm{L,s}})$. Thus, $Bu_0 \in H_{-1} := \mathrm{Dom}(A^*)^*$ is continuous w.r.t. to the $\|\cdot\|_{\mathrm{Dom}(B^*_{\mathrm{L,s}})}$ norm, hence and element of $\mathrm{Dom}(B^*_{\mathrm{L,s}})^*$. By (6.35) and density, this element is equal to $(B^*_{\mathrm{L,s}})^*u_0$.

Of course, we can replace $B^*_{\mathrm{L,s}}$ by any of its extensions (see (d1)) in equation $\langle x_0, Bu_0\rangle_{H_1^*, H_{-1}^*} = \langle B^*_{\mathrm{L,s}}x_0, u_0\rangle_U$ ($x_0 \in \mathrm{Dom}(A^*)$, $u_0 \in U$). (Note that this does not determine Bu_0 as an element of $\mathrm{Dom}(B^*_{\mathrm{L,w}})$ uniquely unless H_1 happens to be dense in $\mathrm{Dom}(B^*_{\mathrm{L,w}})$; situation is the same for B^*_{s} and B^*_{w}.)

(f) For any $x_0 \in H_B$, we have $z_0 := Ax_0 + Bu_0 \in H$ for some $u_0 \in U$, hence $Ax_0 = z_0 - Bu_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})^*$, by (e). Thus, $A[H_B] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})^*$. But $A \in \mathcal{B}(H, H_{-1})$ and $H_B \underset{c}{\subset} H$, hence $A \in \mathcal{B}(H_B, H_{-1})$. Therefore, $A \in \mathcal{B}(H_B, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$, by Lemma A.3.6 and (e). □

Next we present four lemmas that explore the relation between a system and its generators:

Lemma 6.2.9 ($\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ have generators $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ (resp. $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$, and $\mathbb{D} \in \mathrm{WR}$). Then*

(a) The operators $\begin{bmatrix} A & B \\ C & \widehat{\mathbb{D}}(s)\end{bmatrix}$ (resp. $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$) determine Σ uniquely for any $s \in \mathbf{C}^+_{\omega_A}$.

(b) The generators of Σ^{d} are $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$ (resp. $\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^*\end{smallmatrix}\right]$, and Σ^{d} is WR).

(c) The generators of $\mathcal{T}_\omega\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *are* $\left[\begin{smallmatrix}A+\omega I & C\\ B & *\end{smallmatrix}\right]$ *(resp.* $\left[\begin{smallmatrix}A+\omega I & C\\ B & D\end{smallmatrix}\right]$*, and* $\mathcal{T}_\omega\Sigma$ *is WR). Moreover,* $(C_\omega)_{\mathrm{w}} = C_{\mathrm{w}}$ *and* $(C_\omega)_{\mathrm{L,w}} = C_{\mathrm{L,w}}$.

See Proposition 6.6.18 for the generating operators of closed-loop systems.

Proof: (a) This follows from Lemma 6.1.16(d) (resp. and Theorem 6.2.13(a1)).

(b) This holds by Lemma 6.1.16(e) (resp. and Lemma 6.2.2).

(c) For generators, the simple proof is given in Example 4.7.2 of [Sbook]. Because $s(s-A_\omega)^{-1} = \frac{s}{s-\omega}(s-\omega)(s-\omega-A)^{-1}$, the C_{w} claim follows easily. The $C_{\mathrm{L,w}}$ claim as follows from the fact that (here $f := \mathbb{C}x_0 \in \mathrm{L}^2_{\mathrm{loc}} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;Y)$):

$$t^{-1}\int_0^t (1-\mathrm{e}^{-\omega r} f(r)\,dr = t^{-1}\int_0^t \omega r\mathrm{e}^{-\omega r\xi_r} f(r)\,dr \to 0 \tag{6.36}$$

(here $\xi_r \in (0,1)$ for all $r > 0$ (use the Mean Value Theorem)), as $t \to 0+$. □

We shall soon need the following technical lemma:

Lemma 6.2.10 ($\mathbb{B}u = (s-A)^{-1}Bu_0$, $(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in$ WPLS *and let* $\mathrm{Re}\, s > \omega_A$*. Let* $u_0 \in U$ *and* $u(t) = \mathrm{e}^{st}u_0$ *(*$t \in \mathbf{R}$*), so that* $\pi_- u \in \mathrm{L}^2_{\omega_A}(\mathbf{R};U)$*. Then* $\mathbb{B}u = \mathbb{B}\pi_- u = (s-A)^{-1}Bu_0 \in H_B \subset H$*,* $\mathbb{B}\tau^t u = \mathrm{e}^{st}(s-A)^{-1}Bu_0$ *and* $(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$ *for all* $t \in \mathbf{R}$*. Moreover, we have*

$$\mathbb{B}\tau^t\pi_+\mathrm{e}^{s\cdot}u_0 = (\mathrm{e}^{st} - \mathbb{A}^t)(s-A)^{-1}Bu_0 \quad (t \geq 0), \tag{6.37}$$

$$\mathbb{D}\pi_+\mathrm{e}^{s\cdot}u_0 = \pi_+\mathrm{e}^{s\cdot}\widehat{\mathbb{D}}(s)u_0 - \mathbb{C}(s-A)^{-1}Bu_0 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U). \tag{6.38}$$

Proof: The claim about $\mathbb{D}$ is Lemma 2.1.15. Because $\widehat{\mathbb{B}\tau} = (s-A)^{-1}B$ (see Theorem 6.2.11(b1)), claims on $\mathbb{B}u$ and $\mathbb{B}\tau u$ follow from this. But $\mathbb{B}\tau^t\pi_- = \mathbb{A}^t\mathbb{B}$, hence (6.37) follows; (6.38) is obtained analogously, by using $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$.

(An alternative proof of Lemma 2.1.15 would be to obtain $\mathbb{B}u = (s-A)^{-1}Bu_0$ from (6.24) and Lemma A.4.4(f), and then $(\mathbb{D}u)(t) = C_{\mathrm{c}}(s-A)^{-1}B\mathrm{e}^{st}u_0 + D_{\mathrm{c}}\mathrm{e}^{st}u_0 = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$ from Lemma 6.3.10(c).) □

Now we are ready to show that for any $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\beta(U,H,Y)$, $\beta \in \mathbf{R}$, $s \in \mathbf{C}^+_\beta$, $u \in \mathrm{L}^2_\beta(\mathbf{R}_+;U)$ and $x_0 \in H$, we have

$$\begin{cases}\widehat{x}(s) = (s-A)^{-1}x_0 + (s-A)^{-1}B\widehat{u}(s)\\ \widehat{y}(s) = C(s-A)^{-1}x_0 + C_{\mathrm{w}}(s-A)^{-1}B\widehat{u}\end{cases} \quad \text{where} \quad \begin{cases} x = \mathbb{A}x_0 + \mathbb{B}\tau u,\\ y = \mathbb{C}x_0 + \mathbb{D}u,\end{cases} \tag{6.39}$$

(the formula for $\widehat{y}$ requires that $\mathbb{D}$ is WR or $u = 0$). This and further information on the Laplace transforms of the components of a system are given below:

Theorem 6.2.11 ($\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}, \widehat{\mathbb{D}}$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *have generating operators* $\left[\begin{smallmatrix}A & B\\ C & *\end{smallmatrix}\right]$ *and let* $\omega := \omega_A \leq \alpha < \beta$.

(a) $\widehat{\mathbb{A}(\cdot)x_0}(s) = (s-A)^{-1}x_0$ *for all* $x_0 \in H$*,* $s \in \mathbf{C}^+_\omega$.

(b1) $\widehat{\mathbb{B}\tau u}(s) = (s-A)^{-1}B\widehat{u}(s)$ *for* $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$ *and* $s \in \mathbf{C}^+_\alpha$.

(b2) $(s-A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\beta;\mathcal{B}(U,H)) \cap \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_B))$.

(b3) $\|(s-A)^{-1}B\|_{\mathcal{B}(U,H)} \le \|\mathbb{B}\|_{\mathcal{B}(\mathrm{L}^2_\gamma(\mathbf{R};U),H)}/\sqrt{2}\sqrt{\mathrm{Re}\, s-\gamma}$ *for* $s \in \mathbf{C}^+_\gamma$, $\gamma \in \mathbf{R}$.

(c1) $\widehat{\mathbb{C}x_0}(s) = C(s-A)^{-1}x_0$ *for* $s \in \mathbf{C}^+_\omega$, $x_0 \in H$.

(c2) $C(s-A)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_\beta;\mathcal{B}(H,Y))$, *and* $C(s-A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\beta;Y)$ *for* $x_0 \in H$. *Moreover,* $\|C(\cdot-A)^{-1}\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))} = \sqrt{2\pi}\|\mathbb{C}\|_{\mathcal{B}(H,\mathrm{L}^2_\omega)}$.

(c3) $\|C(s-A)^{-1}\|_{\mathcal{B}(H,Y)} \le \|\mathbb{C}\|_{\mathcal{B}(H,\mathrm{L}^2_\gamma(\mathbf{R};Y))}/\sqrt{2}\sqrt{\mathrm{Re}\, s-\gamma}$ *for* $s \in \mathbf{C}^+_\gamma$, $\gamma \in \mathbf{R}$.

(d1) $\widehat{\mathbb{D}}(s) = C_{\mathrm{w}}(s-A)^{-1}B + D$ *on* $\mathbf{C}^+_\omega$ *if* $\mathbb{D}$ *is WR,* [2] *hence then* $C_{\mathrm{w}}(s-A)^{-1}Bu_0 \to 0$ *weakly as* $s \to +\infty$, *for all* $u_0 \in H$. *The same holds also for* C_{s} *in place of* C_{w} *if* $\mathbb{D}$ *is SR.*

(d2) For $s, s_0 \in \mathbf{C}^+_\omega$ *we have*

$$\frac{\widehat{\mathbb{D}}(s) - \widehat{\mathbb{D}}(s_0)}{s-s_0} = -C(s-A)^{-1}(s_0-A)^{-1}B = -(\mathcal{L}(\pi_+\mathbb{D}\pi_-\mathrm{e}^{s_0\cdot}u_0))(s). \tag{6.40}$$

Proof: (a) This is Lemma A.4.4(f).

(b1) This is [W89a, Remark 3.12].

(b2) By (c2) and Lemma 6.1.4, $B^*(s-A^*)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_{\omega'};\mathcal{B}(H,U))$, from which the "$\in \mathrm{H}^\infty$" claim follows.

Set $F(s) := (s-A)^{-1}B$. Then $F(\alpha) \in \mathcal{B}(U,H_B)$ (with $\|F(\alpha)u_0\|_{H_B} \le \|u_0\|_U$), where α is as in Definition 6.1.17, and $F(s) - F(\alpha) = (\alpha - s)(\alpha - A)^{-1}(s-A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_1))$, by Lemma A.4.4(a). But $\mathcal{B}(U,H_1) \subset \mathcal{B}(U,H_B)$ continuously, hence $F(s) \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_B))$, by Lemma D.1.2(b1).

(b3) Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}$ be s.t. $\mathbb{B}$ is stable, $u_0 \in U$, $\mathrm{Re}\, s > \omega_A$. Set $u := \pi_-\mathrm{e}^{st}u_0 \in \mathrm{L}^2(\mathbf{R}_-;U)$. By Lemma 6.2.10, we have

$$\|(s-A)^{-1}Bu_0\| = \|\mathbb{B}u\| \le \|\mathbb{B}\|_{\mathcal{B}(\mathrm{L}^2;H)}\|u\|_2 = \|\mathbb{B}\|\|u_0\|_H/\sqrt{2\,\mathrm{Re}\, s}. \tag{6.41}$$

By shifting, we obtain an analogous claim for an arbitrary $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}$ (note that $\|\mathbb{B}\| < \infty$ at least for $\gamma > \omega_A$).

(The condition "$\|(s-A)^{-1}Bu_0\| \le M/(\mathrm{Re}\, s)^{1/2}$ for all $s \in \mathbf{C}^+$" is also suffient for given $B \in \mathcal{B}(U,H_{-1})$ to generate a WPLS with A if, e.g., A is left-invertible [W91b] or A generates a contraction semigroup and U is finite-dimensional [JP]; however this is not the case in general, see [JZ00] for a counter-example.

(c1) This is (3.6) on p. 26 of [W89b].

(c2) By (c1) and the Paley–Wiener Theorem Theorem 3.3.1(b), we have $\|\widehat{\mathbb{C}x_0}\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;Y)} = \sqrt{2\pi}\|\mathbb{C}x_0\|_{\mathrm{L}^2}$ for all $x_0 \in H$, hence the second and third claim hold. The first claim follows from the third claim and Lemma F.3.2(a).

[2] If $\mathbb{D} \in \mathrm{TIC}_\alpha$ for some $\alpha < \omega_A$, then, by $C_{\mathrm{w}}(s-A)^{-1}B + D$ for $s \in \mathbf{C}_\alpha \setminus \mathbf{C}_{\omega_A}$, we mean the unique analytic extension $\widehat{\mathbb{D}}(\cdot)$ of this function (otherwise the equation would not always hold for $s \in \mathbf{C}^+_\alpha \cap \sigma(A)^c \setminus \mathbf{C}^+_{\omega_A}$, see [W94a, Remark 4.8]).

(c3) This is the dual of (b3).

(d1) This is [WW, Theorem 4.4].

(d2) The first identity is given in [Sal89] (and in Theorem 4.5.9 of [Sbook]; it follows from (6.51)). For the second, we note that

$$(\mathcal{L}(\pi_+\mathbb{D}\pi_- \mathrm{e}^{s_0 \cdot} u_0))(s) = (\mathcal{L}(\mathbb{C}\mathbb{B}\mathrm{e}^{s_0 \cdot} u_0))(s) = (\mathcal{L}(\mathbb{C}(s_0 - A)^{-1} B u_0))(s) \tag{6.42}$$

$$= C(s-A)^{-1}(s_0-A)^{-1}Bu_0, \tag{6.43}$$

by Lemma 6.2.10 and (c1). □

We shall need several different forms of the formula "$\mathbb{C} = C\mathbb{A}$":

Lemma 6.2.12 ($\mathbb{C} = C_{\mathrm{w}}\mathbb{A}$) *Let $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega > \omega_A$, $x_0 \in H$, $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$, $x = \mathbb{A}x_0 + \mathbb{B}\tau u$. Set $C^r_{\mathrm{L,s}}x_1 := C\frac{1}{r}\int_0^r \mathbb{A}^s x_1\,ds$ ($x_1 \in H$). Then*

- *(a) $(\mathbb{C}x_0)(t) = C_{\mathrm{L,s}}\mathbb{A}^t x_0 = C_{\mathrm{w}}\mathbb{A}^t x_0$ for a.e. $t \geq 0$ (for all $t \geq 0$ if $x_0 \in \mathrm{Dom}(A)$). In particular, $\mathbb{A}^t x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e.*
- *(b1) We have $C^r_{\mathrm{L,s}}x \to C_{\mathrm{L,s}}x$ in L^2_ω and pointwise a.e., as $r \to 0+$, (in particular, $x \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e.) if $\mathbb{D}$ is SR.*
- *(b2) We have $C^r_{\mathrm{L,s}}x \to C_{\mathrm{L,w}}x$ weakly in L^2_ω and weakly pointwise a.e., as $r \to 0+$, (in particular, $x \in \mathrm{Dom}(C_{\mathrm{L,w}})$ a.e.) if $\mathbb{D}$ is WR.*
- *(c1) Assume that $\mathbb{D}$ is SR. Then, for any $f \in \mathrm{L}^2_{-\omega}(\mathbf{R}_+)$, we have*

$$\int_{\mathbf{R}_+} f(t)C_{\mathrm{L,s}}x(t)\,dt = C_{\mathrm{L,s}}\int_{\mathbf{R}_+} f(t)x(t)\,dt \tag{6.44}$$

  (in particular, $\int_0^\infty f(t)x(t)\,dt \in \mathrm{Dom}(C_{\mathrm{L,s}})$). Thus, then $\int_{T_1}^{T_2} f(t)C_{\mathrm{L,s}}x(t)\,dt = C_{\mathrm{L,s}}\int_{T_1}^{T_2} f(t)x(t)\,dt$ for any $T_1,T_2 \in \mathbf{R}_+$, $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$.
- *(c2) Claim (c1) also holds with replacements SR$\mapsto$WR and $C_{\mathrm{L,s}} \mapsto C_{\mathrm{L,w}}$.*
- *(c3) In (b1)–(c2), we may replace $\mathbf{R}_+$ by $[T,+\infty)$, and allow for any $u \in \mathrm{L}^2_\omega([T,+\infty);U)$ ($T \in \mathbf{R}$).*
- *(c4) In (b1)–(c2), $\mathbb{D}$ need not be regular if $u = 0$.*

Proof: (a) By [W89b, Theorem 4.5], we have $(\mathbb{C}x_0)(t) = C_{\mathrm{L,s}}\mathbb{A}^t x_0$ for any (right-)Lebesgue point t of $\mathbb{C}x_0$, hence for a.e. $t \geq 0$. Recall that $C_{\mathrm{L,s}} \subset C_{\mathrm{w}}$.

(b1) The L^2_ω claim is Lemma 4.4 of [W94a]; the pointwise claim follows from the fact that $x(t) \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e., by Theorem 5.8 of [W94a].

(b2)&(c2) The proofs of (b1) and (c1) apply mutatis mutandis (which means corresponding slight changes in the proofs of [W94a]; these results were contained in a preprint of [SW01a] and implicitly used in [SW01a]).

(c1) This is essentially given in Theorem 4.6 of [W94a] (substitute $f \mapsto f\chi_{[T_1,T_2)}$ to obtain the second claim).

(Alternatively, claim (c1) follows from (b1) (the proof of (c2) is analogous), since $fC^r_{\mathrm{L,s}}x \to fC_{\mathrm{L,s}}x$ in $\mathrm{L}^1(\mathbf{R}_+;Y)$, $fx, C^r_{\mathrm{L,s}}fx, C_{\mathrm{L,s}}x \in \mathrm{L}^1$, and $C^r_{\mathrm{L,s}}$ commutes with the integral (because $C^r_{\mathrm{L,s}} \in \mathcal{B}(H,Y)$).)

(c3) W.l.o.g., we assume that $x_0 = 0$ (substract (6.44) if necessary). Just shift u (hence x too) and f, and use the original claim.

(c4) Replace B by 0 to obtain a SR (in fact, ULR) WPLS with same $\mathbb{A}$ and $\mathbb{C}$ (see Lemma 6.3.16(b)). $\square$

Now we may extend Lemma 6.1.16 by presenting the standard formulae $x' = Ax + Bu$ and $y = Cx + Du$ with certain limitations:

Theorem 6.2.13 ($x' = Ax + Bu$, $y = Cx + Du$) *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U, H, Y)$ *and either*

(i) $x_0 \in H$, $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$ *and* $J = \mathbf{R}_+$, *or*

(ii) $x_0 = 0$, $u \in \mathrm{L}^2_\omega(\mathbf{R}; U)$ *and* $J = \mathbf{R}$.

Set

$$\begin{cases} x &= \mathbb{A}x_0 + \mathbb{B}\tau u, \\ y &= \mathbb{C}x_0 + \mathbb{D}u. \end{cases} \tag{6.45}$$

Then

(a1) $x' = Ax + Bu \in \mathrm{L}^2_{\mathrm{loc}}(J; H_{-1})$ *(a.e.) and* $x \in \mathcal{C}(J; H) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(J; H_{-1})$.

(a2) If $\mathbb{D}$ *is WR, then the following formulae hold (in particular,* $x(t) \in \mathrm{Dom}(C_{\mathrm{L,w}})$*) for almost every* $t \in J$*, including those* $t \in J$ *where* u *and* y *are right-continuous:*

$$y(t) = C_{\mathrm{L,w}}x(t) + Du(t) \tag{6.46}$$

$$= C_{\mathrm{L,w}} \int_0^t \mathbb{A}(t-s)Bu(s)\,ds + C_{\mathrm{L,w}}\mathbb{A}x_0 + Du(t) \qquad \text{(case (i) only)} \tag{6.47}$$

$$= C_{\mathrm{L,w}} \lim_{T\to-\infty} \int_{-T}^t \mathbb{A}(t-s)Bu(s)\,ds + Du(t) \qquad \text{(case (ii) only)}\,. \tag{6.48}$$

If $\mathbb{D}$ *is SR, then we can use* $C_{\mathrm{L,s}}$ *in place of* $C_{\mathrm{L,w}}$ *in (6.46)–(6.48).*

Stricten (i) and (ii) as follows:

(i) $x_0 \in H$, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+; U)$, $x_0' := Ax_0 + Bu(0) \in H$ *and* $J = \mathbf{R}_+$, *or*

(ii) $x_0 = 0 = x_0'$, $u \in \mathrm{W}^{1,2}_\omega(\mathbf{R}; U)$ *and* $J = \mathbf{R}$.

Then,

(b1)

$$x' = \mathbb{A}x_0' + \mathbb{B}\tau u' = Ax + Bu \in \mathcal{C}(J; H), \quad y' = \mathbb{C}x_0' + \mathbb{D}u', \tag{6.49}$$

$$e^{-\omega\cdot}x, e^{-\omega\cdot}x' \in \mathcal{C}_{\mathrm{b}}(J; H), \quad e^{-\omega\cdot}x \in \mathcal{C}_{\mathrm{b}}(J; H_B), \quad \text{and } y \in \mathrm{W}^{1,2}_\omega. \tag{6.50}$$

(b2) $y \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ *and* $y' = \mathbb{D}u'$. *If* $u \in \mathrm{W}^{1,2}_\omega$, *then* $y \in \mathrm{W}^{1,2}_\omega$. *Moreover, for any* $s_0 \in \mathbf{C}^+_\omega$ *we have*

$$(\mathbb{D}u)(t) = C(x(t) - (s_0 - A)^{-1}Bu(t)) + \widehat{\mathbb{D}}(s_0)u(t) \quad (t \in J). \tag{6.51}$$

(c1) If $x_0 \in H_n := \mathrm{Dom}(A^n)$, $n \in \mathbf{N}$, then $\mathbb{A}x_0 \in \mathcal{C}^k(\mathbf{R}_+;H_{n-k})$ $(k = 0,1,\ldots,n)$ and $y := \mathbb{C}x_0 \in \mathcal{C}^{k-1}(\mathbf{R}_+;H_{n-k})$ $(k = 1,\ldots,n)$ ($y \in \mathcal{C}^k(\mathbf{R}_+;H_{n-k})$ if $C \in \mathcal{B}(H,Y)$).

(c2) If $u \in \mathrm{W}^{n,2}_\omega(\mathbf{R};U)$, $n \in \mathbf{N}$, then $x := \mathbb{B}\tau u$, $\mathrm{e}^{-\omega\cdot}x \in \mathcal{C}^n_\mathrm{b}(\mathbf{R};H) \cap \mathcal{C}^{n-1}_\mathrm{b}(\mathbf{R};H_B) \cap \mathrm{W}^{n-1,2}(\mathbf{R};H_B)$, $y := \mathbb{D}u \in \mathrm{W}^{n,2}_\omega(\mathbf{R};Y) \subset \mathcal{C}^{n-1}$, $x^{(n)} = \mathbb{B}\tau u^{(n)}$, $y^{(n)} = \mathbb{D}u^{(n)}$.

(d) If, instead, $y \in \mathrm{L}^2_\omega(\mathbf{R};Y)$ is arbitrary, and $\mathbb{D} \in \mathrm{WR}$, then

$$(\mathbb{D}^* y)(t) = D^* y(t) + B^*_{\mathrm{L,w}} \lim_{T\to\infty} \int_0^T \mathbb{A}^*(s) C^* y(t+s)\, ds \tag{6.52}$$

a.e. and at every $t \in \mathbf{R}$ at which y and $\mathbb{D}^* y$ are left-continuous. (We can replace $B^*_{\mathrm{L,w}}$ by $B^*_{\mathrm{L,s}}$ if $\mathbb{D}^\mathrm{d} \in \mathrm{SR}$).

We conclude that $\mathbb{D} = D + C\mathbb{B}\tau$ when C is bounded. Recall from Proposition 6.2.8 that $C_{\mathrm{L,s}} \subset C_\mathrm{s} \subset C_\mathrm{w}$ and $C_{\mathrm{L,w}} \subset C_{\mathrm{L,s}} \subset C_\mathrm{w}$.

Proof: (a)&(b) This is well-known (see, e.g., Sections 4.2 and 4.5 of [Sbook] or combine Section 4 of [WW] and Propositions 29 and 33 of [S97b] with Remark 6.1.9 and Lemma 6.1.16).

(c1) This follows from Lemma A.4.2(c5). (Recall that $\mathbb{C}x_0 = C\mathbb{A}x_0$ for $x_0 \in \mathrm{Dom}(A)$.)

(c2) (Ignore $\mathcal{C}^{n-1}$ and $\mathrm{W}^{n-1,2}_\omega$ for $n = 0$.) This follows by applying (b2) subsequently, except for the H_B claims. We have $\tau u \in \mathcal{C}^{n-1}(\mathbf{R};\mathrm{W}^{1,2}_\omega)$, by Lemma B.7.11, and $\mathbb{B} \in \mathcal{B}(\mathrm{W}^{1,2}_\omega, H_B)$, by Lemma 6.3.19. Therefore, $\mathbb{B}\tau u \in \mathcal{C}^{n-1}(\mathbf{R};H_B)$.

Moreover, for $u \in \mathrm{W}^{n,2}_\omega$, $n \in \mathbf{N}+1$, $x := \mathbb{B}\tau u$, we have $(\alpha - A)x = \alpha x - x' + Bu$, and $\alpha x - x' \in \mathrm{L}^2_\omega(\mathbf{R};H)$, $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$, hence $x \in \mathrm{L}^2_\omega(\mathbf{R};H_B)$, because $(\alpha - A)^{-1}\begin{bmatrix} I & B\end{bmatrix} \in \mathcal{B}(H \times U, H_B)$. By induction, we have $x \in \mathrm{W}^{n-1,2}(\mathbf{R};H_B)$.

(d) We have $\mathbb{D}^* y = \text{Я}\mathbb{D}^\mathrm{d}\text{Я}y$, so that this follows from (6.48) and Lemma 6.2.9(b). □

The following standard delay line example (to which we return several times later) illustrates the symbols defined in this section:

Example 6.2.14 (C_w and B^*_w) Let $U = \mathbf{C} = Y$, $H := \mathrm{L}^2(\mathbf{R}_+;Y)$, and

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_{[0,1)}\tau(-1) \\ \hline I & \tau(-1) \end{array}\right] \in \mathrm{WPLS}_0(U,H,Y). \tag{6.53}$$

More details of this system are given in [WZ], [S95] and [Sbook]; the reader might also wish to consult some text book on distributions (e.g., [Rud73]) for H_{-1} and H^*_{-1}, the duals of H_1 and H^*_1 w.r.t. L^2.

Now $\widehat{\mathbb{D}}(s) = \mathrm{e}^{-s} \to 0$, as $\mathrm{Re}\, s \to \infty$, hence $\mathbb{D} \in \mathrm{ULR}$ and $D = 0$ (in fact, $\mathbb{D} \in \mathrm{MTIC_d}$). By Proposition 6.2.8(a2), we have $C_\mathrm{w} \subset \mathcal{B}(H_B,Y)$ and $B^*_\mathrm{w} \subset \mathcal{B}(H^*_C,U)$. (Because $\dim U = 1 = \dim Y$, we have in fact that $C_\mathrm{w} = C_\mathrm{s}$ and $B^*_\mathrm{w} = B^*_\mathrm{s}$.)

By Proposition B.7.12, we have

$$A = \frac{\mathrm{d}}{\mathrm{d}\theta}, \qquad H_1 := \mathrm{W}^{1,2}((0,\infty)) := \{f \in H \,\big|\, f' \in H\} \tag{6.54}$$

$$A^* = -\frac{\mathrm{d}}{\mathrm{d}\theta}, \qquad H_1^* = \mathrm{W}_0^{1,2}((0,\infty)) := \{f \in \mathrm{W}^{1,2}((0,\infty)) \,\big|\, f(0+) = 0\}. \tag{6.55}$$

(Here and below, $\theta \in \mathbf{R}_+$ is the argument of an element (function) of H. Recall $\mathrm{W}^{1,2}((0,\infty)) \subset \mathcal{C}_0(\mathbf{R}_+)$, continuously, by Theorem B.7.4 and Lemma B.7.6.)

Because $\mathbb{C} = I$, we have $x_0(t) = (\mathbb{C}x_0)(t) = C\pi_+\tau^t x_0$ for $x_0 \in H$, and $C = \delta_0^* : f \mapsto f(0)$ is the unique operator $C \in \mathcal{B}(H_1, Y)$ satisfying this. By Lemma 6.2.9(b), we have $(\mathbf{Я}\mathbb{B}^* f)(t) = B^*\mathbb{A}^*(t) f$ for $f \in H_1^*$, hence

$$B^*\tau(-t)\pi_+ f = B^*\mathbb{A}^*(t) f = (\tau^1\pi_{[0,1)} f)(-t) = (\pi_{[0,1)} f)(1-t), \tag{6.56}$$

hence $B^* = \delta_1^* : f \mapsto f(1)$. Summarizing,

$$Bu_0 = u_0\delta_1 \in H_{-1}, \qquad C = \delta_0^* : x_0 \mapsto x_0(0) \in Y = \mathbf{C}, \quad D = 0 = D^*; \tag{6.57}$$

$$B^* = \delta_1^*, \qquad C^* y_0 = y_0\delta_0, \quad (u_0 \in U,\ x_0 \in H_1,\ y_0 \in Y). \tag{6.58}$$

By taking Laplace transforms of $\mathbb{A}x_0$, $\mathbb{A}^*x_0$ and $\mathbb{B}\tau^t\pi_+u_0$ for $x_0 \in H$ and $u_0 \in U$, we obtain

$$H_1 \ni (s-A)^{-1}x_0 : \theta \mapsto \int_\theta^\infty \mathrm{e}^{-s(r-\theta)} x_0(r)\, dr, \tag{6.59}$$

$$H_1^* \ni (s-A^*)^{-1}x_0 : \theta \mapsto \int_0^\theta \mathrm{e}^{-sr} x_0(\theta - r)\, dr, \tag{6.60}$$

$$H_B \ni (s-A)^{-1}Bu_0 = \pi_{[0,1)}(\cdot)\mathrm{e}^{-s(1-\cdot)}u_0 \qquad (s \in \mathbf{C}^+). \tag{6.61}$$

Note that "$\pi_{[0,1)}$" in (6.61) (instead of "$\pi_{[0,1]}$") is just our choice — the elements of H are known just a.e.

By Definition 6.1.17, the space $H_B \subset H$ is given by

$$H_B = H_1 + \mathbf{C}\pi_{[0,1)}(\cdot)\mathrm{e}^{-s(1-\cdot)} = \{f \in H \,\big|\, f' \in H + \mathbf{C}\delta_1\} \tag{6.62}$$

$$= \mathrm{W}^{1,2}((0,1)) + \mathrm{W}^{1,2}([1,\infty)) \tag{6.63}$$

(we used here the choice made above; note that elements of H_B are bounded and continuous from the right on $\mathbf{R}_+$). We can set, e.g., $\alpha = 1$, to make the norm on H_B equal to

$$\|(1-A)^{-1}(x_0 + Bu_0)\|_{H_B} := \|(x_0, u_0)\|_{H\times U}. \tag{6.64}$$

However, this norm serves only as an example, it is enough for us to know that the inclusions $H_1 \subset H_B \subset H$ are continuous. The operator $C_\mathrm{w} \in \mathcal{B}(H_B, U)$ is given by

$$C_\mathrm{w}x_0 = \operatorname*{w-lim}_{s\to+\infty} Cs(s-A)^{-1}x_0 = \operatorname*{w-lim}_{s\to+\infty} s\int_0^\infty \mathrm{e}^{-sr}x_0(r)\, dr = x_0(0+), \tag{6.65}$$

i.e., $C_\mathrm{w} = \delta_{0+}^* \in \mathcal{B}(H_B, U)$. From this and (6.61) we can verify the identity

$$\widehat{\mathbb{D}}(s) = C_\mathrm{w}(s-A)^{-1}B = \mathrm{e}^{-s} \tag{6.66}$$

as required, since $\mathrm{e}^{-s} = \widehat{\tau(-1)}$. Furthermore, $(s-A^*)C^*y_0 = \mathrm{e}^{-s\cdot}y_0 \in H_C^* \subset H$ for $s \in \mathbf{C}^+$, $y_0 \in \mathbf{C}$, hence $(1-A^*)C^*y_0 = y_0\mathrm{e}^{-\cdot}$, hence $H_C^* = H_1^* + \mathbf{C}\mathrm{e}^{-\cdot} = \mathrm{W}^{1,2}$, by

Lemma A.3.4(I1). Finally

$$B^* s(s-A^*)x_0 = s\int_0^1 \mathrm{e}^{-sr} x_0(1-r)\,dr \to x_0(1-), \quad \text{as } s \to +\infty, \tag{6.67}$$

for $x_0 \in H_C^*$, hence $B_\mathrm{w}^* = \delta_1^* : x_0 \mapsto x_0(1)$, so that $B_\mathrm{w}^* \in \mathcal{B}(H_C^*, U)$ as required. In fact, $B_\mathrm{w}^* x_0 = x_0(1-)$ for any $x_0 \in H$ that is weakly continuous to the left (or has a weak left Lebesgue value at 1, by Lemma B.5.10). $\triangleleft$

Notes

Strong and weak regularity and their core theory are due to G. Weiss [W89c], [WW]. The concept ULR is due to [Helton76a], p. 155.

Vertical and half-plane-regularity seem to be the most reasonable transfer function properties that connect the I/O map equalities to corresponding feedthrough operator equalities. Their main advantage is that they can be used to guarantee that the signature operator of an optimal control problem equals the classical one (see, e.g., Lemma 6.3.6(b) and Proposition 9.11.3(c)).

Except for Lemma 6.2.2 (in this generality) and Proposition 6.2.8(e)&(f), almost all results of this section can be found in some form in the literature, mostly due to G. Weiss. The Lemma 6.2.10, the composition part of Example 6.2.6 and some minor results are from the works of O. Staffans.

Further results on regularity are given in the next section and in [W94a], [W94b], [SW00], [SW01a] and [Sw01b] among others; see [Sbook] for an extensive treatment on the subject and for historical remarks (the notes for Chapters 4 and 5).

6.3 Further regularity and compatibility

Be regular and orderly in your life, so that you may be violent and original in your work.
— Gustave Flaubert (1821–1880)

We start this section by studying several types of regular I/O maps (regularity and invertibility, H^p transfer functions and convolution I/O maps).

Not every WPLS is regular, but the output operator C of any WPLS has an extension C_{c} s.t. $D_{\mathrm{c}} := \widehat{\mathbb{D}} - C_{\mathrm{c}}(\cdot - A)^{-1}B$ is constant; such pairs $(C_{\mathrm{c}}, D_{\mathrm{c}})$ are called *compatible output operator pairs* for the system. Also the formula $y = C_{\mathrm{c}}x + D_{\mathrm{c}}u$ holds if the input is smooth enough. We give a few basic results on such pairs.

Further on, we give necessary and sufficient conditions for certain operators to generate a WPLS. Then we present several auxiliary lemmas on the connection between the generators and components or signals of a system until we finish this section by a brief treatment of reachability and observability.

In connection with feedback, it is often important to know whether the inverse of an I/O map is regular and whether its feedthrough operator is invertible; we list here the basic facts on this:

Proposition 6.3.1 (Regularity of $\mathbb{X}^{-1}$) *(a) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ be SR. Then*

(a1) The feedthrough operator $X := \mathbb{X}(+\infty)$ is left-invertible.
(a2) If $\mathbb{X}^{\mathrm{d}}$ is SR, then X is invertible and $\mathbb{X}^{-1}$ is SR.
(a3) $\mathbb{X}^{-1}$ is SR iff X is invertible. If $\mathbb{X}^{-1}$ is SR, then $\mathbb{X}^{-1}(+\infty) = X^{-1}$.

(b1) If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ is UR, then $\mathbb{X}^{-1}$ is UR, $X \in \mathcal{GB}$, and $\mathbb{X}^{-1}(+\infty) = X^{-1}$.

(b2) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $X \in \mathcal{GB}$. Then $\mathbb{X}$ is SR (resp. UR, SVR, UVR, SLR, ULR) iff $\mathbb{X}^{-1}$ is SR (resp. UR, SVR, UVR, SLR, ULR).

(b3) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Then $\mathbb{X}$ is strongly (resp. uniformly) half-plane regular iff $\mathbb{X}^{-1}$ is.

(b4) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $X \in \mathcal{GB}$. Then $\mathbb{D}$ and $\mathbb{X}$ are SR (resp. UR, SVR, UVR, SLR, ULR) iff $\mathbb{D}\mathbb{X}^{-1}$ and $\mathbb{X}$ are SR (resp. UR, SVR, UVR, SLR, ULR).

(c) Let $\mathbb{X} \in \mathrm{TIC}_\infty$ be ULR. *Then $\mathbb{X} \in \mathcal{G}\mathrm{ULR} \Leftrightarrow \mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow X \in \mathcal{GB}$.*

Due to (c), one can work with ULR maps in the same way as with rational maps: a map is invertible iff its feedthrough operator is invertible. Moreover, the class ULR is also closed under inverses, compositions, linear operations and causal adjoints (see Lemma 6.2.5). These properties make uniform line-regularity the most important regularity property in the optimal control theory of Part III.

Proof: (a)–(c) W.l.o.g. (see Lemma 2.2.1(c4)), we assume that $Y = U$.

(a1–3) follow from [W94b, Theorems 4.7 & 4.8] (with $\mathbf{H} := I - \widehat{\mathbb{X}}$ and $K := I$).

(b1) Now $\widehat{\mathbb{X}}^{-1}(s) = \widehat{\mathbb{X}}(s)^{-1}$ is bounded on some $\mathbf{C}^+_\omega$ and $\widehat{\mathbb{X}}(s) \to X$, hence $X \in \mathcal{GB}$, by Lemma A.3.3(A3). The rest follows as in the proof of (b2).

(b2) Let $a \in U$ be arbitrary and set $b := X^{-1}a$. Now

$$\widehat{\mathbb{X}}^{-1}(s)a - X^{-1}a = \widehat{\mathbb{X}}^{-1}(s)Xb - b = \widehat{\mathbb{X}}^{-1}(s)[Xb - \widehat{\mathbb{X}}(s)b] \to 0 \tag{6.68}$$

as $s \to \infty$ (along $\mathbf{R}_+$ or $\omega + i\mathbf{R}_+$ or $\mathrm{Re}\, s \to +\infty$), because $\|\widehat{\mathbb{X}}^{-1}(s)\|_{\mathcal{B}} \le \|\mathbb{X}^{-1}\|_{\mathrm{TIC}_\omega}$ on $\mathbf{C}^+_\omega$. Therefore, $\mathbb{X}^{-1}$ is SR (resp. SVR, SLR) if $\mathbb{X}$ is. The uniform properties follow by removing a and replacing b by X^{-1} in (6.68). The converses follow by exchanging the roles of $\mathbb{X}$ and $\mathbb{X}^{-1}$.

(b3) The proof of (b2) applies here too.

(b4) This follows from (b2) and Lemma 6.2.5 (these properties are preserved under compositions).

(c) This follows from (b1)–(b2). □

The obvious facts (a1)–(b) below are often needed, (c) and (d) less so:

Lemma 6.3.2 (Regular I/O maps) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$.

(a1) If $\dim Y < \infty$*, then* $\mathbb{D}$ *is WR (resp. WLR, WVR) iff* $\mathbb{D}$ *is SR (resp. SLR, SVR).*

(a2) If $\dim U < \infty$*, then* $\mathbb{D}$ *is UR (resp. ULR, UVR) iff* $\mathbb{D}$ *is SR (resp. SLR, SVR).*

(b) $\mathbb{D}^{\mathrm{d}}$ *is WR (resp. any other weak or uniform property from Definition 6.2.3) iff* $\mathbb{D}$ *is.*

(c) If $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *is uniformly (resp. strongly, weakly) regular and* $\widehat{\mathbb{D}}$ *is holomorphic and bounded on the sector* $\{s \in \mathbf{C} \,\big|\, |\arg s| < \frac{\pi}{2} + \varepsilon\}$ *for some* $\varepsilon > 0$*, then* $\widehat{\mathbb{D}}$ *is uniformly (resp. strongly, weakly) half-plane-regular.*

(d) If $\mathbb{D}$ *is UR and* $D \in \mathcal{GB}$*, then there is* $R > 0$ *s.t.* $\widehat{\mathbb{D}}(s) \in \mathcal{GB}$ *for* $s > R$*.*

(e) If $\mathbb{D} \in \mathrm{TIC}_\omega \cap \mathrm{WR}$*, then* $\|D\|_{\mathcal{B}(U,Y)} \le \|\mathbb{D}\|_{\mathrm{TIC}_\omega}$*.*

Proof: (a1)&(a2) See Lemma A.3.1(k1)–(k2).

(b) This is obvious.

(c) By [HP, Theorem 3.14.3], $\|\widehat{\mathbb{D}} - D\|_{\mathcal{B}(U,Y)} \to 0$ as $|s| \to \infty$, on any closed subsector of $\{s \in \mathbf{C} \,\big|\, |\arg s| < \frac{\pi}{2} + \varepsilon\}$, in particular, on $\mathbf{C}^+$. (By [HP], we have the above convergence even if a disc $\{|s| \le R\}$ were excluded from the sector.)

The strong [weak] claim is proved by replacing $\widehat{\mathbb{D}}$ by $\widehat{\mathbb{D}}u_0$ [by $\langle \widehat{\mathbb{D}}u_0, y_0\rangle$] for $u_0 \in U$ [and $y_0 \in Y$].

(d) This follows from Lemma A.3.3(A2) and the continuity of $\widehat{\mathbb{D}}$ on $(\omega, +\infty]$, where ω is s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$.

(N.B. There is $\mathbb{D} = \mathbb{D}^{\mathrm{d}} \in \mathrm{SR} \cap \mathrm{TIC}$ s.t. $D = I$ but $\widehat{\mathbb{D}}(n)e_n = 0$ ($n \in \mathbf{N}$): Let $U := \ell^2(\mathbf{N})$ and define $\mathbb{D}$ by $\widehat{\mathbb{D}}(s)e_n := (1 - 2\mathrm{e}^{-s\log 2/n})e_n$ ($n \in \mathbf{N}, s \in \mathbf{C}^+$) (obviously, $\|\widehat{\mathbb{D}}\|_{\mathcal{B}(U)} \le 1$; by Lemma D.1.1(b) we have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U))$).) □

Recall that $\mathrm{H}^p_\omega := \mathrm{H}^p(\mathbf{C}^+_\omega; *) = \tau(-\omega)\mathrm{H}^p$ $\quad(1 \le p \le \infty,\ \omega \in \mathbf{R})$ and that $\mathrm{H}^p_\infty := \cup_{\omega \in \mathbf{R}} \mathrm{H}^p_\omega$. By Theorem 6.2.1, the set of transfer functions of WPLSs (or of TIC_∞ maps) equals H^∞_∞. If a transfer function belongs to H^p_∞ (or to weak H^p_∞, see Definition F.3.1), for any $p < \infty$, then it is necessarily uniformly line-regular and much more:

Proposition 6.3.3 ($\widehat{\mathbb{D}} \in \mathrm{H}^p_{[\mathrm{strong}]}$) *Let $-\infty < \omega < \alpha < \infty$, $1 \le p < \infty$.*

(a) **($\mathrm{H}^p_{\mathrm{weak},\infty} \subset \widehat{\mathrm{ULR}}$)** *Let $\widehat{\mathbb{D}} \in \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. Then $\mathbb{D} \in \mathrm{TIC}_\alpha(U,Y) \cap \mathrm{ULR} \cap \mathrm{WVR}$, and $D = 0$. If $\omega < 0$, then $\mathbb{D} \in \mathrm{WHPR}$.*

The above claim also holds with replacements $\mathrm{H}^p_{\mathrm{weak}} \mapsto \mathrm{H}^p_{\mathrm{strong}}$ and "W"$\mapsto$"S", as well as with replacements $\mathrm{H}^p_{\mathrm{weak}} \mapsto \mathrm{H}^p$ and "W"$\mapsto$"U".

(b1) **($\mathcal{B} + \mathrm{H}^p_{\mathrm{strong},\infty}$ is inverse-closed in H^∞_∞)** *Let $\widehat{\mathbb{D}} \in \mathcal{B}(U,Y) + \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. Then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\infty \Leftrightarrow \widehat{\mathbb{D}} \in \mathcal{G}(\mathcal{B} + \mathrm{H}^p_{\mathrm{strong},\infty})$. Moreover, if $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\omega$, then*

$$\|\widehat{\mathbb{D}}^{-1} - D^{-1}\|_{\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \le \|\widehat{\mathbb{D}}^{-1}\|_{\mathrm{H}^\infty_\omega} \|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))} \|D^{-1}\|. \tag{6.69}$$

Part (b1) also holds with "strong" removed.

(b2) If $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathcal{B}(U,Y) + \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\infty \Leftrightarrow \widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathcal{G}(\mathcal{B} + \mathrm{H}^p_{\mathrm{strong},\infty})$ etc., as in (b1), by duality.

(c) If $\widehat{\mathbb{D}} \in \mathcal{B}(U,Y) + \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$ and $\widehat{\mathbb{F}} \in \mathcal{B}(Y,Z) + \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,Z))$, then $\widehat{\mathbb{F}}\widehat{\mathbb{D}} \in \mathcal{B}(U,Z) + \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Z))$.

(This also holds with "strong" removed.)

(d) Let $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, $p \in [2,\infty]$. Then $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M_{\mathbb{D},p,\varepsilon} \|u\|_{\mathrm{L}^2_{\omega-\varepsilon}}$ for any $\varepsilon > 0$, $u \in \mathrm{L}^2_{\omega-\varepsilon}(\mathbf{R}_+; U)$. In particular, $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2_\omega$.

(e) Let $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$. Then $\mathbb{D}u \in \mathcal{C}(\mathbf{R}; U)$ and

$$\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} \mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \|u\|_{\mathrm{L}^2_\omega} \qquad (u \in \mathrm{L}^2_\omega(\mathbf{R}; U)). \tag{6.70}$$

Note for (e) that $\|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}$ for any $\widehat{\mathbb{D}} \in \mathrm{H}^2_\omega$. By (c), $\mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$ is a subalgebra of H^∞_∞ when $U = Y$.

In Theorem 6.9.1 we shall show that $\widehat{\mathbb{D}}(\cdot - \omega) \in \mathrm{H}^2_{\mathrm{strong}}$ for some $\omega \in \mathbf{R}$ iff $\mathbb{D}$ has a realization with a bounded B and $D = 0$ (and a dual claim holds for bounded C).

Proof: (a) 1° *$\mathbb{D} \in \mathrm{ULR} \cap \mathrm{TIC}_\alpha(U,Y)$:* By Lemma F.3.2(a), $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$, hence it is the transfer function of some $\mathbb{D} \in \mathrm{TIC}_\alpha$. By Lemma F.3.2(b), $\mathbb{D}$ is ULR.

2° *Half-plane-regularity:* We give the proof for $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$; (add $u_0 \in U$ [and $\Lambda \in Y^*$] for $\mathrm{H}^p_{\mathrm{strong}}$ [or $\mathrm{H}^p_{\mathrm{weak}}$]).

Assume that $\omega < 0$. Then $\mathbb{D}$ is uniformly half-plane-regular, by Theorem 6.4.2 of [HP] (use $\alpha := \omega/2 =: -\delta < 0$; note that $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}_{\omega/2}; \mathcal{B})$ in the sense of [HP], i.e.,the extra assumption (iii) of Definition 6.4.1 of [HP] is satisfied for $\alpha = \omega/2$).

3° *Vertical regularity:* This follows from the half-plane-regularity of $\widehat{\mathbb{D}}(\cdot - \omega - 1)$.

(b1) This follows from Theorem 4.1.1(j). The norm estimate is obtained as in the proof of Lemma 4.1.2.

(b2) Apply (b1) to $\widehat{\mathbb{D}}(\bar{\cdot})^*$, and note that $\widehat{\mathbb{D}} \mapsto \widehat{\mathbb{D}}(\bar{\cdot})^*$ is an isometric isomorphism of H^∞_α onto H^∞_α.

(c) Set $g := \widehat{\mathbb{D}} - D$, $f := \widehat{\mathbb{F}} - F$. Obviously, we have $FD \in \mathcal{B}(U,Z)$ and $Fg, fD \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Z))$. But $f \in \mathrm{H}^\infty_\alpha$, by (a), hence $fg \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Z))$. (We can also remove "strong" from the statement and the proof of (c).)

(d) In fact, it suffices that $u \in \mathrm{L}^q_\omega(\mathbf{R}_+;U)$, where $q = (1/2+1/p)^{-1}$. Indeed, $q \in [1,2]$, hence $q' := (1/2-1/p)^{-1} \in [2,\infty]$ and $\|\widehat{u}\|_{\mathrm{H}^{q'}_\omega} \le M_q \|u\|_{\mathrm{L}^q_\omega}$, by Theorem E.1.7. On the other hand, $\|\widehat{\mathbb{D}}\widehat{u}\|_{\mathrm{H}^2_\omega} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}\|\widehat{u}\|_{\mathrm{H}^{q'}_\omega}$, by Lemma B.3.13.

Thus, $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M'_p\|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}\|u\|_{\mathrm{L}^q_\omega}$, where $M'_p := \sqrt{2\pi}M_q$. Because $\|u\|_{\mathrm{L}^q_\omega} \le M'_{q,\varepsilon}\|u\|_{\mathrm{L}^2_{\omega-\varepsilon}}$, by Lemma D.1.4(b4), we can set $M_{\mathbb{D},p,\varepsilon} := M'_{q,\varepsilon}M'_p\|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}$.

(e) See Lemma F.3.7(b2). (N.B. although $\mathbb{D} \in \mathrm{TIC}_\alpha$ and $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega, \mathcal{C})$ (where $\mathbb{D}$ stands for its unique continuous extension), we do not have $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega,\mathrm{L}^2_\omega)$ unless $\widehat{\mathbb{D}} \in \mathrm{H}^\infty_\omega$.) $\square$

If $F \in \mathrm{L}^1(\mathbf{R}_+;\mathcal{B}(U,Y))$, then $\mathbb{D}u := F * u$ defines a TIC map, and we have $\widehat{\mathbb{D}}\widehat{u} = \widehat{F}\widehat{u}$ (see Lemma D.1.11(c')). If merely $F \in \mathrm{L}^1_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$ (Definition F.1.4), i.e., $F : \mathbf{R}_+ \to \mathcal{B}(U,Y)$ is s.t. $Fu_0 \in \mathrm{L}^1(\mathbf{R}_+;Y)$ $(u_0 \in U)$, then $\mathbb{D}u := F * u$ can in general be written as an integral for finite-dimensional $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ only, but still $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, hence $\widehat{\mathbb{D}}\widehat{u} := \widehat{F}\widehat{u}$ still defines a map $\mathbb{D} \in \mathrm{TIC}(U,Y)$.

We list below further properties of convolution maps:

Proposition 6.3.4 ($\mathbb{D} \in \mathrm{L}^p_{[\mathrm{strong}]}*$) *Let $-\infty < \omega < \alpha < \infty$ and $1 \le p \le \infty$.*

(a1) Let $F \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$.

Then $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\alpha \subset \mathrm{ULR} \cap \mathrm{UVR}$, $\mathbb{D}u := F * u$ for all $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$, and $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{F^*}$.

If $\omega < 0$, then $\mathbb{D}$ is UHPR. If $E \in \mathcal{B}(U,Y)$ and $E + \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\alpha$, then $(E+\mathbb{D})^{-1} \in \mathcal{GB}(Y,U) + \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(Y,U))$. Finally, $\frac{1}{r}\mathbb{D}\chi_{[-r,0)} \to F$ in $\mathcal{B}(U,Y)$ at every Lebesgue point of F, hence a.e. (See (a2) and (a3) for further results.)

(a2) Let $\mathrm{e}^{-\omega\cdot}F \in \mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$. Then $\mathbb{D}^{\mathrm{d}} = \widehat{F^*}$ on $\mathbf{C}^+_\omega$, and (a3) applies.

(a3) Let $F \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$.

Then $\mathbb{D} \in \mathrm{TIC}_\alpha \cap \mathrm{SLR} \cap \mathrm{SVR}$ (and $\mathbb{D} \in \mathrm{ULR}$ if $p > 1$). If $\omega < 0$, then $\mathbb{D}$ is strongly half-plane-regular. Moreover, $\mathbb{D}u = F * u$ for finite-dimensional $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$.

If $1 \le p \le 2$ and $1/p + 1/q = 1$, then $\widehat{\mathbb{D}} \in \mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$; if $p \ge 2$, then $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$. Finally, $\frac{1}{r}\mathbb{D}\chi_{[-r,0)}u_0 \to Fu_0$ in Y at every Lebesgue point of Fu_0, hence a.e. $(u_0 \in U)$.

(a4) Let $\mathrm{e}^{-\omega\cdot}F \in \mathrm{L}^p_{\mathrm{weak}}(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$. Then $\mathbb{D}^{\mathrm{d}} = \widehat{F^*}$ on $\mathbf{C}^+_\omega$, and (a5) applies.

(a5) Let $F \in \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\omega(\mathbf{R}_+)))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$.

Then $\mathbb{D} \in \mathrm{TIC}_\alpha \cap \mathrm{WLR} \cap \mathrm{WVR}$ (and $\mathbb{D} \in \mathrm{ULR}$ if $p > 1$). If $\omega < 0$, then $\mathbb{D}$ is weakly half-plane-regular. Moreover, $\Lambda\mathbb{D}u = \Lambda F * u$ for finite-dimensional $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$ and all $\Lambda \in Y^*$.

If $1 \le p \le 2$ and $1/p + 1/q = 1$, then $\widehat{\mathbb{D}} \in \mathrm{H}^q_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$; if $p \ge 2$, then $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$. Finally, $\Lambda \frac{1}{r}\mathbb{D}\chi_{[-r,0)}u_0 \to \Lambda F u_0$ in $\mathbf{C}$ at every Lebesgue point of $\Lambda F u_0$, hence a.e. ($u_0 \in U$, $\Lambda \in Y^*$).

(b) For $p = 1$, we can take $\alpha = \omega$ in (a1)–(a4).

See Theorem 2.6.4 for more on the classes in (a1) and (a3). See Lemma F.2.2(a) for the "strong convolutions" appearing in (a3); they coincide with ordinary ones when $F \in \mathrm{L}^p_\omega$.

A convolution kernel ("F" above) corresponding to an I/O map of a system is often called the impulse response (since it equals "$F * \delta_0 = \mathbb{D}\delta_0$") or the weighting pattern of the system.

Proof: (a1) We have $F \in \mathrm{L}^1_\alpha$, hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\alpha$, by definition, and $\widehat{F * u} = \widehat{F}\widehat{u} = \widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ for $u \in \mathrm{L}^2_\alpha$, by Lemma D.1.11(c'). By causality (replace u by $\pi_{[0,T)}u$, for arbitrary $T > 0$), we have $F * u = \mathbb{D}u$ also for $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$ (cf. Lemma D.1.7).

By Lemma D.1.12(d), we have $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{F^*}$ (hence $\mathbb{D}^{\mathrm{d}} = F^* *$).

If $\omega < 0$, then we can take $\alpha = 0$ to see that $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$ (and hence $\widehat{\mathbb{D}}$ is uniformly half-plane-regular, by Theorem 2.6.4). By Lemma D.1.11(b'), we have $\mathbb{D} \in \mathrm{ULR} \cap \mathrm{UVR}$ (and $\mathbb{D}$ is uniformly half-plane-regular if $\omega < 0$ or $\omega = 0$ and $p = 1$). Apply Theorem 4.1.1(b) to $\mathcal{T}_{-\alpha}(E + F*)$ to obtain that also the inverse lies in $\mathcal{B} + (\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*$ (recall that $F \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha$).

At each Lebesgue point t of F, we have

$$\frac{1}{r}\mathbb{D}\chi_{[-r,0)}\cdot = \frac{1}{r}\int_{-r}^{0} F(t-s)\,ds = \frac{1}{r}\int_0^r F(t+s)\,ds \to F, \tag{6.71}$$

as $r \to 0+$. (N.B. $\frac{1}{r}\int_0^r F(t+s)\,ds$ is a continuous function of r and t, by Corollary B.3.8.)

(a2) (Note that if, instead, $\mathrm{e}^{-\omega\cdot}F^* \in \mathrm{L}^p_{\mathrm{strong}}$, then $\mathbb{D}$ is "strong* half-plane-regular" etc., by duality.)

As in (a1), one can show that $\mathbb{D} \in \mathrm{SLR} \cap \mathrm{SVR}$ and that $\mathbb{D}$ is strongly half-plane-regular if $\omega < 0$ (or $\omega = 0$ and $p = 1$).

By Lemma F.3.4(c2)&(a1), we have $\widehat{F} \in \mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y)) \cap \mathrm{H}^\infty_\alpha$ for $p \in [1,2]$; for $p > 2$ we have $\mathrm{e}^{-\omega'\cdot}F \in \mathrm{L}^2_{\mathrm{strong}}$ for any $\omega' > \omega$. It follows that $\mathbb{D} \in \mathrm{ULR}$ if $p > 1$, by Proposition 6.3.3(a1).

For $u = \phi u_0$, $\phi \in \mathrm{L}^2_\alpha(\mathbf{R})$, $u_0 \in U$, we have $\widehat{F * u} = \widehat{F u_0 * \phi} = \widehat{F}u_0\widehat{\phi} = \widehat{\mathbb{D}}\widehat{u}$, i.e., $F * u = \mathbb{D}u$; hence the same holds for finite-dimensional u (see the proof for $\pi_+\mathrm{L}^2_{\mathrm{loc}}$).

By Lemma 6.2.2, we have $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{\mathbb{D}}(\bar{s})^* = \widehat{F}(\bar{s})^*$; by Lemma F.3.3(c), $\widehat{F}(\bar{s})^* = \widehat{F^*}(s)$, for $s \in \mathbf{C}^+_\omega$.

(a3) Most of this follows from Lemma F.2.2(d1)–(d3); the rest can be obtained as in (a2).

(a4) The proofs of (a1)–(a3) apply mutatis mutandis. (Recall from Remark A.3.22 that Y^{B} refers to the "linear" dual, where the scalar multiplication is defined by $(\beta\Lambda)y_0 := \beta(\Lambda y_0)$ $\alpha \in \mathbf{C}$, $\Lambda \in Y^*$, $y_0 \in Y$, as in Theorem F.2.1(f).)

(Obviously, $\mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\omega(\mathbf{R}_+)))$ is the space of bilinear mappings $U \times Y^{\mathrm{B}} \to \mathrm{L}^p_\omega(\mathbf{R}_+)$, i.e., the space of sesquilinear mappings $U \times Y \to \mathrm{L}^p_\omega(\mathbf{R}_+)$, with norm $\|T\| \le \sup_{\|u\|,\|y\|\le 1} \|T(u_0, y)\|_{\mathrm{L}^p_\omega}$.)

(a5) One can observe this from the proofs of (a1)–(a3). $\square$

In classical Riccati equation theory, one often needs to make conclusions such as $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \implies D^*D = X^*X$. This may fail even if $\mathbb{D}, \mathbb{X} \in$ ULR (Example 6.3.7), but the following two lemmas give us important sufficient conditions:

Lemma 6.3.5 ($\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X} \implies D^*JD = X^*SX$) *Let* $\widehat{\mathbb{D}}_k \in \mathrm{H}^{p_k}_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U, Y_k)) + \mathcal{B}(U, Y_k)$, $p_k \in [2, \infty)$ *(*$k = 1, 2, 3, 4$*) and* $Y_1 = Y_2$, $Y_3 = Y_4$. *Assume that* $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle \ge \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ *for all* $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U)$. *Then* $D_1^* D_2 \ge D_3^* D_4$.

In particular, $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle = \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ ($u \in \mathrm{L}^2_{\mathrm{c}}$) implies that $D_1^* D_2 = D_3^* D_4$. We may allow for any $p_k \in [1, \infty)$ under slightly stronger stability, by Lemma F.3.2(a2).

For $\mathbb{D}_k \in \mathrm{MTIC}^{\mathrm{L}^1}$ (or $\mathrm{L}^1_{\mathrm{strong}} * + \mathcal{B}$), we obtain the same from Lemma 6.3.6(b).

Proof: By Proposition 6.3.3(a)&(d), we have $\widehat{\mathbb{D}}_k \in \mathcal{B}(\mathrm{L}^2_{-\delta}, \mathrm{L}^2)$, $\mathbb{D}_k \in$ ULR, hence $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle \ge \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ (use continuity and Corollary B.3.8) for all $u \in \mathrm{L}^2_{-\delta}(\mathbf{R}_+; U)$, $k = 1, 2, 3, 4$, $\delta > 0$.

We shall assume that $\mathbb{D}_3 = 0 = \mathbb{D}_4$ to simplify the notation; the general case is analogous (just the number of terms is doubled).

Let $u_0 \in U$. Set $\alpha := \langle D_1 u_0, D_2 u_0\rangle_Y \in \mathbf{C}$, $F := \widehat{\mathbb{D}}_1 - D_1$, $G := \widehat{\mathbb{D}}_2 - D_2$. Because $F u_0 \in \mathrm{H}^{p_1}(\mathbf{C}^+; Y_1)$ and $G u_0 \in \mathrm{H}^{p_2}(\mathbf{C}^+; Y_1)$, they have L^{p_k} boundary functions, by Theorem 3.3.1(a2). Moreover, $g_1 := \langle G u_0, F u_0\rangle_{Y_1} \in \mathrm{L}^q(i\mathbf{R})$, where $q^{-1} = p_1^{-1} + p_2^{-1} \in [1, \infty)$, by Lemma B.3.13.

Given $\varepsilon > 0$, there $R_1 > 0$ s.t. $f_r := f_{1,r} \in \mathrm{L}^2_{-1/2}(\mathbf{R}_+)$ satisfies $\|f_r\|_2 = 1$ and $\int_{i\mathbf{R}} |\widehat{f}_r|^2 |g_1|\, dm < \varepsilon/3$ for all $r > R_1$, by Lemma D.1.24. Choose, analogously, numbers R_2 and R_3 for functions $g_2 := \langle F u_0, D_2 u_0\rangle_Y \in \mathrm{L}^{p_1}(i\mathbf{R})$ and $g_3 := \langle D_1 u_0, G u_0\rangle_Y \in \mathrm{L}^{p_2}(i\mathbf{R})$, respectively, in place of g_1. Set $r_\varepsilon := \max\{R_1, R_2, R_3\}$, $\beta_\varepsilon := \langle \mathbb{D}_1 f_r u_0, \mathbb{D}_2 f_r u_0\rangle_Y \ge 0$. Then, by (D.36), we have

$$2\pi|\beta_\varepsilon - \alpha| = \langle \widehat{\mathbb{D}}_1 f_{r_\varepsilon} u_0, \widehat{\mathbb{D}}_2 f_{r_\varepsilon} u_0\rangle_{\mathrm{L}^2(i\mathbf{R}; Y_1)} - \langle D_1 f_{r_\varepsilon} u_0, D_2 f_{r_\varepsilon} u_0\rangle_{\mathrm{L}^2(i\mathbf{R}; Y_1)} \tag{6.72}$$

$$\le \int_{i\mathbf{R}} |f_{r_\varepsilon}|^2 (|g_1| + |g_2| + |g_3|)\, dm < 3\varepsilon/3, \tag{6.73}$$

hence $|\beta_\varepsilon - \alpha| < \varepsilon/2\pi$. Because ε was arbitrary, we have $\inf_{\beta \in \mathbf{R}_+} |\beta - \alpha| = 0$, i.e., $\alpha \in \mathbf{R}_+$. Because u_0 was arbitrary, we have $\langle D_1 u_0, D_2 u_0\rangle_Y \ge 0$ for all $u_0 \in U$. $\square$

Since convolutions with L^1 are uniformly half-plane-regular (UHPR), we can often use the following lemma for the purpose described above:

Lemma 6.3.6 (Half-plane-regularity) *Assume that* $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$.

(a1) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *be SHPR. Then for each* $u_0 \in U$ *there is a null set* $N_{\mathbb{D},u_0} \subset \mathbf{R}$ *s.t.* $\widehat{\mathbb{D}}(ir)u_0 \to Du_0$ *as* $N^c_{\mathbb{D},u_0} \ni r \to \pm\infty$.

(a2) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *be UHPR, and let U be separable. Then there is a null set* $N_{\mathbb{D}} \subset \mathbf{R}$ *s.t.* $\|\widehat{\mathbb{D}}(ir) - D\| \to 0$ *as* $N^c_{\mathbb{D}} \ni r \to \pm\infty$.

(b) $(\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X} \Rightarrow D^*JD = X^*SX)$ *Let* $\mathbb{D},\mathbb{E} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{X},\mathbb{Z} \in \mathrm{TIC}(U,H)$ *be strongly half-plane-regular. Then* $\mathbb{E}^*\mathbb{D} = \mathbb{Z}^*\mathbb{X} \implies E^*D = Z^*X$, *and* $\mathbb{E}^*\mathbb{D} \geq \mathbb{Z}^*\mathbb{X} \implies E^*D \geq Z^*X$.

(c1) $(\exists(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1} \Rightarrow \exists(D^*JD)^{-1})$ *Let* $\mathbb{D},\mathbb{D}^{\mathrm{d}},\mathbb{E},\mathbb{E}^{\mathrm{d}} \in \mathrm{SHPR} \cap \mathrm{TIC}$ *and* $\pi_+\mathbb{D}^*\mathbb{E}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$. *Then* $D^*E \in \mathcal{GB}(U)$.

(c2) $(\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0)$ *Let* $\mathbb{D},\mathbb{E} \in \mathrm{SHPR} \cap \mathrm{TIC}(U,Y)$ *and* $\mathbb{D}^*\mathbb{E} \gg 0$. *Then* $D^*E \gg 0$.

(d1) $(\exists(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1} \Rightarrow \exists(D^*JD)^{-1})$ *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be s.t.* $\mathbb{B}\tau \in \mathrm{UHPR} \cap \mathrm{TIC}$ *and* $\mathbb{D} \in \mathrm{ULR} \cap \mathrm{TIC}$ *(or* $\mathbb{B}\tau, \mathbb{C}^{\mathrm{d}}\tau \in \mathrm{SHPR} \cap \mathrm{TIC}$ *and* $\mathbb{D},\mathbb{D}^{\mathrm{d}} \in \mathrm{SLR} \cap \mathrm{TIC}$*). If* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$, *then* $D^*JD \in \mathcal{GB}(U)$.

(d2) $(\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0)$ *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be s.t.* $\mathbb{B}\tau \in \mathrm{SHPR} \cap \mathrm{TIC}$ *and* $\mathbb{D} \in \mathrm{SLR} \cap \mathrm{TIC}$. *If* $\mathbb{D}^*J\mathbb{D} \geq \varepsilon I$, $\varepsilon > 0$, *then* $D^*JD \geq \varepsilon I$.

Parts (d1) and (d2) are mainly applied to exponentially stable (or stabilizable) systems of the type studied in Sections 6.8 and 9.2. See Lemma 9.2.17 for the unstable case.

Proof: (a1) For any $\varepsilon > 0$ choose $R_\varepsilon > 0$ s.t. $\|[\widehat{\mathbb{D}}(s) - D]u_0\| < \varepsilon$ for all $s \in \mathbf{C}^+$ with $|s| > R_\varepsilon$. Then $\|[\widehat{\mathbb{D}}(ir) - D]u_0\| = \lim_{t\to 0+} \|[\widehat{\mathbb{D}}(ir+t) - D]u_0\| \leq \varepsilon$, when $r \in \mathbf{R} \setminus N_{\mathbb{D},u_0}$ and $|r| > R_\varepsilon$, for some null set $N_{\mathbb{D},u_0}$, by Theorem 3.3.1(c1).

(a2) Modify the proof of (a1) suitably (use Theorem 3.3.1(c2)).

(b) We only prove that $\mathbb{E}^*\mathbb{D} \geq 0 \implies E^*D \geq 0$, because the latter claim follows by applying this to $\begin{bmatrix}\mathbb{E} & -\mathbb{Z}\end{bmatrix}^* \left[\begin{smallmatrix}\mathbb{D}\\ \mathbb{X}\end{smallmatrix}\right] \geq 0$, and the former claim follows from the latter.

We have $\langle \widehat{\mathbb{E}}u_0, \widehat{\mathbb{D}}u_0 \rangle \geq 0$ a.e., by Theorem 3.1.3(e2). Letting $(N_{\mathbb{D},u_0} \cup N_{\mathbb{E},u_0})^c \ni r \to +\infty$, we get that $\langle Eu_0, Du_0 \rangle_U \geq 0$, by (a). Because u_0 was arbitrary, we have $E^*D \geq 0$.

(c1) (N.B. $\mathbb{D} \in \mathrm{UHPR} \Rightarrow \mathbb{D},\mathbb{D}^{\mathrm{d}} \in \mathrm{UHPR} \subset \mathrm{SHPR}$.) Choose $\eta > 0$ s.t. $\|\pi_+\mathbb{D}^*\mathbb{E}u\|_2 \geq \eta\|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$.

Let $\|u_0\|_U = 1$ and $\varepsilon > 0$ be arbitrary. By (a), there are $R > 0$ and a null set N s.t. $\|(\widehat{\mathbb{E}}(ir) - E)u_0\|, \|(\widehat{\mathbb{D}}^*(ir) - D^*)Eu_0\| < \varepsilon$ for $r > R$ s.t. $r \in N$. By Lemma D.1.24(a), there are $\delta > 0$ and $R' > 0$ s.t. for all $r > R'$ and $t \in (0,\delta)$ we have $\|\widehat{f}_{t,r}\|_{H^2(\mathbf{C}^+)} = \sqrt{2\pi}$ and $\|\chi_{i[-R,R]}\widehat{f}\|_{\mathrm{H}^2}$ is arbitrarily small, hence for suitable $\delta > 0$ and $R' > 0$ we have

$$|\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle| \le |\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle - \langle \widehat{v}, D^*E\widehat{f}_{1,r}u_0\rangle| + \sqrt{2\pi}\|\widehat{v}\|_{H^2}\|D^*Eu_0\|_U \quad (6.74)$$
$$\le (2\pi)\varepsilon + (2\pi)\|v\|_2\|D^*Eu_0\|_U \quad (v \in \mathrm{L}^2(\mathbf{R}_+;U)). \quad (6.75)$$

But for a fixed r, there is v s.t. $\|v\|_2 = 1$ and $|\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle| > 2\pi\eta$, by Lemma A.3.1(c1)(xi), hence $\|D^*Eu_0\| > \eta - \varepsilon$. Since ε and u_0 were arbitrary, we have $\|D^*Eu_0\| \ge \eta$ whenever $\|u_0\|_U = 1$.

By exchanging the roles of $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{D}}$, we obtain that $\|E^*Du_0\| \ge \eta\|u_0\|_U$ for all $u_0 \in U$. Consequently, $D^*E \in \mathcal{GB}(U)$, by Lemma A.3.1(c3)(v)&(i).

(c2) Now $\mathbb{D}^*\mathbb{E} \gg \varepsilon I$ for some $\varepsilon > 0$, hence $D^*E \gg \varepsilon I$, by (b).

(d1) By Lemma 6.3.23, we have $\mathbb{D} \in$ UHPR (or $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in$ SHPR), hence $\mathbb{D}^{\mathrm{d}}, J\mathbb{D}, \mathbb{D}^{\mathrm{d}}J \in$ UHPR $\subset$ SHPR (or $J\mathbb{D}, J\mathbb{D}^{\mathrm{d}} \in$ SHPR), hence this follows from (c1).

(d2) By Lemma 6.3.23, we have $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in$ SHPR, hence this follows from (c2). □

For non-half-plane-regular $\mathbb{D}$ and $\mathbb{X}$ (even for $\mathbb{D}, \mathbb{X} \in$ MTIC $\subset$ ULR), we really may have $D^*D \ne X^*X$ although $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X}$:

Example 6.3.7 **($S \ne D^*JD$)** In the system of Example 6.2.14, we have $U = \mathbf{C} = Y$, $\mathbb{D} = \tau(-1)$. Take $J = I$, so that $\mathbb{D}^*J\mathbb{D} = I$ has the spectral factorization $\mathbb{X}^*S\mathbb{X}$ with $S = I = \mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ (or $\mathbb{X} = E,\ S = (EE^*)^{-1}$ with $E \in \mathcal{B}(U)$).

Clearly $D = 0$ and $X = I$, hence $D^*JD = 0 \ne I = X^*SX$. Note that $\mathbb{D}, \mathbb{X} \in$ ULR, but, of course, $\mathbb{D}$ is not (even weakly) half-plane-regular. Moreover, for $s = i\omega \in i\mathbf{R}$ we have $\mathbb{D}^*(s)J\mathbb{D}(s) = |\mathrm{e}^{-s}|^2 = 1 = \mathbb{X}^*(s)S\mathbb{X}(s)$, but $\mathbb{D}(i\omega) = \mathrm{e}^{-i\omega} \not\to D$ as $\omega \to \pm\infty$). ◁

(In Example 9.13.8 (with Σ divided by $\sqrt{2}$), we have $X^*X \ne D^*D$ even though $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X}$, C is bounded and $\mathbb{X} = I$.)

This unfortunate fact makes the WPLS Riccati theory more complicated than the earlier theories for smoother classes and forces us to introduce the signature operator S (in general, X^*SX) that replaces the standard term D^*JD in Riccati equations in the general case, cf. (9.3).

We shall show in Theorem 6.3.9 that the output operator of an arbitrary WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ has an extension $C_{\mathrm{c}} \in \mathcal{B}(H_B, Y)$ s.t. $\widehat{\mathbb{D}}(s) = D_{\mathrm{c}} + C_{\mathrm{c}}(s-A)^{-1}B$ for some D_{c}. Such operators are the compatible generators of Σ:

Definition 6.3.8 (Compatibility) *We call* $(C_{\mathrm{c}}, D_{\mathrm{c}})$ *a* compatible (output operator) pair *for* Σ *(or for* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$*), and we call* $\left[\begin{array}{c|c} A & B \\ \hline C_{\mathrm{c}} & D_{\mathrm{c}} \end{array}\right]$ compatible generators *of* $\Sigma \in$ WPLS$_\omega(U, H, Y)$ *if* $\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$ *are the generators of* Σ*,* $C_{\mathrm{c}} \in \mathcal{B}(W, Y)$ *is an extension of* C*, where* W *is a Banach space s.t.* $H_B \subset W \underset{c}{\subset} H$*, and* $D_{\mathrm{c}} = \widehat{\mathbb{D}}(\alpha) - C_{\mathrm{c}}(\alpha - A)^{-1}B$ *for some* $\alpha \in \mathbf{C}^+_\omega$.

As before, "generate" means "are the generators of". We consider pairs $(C_{\mathrm{c}}, D_{\mathrm{c}})$ and $(\widetilde{C}_{\mathrm{c}}, D_{\mathrm{c}})$ for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ equal iff $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{array}\right]$, i.e., iff

$C_c = \widetilde{C}_c$ on H_B (the extension of C_c outside H_B is irrelevant, see below); cf. Lemma 6.3.10(d3).

The operator $D_c \in \mathcal{B}(U,Y)$ is independent of α, by (6.40) and the Resolvent Equation. By Corollary A.3.7, we have $H_B \underset{c}{\subset} W$, hence $C_c|_{H_B} \in \mathcal{B}(H_B,Y)$, so that we could always replace W by H_B (or by any other Banach space $V \underset{c}{\subset} W$ s.t. $H_B \subset V$), and so we usually do. However, sometimes we wish to choose W s.t. H_1 is dense in W (it need not be dense in H_B even if $\mathbb{D} = 0$, by Lemma 6.3.10(f)), and this can be done by taking $W := \mathrm{Dom}(C_{\mathrm{L,s}})$ whenever $\mathbb{D}$ is SR, by Lemma 6.3.10(e) and Proposition 6.2.8(c6). Also physical considerations might sometimes make some other W more convenient.

Theorem 6.3.9 *Every WPLS has a compatible pair.*

In the proof we show that C is bounded in the H_B norm, hence it has a unique bounded extension to the closure of $\mathrm{Dom}(A)$ in H_B. We could take the zero extension of this to extend C to H_B, but this is not always the most natural choice. E.g., if C is bounded (i.e., it coincides with some $C_0 \in \mathcal{B}(H,Y)$), then $C_w = C$ is in general different from the zero extension.

Proof: Let $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. Let H_{1B} be H_1 with the topology inherited from H_B. Assume, w.l.o.g., that $\alpha > \omega_A$ (in Lemma 6.1.16).

1° *If $z_n \to 0 \in H_{1B}$, then $Cz_n \to 0$ in Y:* Assume that $\{z_n\} \subset \mathrm{Dom}(A)$ is s.t. $\|z_n\|_{H_B} \to 0$ as $n \to \infty$. By definition of $\|z_n\|_{H_B}$, there must be $\{x_n\} \subset H_1$ and $\{u_n\} \subset U$ s.t.

$$z_n = x_n + (\alpha - A)^{-1}Bu_n, \quad \|x_n\|_{H_1} \to 0, \quad \|u_n\|_U \to 0. \tag{6.76}$$

It follows that $Cx_n \to 0$. Now $z_n, x_n \in H_1$ imply that $(\alpha - A)^{-1}Bu_n \in H_1$. Choose $\omega > \omega_A$ and set $M := \|\widehat{\mathbb{D}}\|_{\mathrm{H}^\infty_\omega} < \infty$. Then

$$\widehat{\mathbb{D}}(s)u_n = \widehat{\mathbb{D}}(\alpha)u_n - C(s-\alpha)(s-A)^{-1}(\alpha-A)^{-1}Bu_n \to \widehat{\mathbb{D}}(\alpha)u_n - C(\alpha-A)^{-1}Bu_n =: y_n, \tag{6.77}$$

and $\|y_n\|_Y \le M\|u_n\|_U$, hance $\|C(\alpha-A)^{-1}Bu_n\|_Y \le 2M\|u_n\|_U$. Consequently, $Cz_n \to 0$ as $n \to \infty$.

2° *Σ is compatible:* By density (see Lemma A.3.10), C has a continuous extension to the closure $\bar{H}_{1B}$ of H_{1B} in H_B. Because H_B is a Hilbert space, this extension has an extension $C_c \in \mathcal{B}(H_B,Y)$, by Lemma A.3.11. Thus, C_c and $D_c := \widehat{\mathbb{D}}(\alpha) - C_c(\alpha - A)^{-1}B$ form a compatible pair for Σ. □

Next we list the basic properties of compatible pairs:

Lemma 6.3.10 *Let Σ, C_c and D_c be as in Definition 6.3.8. Then*

(a) $\widehat{\mathbb{D}}(s) = D_c + C_c(s-A)^{-1}B$ *for all* $s \in \mathbf{C}^+_\omega$.

(b) If $x_0 \in H$, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;U)$, *and* $Ax_0 + Bu(0) \in H$, *then* $\mathbb{C}x_0 + \mathbb{D}u = C_c x + D_c u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;Y)$, *where* $x := \mathbb{A}x_0 + \mathbb{B}\tau u$.

(c) If $u \in \mathrm{W}^{1,2}_\omega(\mathbf{R};U)$, *then* $\mathbb{D}u = C_c x + D_c u \in \mathrm{W}^{1,2}_\omega(\mathbf{R};Y)$, *where* $x := \mathbb{B}\tau u$.

(d1) Σ *is uniquely determined by* $\left[\begin{array}{c|c} A & B \\ \hline C_c & D_c \end{array}\right]$ *(the converse holds iff* H_1 *is dense in* H_B*).*

(d2) Conversely, Σ *determines* $\left[\begin{array}{c|c} A & B \\ \hline C & * \end{array}\right]$ *uniquely, and* C_c *can be chosen to be any continuous extension of the closure of* C *(on the closure of* H_1 *in* W*); this choice determines* D_c *uniquely.*

(d3) Let also $(\widetilde{C_c}, \widetilde{D_c})$ *be a compatible pair for* Σ*. Then* $\mathcal{C} = \widetilde{C_c}$ *on* H_B *iff* $D_c = \widetilde{D_c}$*.*

(e) If $\mathbb{D}$ *is WR (resp. SR), then* $(C_{L,w}, D)$ *and* (C_w, D) *(resp.* $(C_{L,s}, D)$ *and* $(C_{L,s}, D)$*) are compatible pairs for* Σ*.*

(f) C_c *and* D_c *may be nonunique even if* Σ *is very regular (e.g.,* $\mathbb{D} = 0 = \mathbb{C}$*).*

(g) $(\alpha - A)^{-1}B \in \mathcal{B}(U, H_B) \subset \mathcal{B}(U, W)$ *for any* $\alpha \in \sigma(A)^c$*, and* $H_B \underset{c}{\subset} W$*.*

By causality, in (c) it is enough that $\pi_{(-\infty,T)}u \in \mathrm{W}^{1,2}_{\omega}(\mathbf{R}; U)$ for each $T > 0$.

Proof: (a) This follows from (6.40) and the Resolvent Equation.

(b) (Note that, given just u, we can always take $x_0 := (s - A)^{-1}Bu(0)$.) Assume that $u \in \mathrm{W}^{1,2}(\mathbf{R}_+; U)$ (for the general case, choose some $T > 0$ and extend u on $(T, +\infty)$ so that $u \in \mathrm{W}^{1,2}$; the values of x and $\mathbb{C}x_0 + \mathbb{D}u$ on $[0, T]$ remain unchanged, by the causality of $\mathbb{D}$).

Because $\mathrm{e}^{-\omega\cdot}x \in \mathcal{C}_b(\mathbf{R}_+; H_B)$, by Theorem 6.2.13(b1), the Laplace transform of $f := C_c x + D_c u$ is

$$\hat{f}(s) = C_c\widehat{x}(s) + D_c\widehat{u}(s) = C_c[(s - A)^{-1}x_0 + (s - A)^{-1}B\widehat{u}(s)] + D_c\widehat{u}(s), \quad (6.78)$$

which equals the Laplace transform of $\mathbb{C}x_0 + \mathbb{D}u$. Both functions are continuous, hence equal.

(c) See the proof of (b).

(d2) The first claim follows from Lemma 6.1.16(d). Conversely, let H' be the closure of H_1 in W, and let H'' be its orthogonal complement in W. Obviously, C_c is uniquely determined on H', and $C_c|_{H''}$ can be chosen to be any $\mathcal{B}(H'', U)$ operator; this choice determines D_c uniquely, by Definition 6.3.8.

(d1) The uniqueness follows from (a) and Lemma 6.2.9(a). The converse holds iff H_1 is dense in W, by (d2).

(d3) If $C_c = \widetilde{C_c}$ on H_B, then $D_c = \widetilde{D_c}$, by definition. Conversely, let $D_c = \widetilde{D_c}$. Then $C_c = \widetilde{C_c}$ on $(\alpha - A)^{-1}B[U]$. But $C_c = C = \widetilde{C_c}$ on H_1, hence $C_c = \widetilde{C_c}$ on $H_1 + (\alpha - A)^{-1}B[U] = H_B$, by linearity.

(e) This follows from Proposition 6.2.8.

(f) Let $\mathbb{C} = 0 = \mathbb{D}$ (so that $\mathbb{D}$ is uniformly half-plane-regular etc.), $BU \cap H = \{0\}$ and $\mathrm{Ker}(B) = \{0\}$ (so that $(\alpha - A)^{-1}B \in \mathcal{B}(U, H'')$ is an isometric isomorphism onto, in terms of the proof of (d2); here α is as in Lemma 6.1.16).

Then we can choose an arbitrary $D_c \in \mathcal{B}(U, Y)$, and define C_c on $H' = H_1$ by $C_c|_{H'} = C$, and on H'' by $C_c(\alpha - A)^{-1}B := \widehat{\mathbb{D}}(\alpha) - D_c \in \mathcal{B}(U, Y)$, so that $C_c \in \mathcal{B}(H' \times H'', Y)$, and (C_c, D_c) is a compatible output operator pair for Σ.

(g) By Corollary A.3.7, we have $H_B \underset{c}{\subset} W$, hence $\mathcal{B}(U, H_B) \subset \mathcal{B}(U, W)$. Trivially, $\|(alpha - A)^{-1}Bu_0\|_{H_B} \le \|u_0\|_U$ for all $u_0 \in U$ if α is as in Definition 6.1.17. Since a different α leads to an equivalent norm on H_B, as noted at the end of Definition 6.1.17, we have $(\alpha-A)^{-1}B \in \mathcal{B}(U, H_B)$ for any α. □

If B is bounded, then $H_B = H_1$, so that $C_c = C$ is essentially the only possible choice of C_c. If B is unbouded and $U = \mathbf{C}$ (even if C were bounded), then D_c can be chosen arbitrarily:

Lemma 6.3.11 *Let $\Sigma \in \mathrm{WPLS}(\mathbf{C}, H, Y)$ have B unbounded, i.e., $B1 \notin H$. Choose $r > \omega_A$ and set $x_B := (r-A)^{-1}B1$. Then $\|x_1 + \alpha x_B\|_{H_B'} := |\alpha|^2 + \|x_1\|_{H_1}^2$ is an equivalent norm for $H_B = \{x_1 + \alpha x_B \,|\, x_1 \in H_1,\ \alpha \in \mathbf{C}\}$.*

Let $y_0 \in Y$ be arbitrary. Define, $C_c(x_1 + \alpha x_B) := Cx_1 + \alpha y_0$, so that $C_c \in \mathcal{B}(H_B, Y)$ becomes an extension of $C \in \mathcal{B}(H_1, Y)$ and hence (C_c, Dc) are compatible generators of Σ, where

$$D_c := \widehat{\mathbb{D}}(r) - C_c(r-A)^{-1}B = \widehat{\mathbb{D}}(r) - C_c x_B = \widehat{\mathbb{D}}(r) - \alpha y_0 \in \mathcal{B}(\mathbf{C}, Y) = Y. \quad (6.79)$$

Thus, $D_c \in \mathcal{B}(\mathbf{C}, Y)$ can be chosen arbitrarily. □

(All this is straightforward.)

In particular, the "compatible feedthrough operator" $D_c \in \mathcal{B}(\mathbf{C})$ can be chosen arbitrarily in Example 6.2.14. If we take $D_c \neq 0$, then it becomes compensated in C_c. If we replace C by 0 in Example 6.2.14, then $D_c \neq 0$ implies that $C_c \neq 0$ — thus, the compatible generators are not unique even for bounded C.

How to recognize a suitable pair (C_c, D_c)? Quite easily:

Lemma 6.3.12 *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U, H, Y)$. If (1.) or (2.) holds, then (C_c, D_c) is a compatible pair for Σ.*

(1.) $C_c : H_B \to Y$ is linear, $D_c \in \mathcal{B}(U, Y)$, $C \subset C_c$ and $\widehat{\mathbb{D}}(s) = D_c + C_c(s-A)^{-1}B$ for some $s \in \mathbf{C}^+_\omega$;

(2.) $C_c \in \mathcal{B}(W, Y)$, $D_c \in \mathcal{B}(U, Y)$, $C \subset C_c$, and $(\mathbb{D}u)(0) = D_c u(0) + C_c x(0)$ whenever $u = \phi u_0$, $u_0 \in U$ and $\phi \in C_c^\infty(\mathbf{R})$.

Recall that "$C \subset C_c$" means that C_c is an extension of C.

Proof:

(1.) Now $C_c(\alpha - A)^{-1}(x_0 + Bu_0) = C(\alpha - A)^{-1}x_0 + [\widehat{\mathbb{D}}(\alpha) - D_c]u_0 \leq M[\|x_0\| + \|u_0\|]$ for all $x_0 \in H$, $u_0 \in U$, where $M := \max\{\|C(\alpha - A)^{-1}\|, \|\widehat{\mathbb{D}}(\alpha) - D\|\}$, hence $C_c \in \mathcal{B}(H_B, Y)$, hence (C_c, D_c) is a compatible pair for Σ.

(2.) Because $C_c^\infty(\mathbf{R})$ is dense in $\mathrm{W}^{1,2}_\omega(\mathbf{R})$, by Theorem B.7.3 and Lemma B.7.10, we may choose any $\phi \in \mathrm{W}^{1,2}_\omega(\mathbf{R})$, by continuity (indeed, by Lemma 6.3.19, we have $\mathbb{B} \in \mathcal{B}(\mathrm{W}^{1,2}_\omega; H_B)$; by Theorem B.7.4 and Lemma B.7.10, we also have $(\phi u_0 \mapsto \phi(0)u_0) \in \mathcal{B}(\mathrm{W}^{1,2}_\omega, U)$; by Theorem 6.2.13(b1), $y \in \mathrm{W}^{1,2}_\omega$, hence also $y \mapsto y(0)$ is continuous).

By causality, we may take $\phi = e^{st}$ for any $s \in \mathbf{C}^+_\omega$ (since $\pi_{(-\infty,0]}\phi \in \mathrm{W}^{1,2}_\omega$). From Lemma 6.2.10 we obtain that

$$\widehat{\mathbb{D}}(s)u_0 = D_c u_0 + C_c \mathbb{B}u = (D_c + C_c(s-A)^{-1}B)u_0. \quad (6.80)$$

Since $u_0 \in U$ was arbitrary, we have $D_c = \widehat{\mathbb{D}}(s) - C_c(s-A)^{-1}B$. □

Next few lemmas will give necessary and sufficient conditions for given operators to be the generators of a WPLS:

Lemma 6.3.13 (Generating a WPLS) *Operators* $\left[\begin{array}{c|c} A & B \\ \hline C_c & D_c \end{array}\right]$ *are compatible generators of a WPLS* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *iff the following conditions hold:*

(1.) A is the generator of a C_0-semigroup $\mathbb{A}$ on H.

(2.) $B \in \mathcal{B}(U,H_{-1})$, and there is $T>0$ s.t. for all $u \in \mathrm{L}^2([-T,0);U)$ we have

$$\mathbb{B}u := \int_{-T}^{0} \mathbb{A}(-s)Bu(s)\,ds \in H. \tag{6.81}$$

(3.) $C_c \in \mathcal{B}(W,Y)$, $D_c \in \mathcal{B}(U,Y)$ and $H_B \subset W \underset{c}{\subseteq} H$.

(4.) The map $\mathbb{C} : H_1 \to \mathcal{C}(\mathbf{R}_+;Y)$ defined by

$$(\mathbb{C}x)(t) := C_c\mathbb{A}(t)x, \quad (x \in H_1,\ t \geq 0) \tag{6.82}$$

can be extended to a continuous map $H \mapsto \mathrm{L}^2([0,T);Y)$ for some $T>0$.

(5.) For some $\omega > \omega_A$ and $T>0$, the map $\mathbb{D} : \mathcal{C}_c^\infty((0,T);U) \to \mathcal{C}_b((0,T);Y)$ defined by

$$(\mathbb{D}u)(t) := C_c\mathbb{B}\tau(t)u + D_c u(t) \quad (t \in \mathbf{R}), \tag{6.83}$$

can be extended to a continuous map $\mathrm{L}^2_\omega([0,T);U) \to \mathrm{L}^2_\omega([0,T);Y)$ for some $T>0$.

If this is the case, then $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *for any $\omega > \omega_A$; consequently, $\widehat{\mathbb{D}}(s) = D_c + C_c(s-A)^{-1}B$ for $s \in \mathbf{C}^+_{\omega_A}$. (Here $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are the unique continuous extensions of operators defined in (6.81)–(6.83).)*

By the last claim, $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ is WR iff $C_c(s-A)^{-1}B$ converges weakly as $s \to +\infty$ (the limit is then $D - D_c$, which need not be zero, cf. Lemma 6.3.10(e)).

If we know that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}$, then (1.) and (2.) are redundant; if $\left[\begin{array}{c} \mathbb{A} \\ \hline \mathbb{C} \end{array}\right] \in \mathrm{WPLS}$, then (1.) and (4.) are redundant.

Proof: 1° *"Only if":* This follows from Lemma 6.1.16 and Lemma 6.3.10(c).

2° *"If":* Let $\omega_A < \alpha < \omega$, and choose M s.t. $\|\mathbb{A}(t)\| \leq Me^{\alpha t}$ $(t \geq 0)$. Note first that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H_{-1},\{0\})$, which follows from the α-boundedness of $\mathbb{A}$ and the extended $\mathbb{B}$ defined by $\mathbb{B}u := \int_{\mathbf{R}_-} \mathbb{A}(-\cdot)Bu$. But the range of $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2([-T,0);U),H_{-1})$ is in H, hence $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2([-T,0);U),H)$, by Lemma A.3.6.

Then $\left(\begin{array}{c|c} \mathbb{A}^T & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right)$ generate a e^ω-stable wpls, say $\Delta^S\widetilde{\Sigma}$ (see Section 13.4); let $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{array}\right] := \Delta^{S^{-1}}(\Delta^S\widetilde{\Sigma})$ be the corresponding "WPLS". Then the quadruple $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{array}\right]$ is "ω-stable", by Lemma 13.3.8.

One easily verifies that this quadruple is a WPLS and has generators $\left[\begin{smallmatrix} A & B \\ C & * \end{smallmatrix}\right]$ (hence compatible generators $\left[\begin{smallmatrix} A & B \\ C_c & D_c \end{smallmatrix}\right]$).

3° The last two claims follow from Lemma 6.1.10 and Theorem 6.2.11(d1). $\square$

Corollary 6.3.14 ($\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right]$) *Assume that $\mathbb{A}$ is a C_0-semigroup and that $C \in \mathcal{B}(H_1, Y)$. Then $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ generate a WPLS iff there are $T > 0$, $M < \infty$ s.t. $\|C\mathbb{A}x_0\|_{L^2([0,T);Y)} \le M\|x_0\|_H$ for all $x_0 \in H_1$.* $\square$

(Apply Lemma 6.3.13 with $B = 0$, $D = 0$ and $W = H_1$.)

We sometimes need the following frequency-domain variant of Lemma 6.3.13:

Lemma 6.3.15 (Generating a $\widehat{\text{WPLS}}$) *Operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times U, H \times Y)$ generate a WPLS iff (1.) A generates a C_0-semigroup on H (see Theorem A.4.3), and there are $\varepsilon > 0$ and $\omega > \omega_A$ and $s_0 \in \mathbf{C}^+_{\omega_A}$ s.t. for each $x_0 \in H$ we have*

(2.) $B^*(\cdot - A^*)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega; U)$;

(4.) $C(\cdot - A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega; Y)$;

(5.) $(\cdot - s_0)C(\cdot - A)^{-1}(s_0 - A)^{-1}B \in \mathrm{H}^\infty_\infty$

For $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to generate a WR WPLS, we may replace (5.) by the assumption that $D + C_{\mathrm{w}}(\cdot - A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, but we must require that $H_B \subset \mathrm{Dom}(C_{\mathrm{w}})$, i.e., that $Cr(r-A)^{-1}(\omega - A)^{-1}Bu_0$ converges weakly for all $u_0 \in U$.

One may replace (5.) by the assumption that the map (6.51) extends to a continuous operator $\mathrm{L}^2([0,T);U) \to \mathrm{L}^2([0,T);Y)$. Naturally, the WPLS is ω-stable for any $\omega > \omega_A$.

Proof: (Analogous results are given in Chapter 9 of [Sbook], so we only sketch a proof.) Necessity follows from Theorem 6.2.11. For the converse, assume (1.)–(5.). The operator $\mathbb{C}$ of (6.82) satisfies $\widehat{\mathbb{C}x_0}(s) = C(s-A)^{-1}x_0$ for all $x_0 \in H_1$, hence (1.), (4.) and Lemma 6.3.13 apply that $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in$ WPLS. Analogously, $\left[\begin{smallmatrix} \mathbb{A}^{\mathrm{d}} \\ \mathbb{B}^{\mathrm{d}} \end{smallmatrix}\right] \in$ WPLS, hence $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \end{smallmatrix}\right] \in$ WPLS. By (d2) (see also (d1)) of Theorem 6.2.11, Lemma D.1.26 and Lemma 6.2.10, we have that $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$, hence $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS. $\square$

A bounded operator will always do as a generator:

Lemma 6.3.16 (Bounded *B* or *C*)

(a) Let $B \in \mathcal{B}(U,H)$, $C \in \mathcal{B}(H,Y)$ and $D \in \mathcal{B}(U,Y)$, and let A generate a C_0-semigroup on H (e.g., $A \in \mathcal{B}(H)$). Then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate a WPLS that is ULR.

(b) **(Bounded *B*)** *Let* $\left[\frac{A}{C}\right]$ *generate* $\left[\frac{\mathbb{A}}{\mathbb{C}}\right] \in \mathrm{WPLS}(\{0\},H,Y)$*, i.e., let* A *generate a* C_0*-semigroup* $\mathbb{A}$*, and let* $C \in \mathcal{B}(\mathrm{Dom}(A),Y)$ *be s.t. the map* $\mathbb{C} : \mathrm{Dom}(A) \to \mathcal{C}(\mathbf{R}_+;Y)$ *defined by*

$$(\mathbb{C}x)(t) := C\mathbb{A}(t)x, \quad (x \in \mathrm{Dom}(A),\ t \ge 0) \tag{6.84}$$

can be extended to a continuous operator $H \mapsto \mathrm{L}^2([0,\varepsilon);Y)$ *for some* $\varepsilon > 0$.

Assume that $B \in \mathcal{B}(U,H)$ *and* $D \in \mathcal{B}(U,Y)$*. Then* $\left[\frac{A \mid B}{C \mid D}\right]$ *generate a WPLS,* $\mathbb{D}$ *is ULR, and* $H_B = \mathrm{Dom}(A)$ *with equivalent norms.*

If $\omega_A < 0$*, then* $\mathbb{D}$ *is strongly half-plane-regular. If* $\mathbb{C}$ *is* ω*-stable, then* $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$.

(c) **(Bounded *C*)** *Let* $\left[\ A \mid B\ \right]$ *generate* $\left[\ \mathbb{A} \mid \mathbb{B}\ \right] \in \mathrm{WPLS}(U,H,\{0\})$*, i.e., let (1.)–(2.) of Lemma 6.3.13 hold.*

Assume that $C \in \mathcal{B}(H,Y)$ *and* $D \in \mathcal{B}(U,Y)$*. Then* $\left[\frac{A \mid B}{C \mid D}\right]$ *generate a WPLS* Σ*, and* $\mathbb{D}$ *is ULR.*

(d) In (a)–(c), Σ *and* Σ^{d} *are ULR and* ω*-stable for any* $\omega > \omega_A$*, and* $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ $(s \in \mathbf{C}^+_{\omega_A})$*. If* $\omega_A < 0$*, then* $\mathbb{D}$ *and* $\mathbb{D}^{\mathrm{d}}$ *are weakly half-plane-regular.*

Thus, any WPLS with a bounded B or C is ULR; in particular, $\mathbb{B}\tau$ is always ULR (with zero feedthrough!), since it has the realization $\left(\frac{A \mid B}{I \mid 0}\right)$. Since any bounded A ($\in \mathcal{B}(H)$) generates the (uniformly continuous) C_0-semigroup e^{At} with $H_1 = H = H_{-1}$, the operators B, C and D are necessarily bounded if A is bounded.

Proof: (a) It follows from, e.g., (c), that $\left[\frac{A \mid B}{C \mid D}\right]$ generate a WPLS. By Lemma A.4.4(c3), this WPLS is ULR.

(b) Apply (c) to Σ^{d} (see Lemma 6.1.4) to see that $\Sigma^{\mathrm{d}} \in \mathrm{WPLS}$ and Σ^{d} is ULR (hence so is Σ). If $\omega_A < r < 0$, then $C(s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_r;\mathcal{B}(U,Y))$, by Theorem 6.2.11(c2), hence $\mathbb{D} - D$ is strongly half-plane-regular (hence so is $\mathbb{D}$), by Proposition 6.3.3(a).

Obviously, $H_B = \{x_0 \in H \,|\, Ax_0 \in H\} = \mathrm{Dom}(A)$. Because $H_B \subset H$ and $\mathrm{Dom}(A) \subset H$, continuously, their topologies coincide, by Corollary A.3.7, i.e., they have equivalent norms.

If $\mathbb{C}$ is ω-stable, then $C(s-A)^{-1} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(H,Y))$, by Theorem 6.2.11(c2), hence also the last claim holds.

(c) 1° *WPLS:* Let $\omega > \omega_A$ so that $\left[\ \mathbb{A} \mid \mathbb{B}\ \right] \in \mathrm{WPLS}_\omega$, by Lemma 6.1.10. Set $\mathbb{C}x_0 := \pi_+ C\mathbb{A}(\cdot)x_0$ for $x_0 \in H$. Obviously, $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$, hence (1.)–(4.) of Lemma 6.3.13 hold.

Set $(\mathbb{D}u)(t) := D + C\mathbb{B}\tau(t)u$ (i.e., "$y = Cx + Du$") for $u \in \mathrm{L}^2_\omega$, $t \in \mathbf{R}$. By Theorem 6.2.11(a)&(b1), we have $\widehat{\mathbb{D}u} = D\widehat{u} + C(s-A)^{-1}B\widehat{u}$ for $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. But $C(\cdot - A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$, by Theorem 6.2.11(b2), hence $D + C(s-A)^{-1}B$ defines some $\mathbb{D}' \in \mathrm{TIC}_\omega(U,Y)$. By the obvious time-invariance of $\mathbb{D}$, we have $\mathbb{D}u = \mathbb{D}'u$ for $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, hence also (5.) of Lemma 6.3.13 is satisfied.

2° *ULR:* By Theorem 6.2.11(b3), we have $\|C(s-A)^{-1}B\|_{\mathcal{B}(U,Y)} \le M/\sqrt{\operatorname{Re} s - \gamma}$ for $\gamma > \omega_A$, $s \in \mathbf{C}^+_\gamma$, $M := \|\mathbb{B}\|\|C\|/\sqrt{2}$, hence $\mathbb{D} \in \mathrm{ULR}$.

(d) See the proofs of (b) and (c) (and use duality). □

The state feedback operator corresponding to a Riccati operator is of the following form (except that T is not always bounded):

Lemma 6.3.17 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $K = SC + T$, $S \in \mathcal{B}(Y,Z)$, $T \in \mathcal{B}(H,Z)$.

Then $\left[\begin{array}{c|c} A & B \\ \hline K & \end{array}\right]$ *generate* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Z)$. *Moreover,* $\mathbb{F} = S\mathbb{D} + T\mathbb{B}\tau + E$, *where* $E \in \mathcal{B}(U,Z)$ *is arbitrary, and* $H^*_{C,K} = H^*_C$.

If $\mathbb{D}$ *is WR (resp. SR, UR, WLR, SLR, ULR, WVR), then so is* $\mathbb{F}$, *and* $F = SD + E$, *i.e.,* $\mathbb{F} = S(\mathbb{D} - D) + T\mathbb{B}\tau + F$. *In particular, if* $\mathbb{D} \in$ ULR, *then* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow I - F \in \mathcal{GB}(U)$,

Thus, if $\mathbb{D} \in$ ULR and $Z = U$, then K is an ULR admissible state-feedback operator (see Definition 6.6.10).

Proof: 1° $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}$*:* Now $K\mathbb{A}x_0 = S\mathbb{C}x_0 + T\mathbb{A}x_0$ for $x_0 \in H_1$, hence $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{K}\end{smallmatrix}\right] \in$ WPLS. But if we set $\mathbb{F} = S\mathbb{D} + T\mathbb{B}\tau \in \mathrm{TIC}_\infty(U,Z)$, then

$$\mathbb{KB} = \pi_+(S\mathbb{D} + T\,\mathbb{AB})\pi_- = \pi_+(S\mathbb{D} + T\,\mathbb{B}\tau)\pi_- = \pi_+\mathbb{F}\pi_-, \tag{6.85}$$

hence $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right]$ is a WPLS. By Lemma 6.1.16(d), $\mathbb{F}$ is unique modulo a constant $E \in \mathcal{B}(U,Z)$.

2° Now $\mathrm{Ran}(K^*) = \mathrm{Ran}(C^*S^* + T^*) \subset H + \mathrm{Ran}(C^*)$, hence $(\bar{\alpha} - A^*)^{-1}K^* \subset H^*_C$ (see Definition 6.1.17).

3° Because $\mathbb{B}\tau \in \mathrm{ULR} \cap \mathrm{WVR}$ with feedthrough zero, by Lemma 6.3.16(c)&(d), $\mathbb{F}$ inherits the regularity properties of $\mathbb{D}$ up to ULR and WVR, and $F = SD + E$. The last claim follows from Proposition 6.3.1(c). □

We still need several more auxiliary technical lemmas on generators for the needs of later chapters. The following equivalent norm on H_B makes their proofs simpler:

Lemma 6.3.18 ($\|x_0\|_{H_B}$) *The norm* $\|\cdot\|'_{H_B}$, *defined by*

$$\|x_0\|'_{H_B} := \|x_0\|_H + \inf_{u_0 \in U}\left(\|Ax_0 + Bu_0\|_H + \|u_0\|_U\right), \tag{6.86}$$

is equivalent to $\|\cdot\|_{H_B}$.

In particular, if $u_n \to u_\infty$ *in* U, $x_n \to x_\infty$ *in* H *and* $Ax_n + Bu_n \to Ax_\infty + Bu_\infty$ *in* H, *as* $n \to \infty$, *then* $x_n \to x_\infty$ *in* H_B.

Even $\|x_0\|''_{H_B} := \|x_0\|_H + \inf_{u_0 \in U}\left(\|Ax_0 + Bu_0 + sx_0\|_H + \|u_0\|_U\right)$, where $s \in \mathbf{C}$ is fixed, is equivalent to $\|x_0\|_{H_B}$, as one can see from the proof below.

Proof: (Recall that $\|x_1\|_H := \infty$ for $x_1 \notin H$ etc.) Let α be as in Definition 6.1.17.

1° *The equivalence:* One can rewrite the definition of $\|x_0\|_{H_B}$ by

$$\|x_0\|_{H_B} := \inf\{(\|Ax_0 + Bu_0 - \alpha x_0\|^2_H + \|u_0\|^2_U)^{1/2} \mid u_0 \in U,\ Ax_0 + Bu_0 \in U\}. \tag{6.87}$$

Thus, $\|x_0\|_{H_B} \le \max\{|\alpha|,1\}\|x_0\|'_{H_B}$. It follows that $\|x_0\|'_{H_B} = 0 \Rightarrow x_0 = 0$. Now the reader can easily verify that $\|\cdot\|'_{H_B}$ is a norm.

Choose M s.t. $\|\cdot\|_H \le M\|\cdot\|_{H_B}$. Assume that $\|x_0\|_{H_B} < 1$, so that $\|Ax_0 + Bu_0 - \alpha x_0\|_H < 1$ and $\|u_0\|_U < 1$ for some $u_0 \in U$, by (6.87). It follows that $\|x_0\|'_{H_B} < M + (1 + \alpha M + 1)$. Thus, $\|\cdot\|'_{H_B} \le (2 + (|\alpha|+1)M)\|\cdot\|_{H_B}$. Consequently, the two norms are equivalent.

2° *"$x_n \to x_\infty$":* Obviously, $\|x_n - x_\infty\|'_{H_B} \le \|A(x_n - x_\infty) + B(u_n - u_\infty)\|_H + \gamma\|x_n - x_\infty\|_H + \|u_n - u_\infty\|_U \to 0$ as $n \to \infty$. □

By definition, an input map $\mathbb{B}$ maps $\mathrm{L}^2_\omega \to H$ continuously. Smoother inputs give "smoother states", as noted in Theorem 6.2.13(c1)&(c2) and below:

Lemma 6.3.19 ($\mathbb{B} : \mathrm{W}^{1,2}_\omega \to H_B$) *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,\{0\})$. *Then there is* $M < \infty$ *s.t.* $\|\mathbb{B}\tau^t u\|_{H_B} \le M e^{\omega t}\|u\|_{\mathrm{W}^{1,2}_\omega((-\infty,t);U)}$ *and* $\|(\mathbb{B}\tau u)^{(n)}(t)\|_H \le M e^{\omega t}\|u\|_{\mathrm{W}^{n+1,2}}$ *for all* $u \in \mathrm{W}^{n,2}_\omega((-\infty,t);U)$, $n \in \mathbf{N}$. *Moreover,* $\|(\alpha - A)^{-1}\begin{bmatrix} I & B \end{bmatrix}\|_{\mathcal{B}(H\times U, H_B)} \le 1$.

Proof: The last claim follows from the definition of H_B. For the others, we take $t = 0$ w.l.o.g. (note that $\|\tau^t u\|_{\mathrm{W}^{n+1,2}_\omega((-\infty,0);U)} = e^{\omega t}\|u\|_{\mathrm{W}^{n+1,2}_\omega((-\infty,t);U)}$). Let $x := \mathbb{B}\tau u$, so that on $(-\infty,t)$ we have $x' = Ax + Bu$, i.e.,

$$x = (\alpha - A)^{-1}(\alpha x - x' + Bu), \tag{6.88}$$

(let α be the number in Definition 6.1.17). Therefore,

$$\|x(t)\|_{H_B} \le \|u(t)\|_U + |\alpha|\|x(t)\|_H + \|x'(t)\|_H. \tag{6.89}$$

In particular, $\|x(0)\|_{H_B} \le M_1\|u\|_{\mathrm{W}^{1,2}_\omega} + |\alpha|\|\mathbb{B}\|\|u\|_{\mathrm{L}^2_\omega} + \|\mathbb{B}\|\|u'\|_{\mathrm{L}^2_\omega}$, by Theorem B.7.4 (recall that $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega, H)$ and that $x' = \mathbb{B}\tau u'$). Use induction for the nth derivative. □

If A is not exponentially stable, then $(s-A)^{-1}$ might not exist for $s \in i\mathbf{R}$. Nevertheless, equations $\widehat{x} = (\cdot - A)^{-1}B\widehat{u}$ and $\widehat{y} = C\widehat{x} + D\widehat{u}$ can be partially recovered on $i\mathbf{R}$ when u, x and y are stable (take $\omega = 0$):

Lemma 6.3.20 *Let* (C_c, D_c) *be a compatible pair for* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. *Let* $\omega \in \mathbf{R}$, $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ *and* $x := \mathbb{B}\tau u \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$. *Then* $(s-A)\widehat{x}(s) = B\widehat{u}(s) \in H_{-1}$ *(in particular,* $\widehat{x}(s) \in H_B$*) for a.e.* $s \in \omega + i\mathbf{R}$.

Assume, in addition, that $y := \mathbb{D}u \in \mathrm{L}^2_\omega$. *Then* $\widehat{y} = C_c\widehat{x} + D_c\widehat{u} \in Y$ *a.e. on* $\omega + i\mathbf{R}$. *In particular, for* $\omega = 0$ *and* $J \in \mathcal{B}(Y)$ *we have*

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle_{\mathrm{L}^2(\mathbf{R};Y)} = (2\pi)^{-1}\left\langle \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix}, \kappa \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix} \right\rangle_{\mathrm{L}^2(i\mathbf{R};Y)}, \tag{6.90}$$

where $\kappa := \begin{bmatrix} C_c & D_c \end{bmatrix}^* J \begin{bmatrix} C_c & D_c \end{bmatrix}$.

Note that we have $\langle \mathbb{D}u, J\mathbb{D}u\rangle = \langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \kappa \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\rangle$ whenever $u \in \mathrm{W}^{1,2}(\mathbf{R}_+;U)$, $u(0) = 0$, $\mathbb{D}u \in \mathrm{L}^2$, by Lemma 6.3.10(b) (for general $u \in \mathrm{L}^2$ having $\mathbb{D} \in \mathrm{L}^2$, an analogous result holds when $\mathbb{D}$ is WR). Here the integrands are continuous, whereas those of (6.90) are defined only a.e. In Lemma 6.7.8, we shall show that the assumption that $y \in L^2_\omega$ is redundant.

Proof: 1° *"$(\cdot - A)\widehat{x} = B\widehat{u}$ a.e. on $\omega + i\mathbf{R}$":* Recall that $A \in \mathcal{B}(H, H_{-1})$ and $B \in \mathcal{B}(U, H_{-1})$. We have $(\cdot - A)\widehat{x} = B\widehat{u}$ on $\mathbf{C}^+_{\omega_A}$, by Theorem 6.2.11(b1), hence on $\mathbf{C}^+_\omega$, by Lemma D.1.2(e) (both functions are holomorphic $\mathbf{C}^+_\omega \to H_{-1}$). Because both $\widehat{x}$ and $\widehat{u}$ converge a.e. to their boundary functions, by Theorem 3.3.1(a), also the boundary functions must satisfy $(\cdot - A)\widehat{x} = B\widehat{u}$ a.e.

2° *"$\widehat{y} = C_c\widehat{x} + D_c\widehat{u} \in Y$ a.e. on $\omega + i\mathbf{R}$":* Choose a null set $N \subset \mathbf{R}$ s.t. $\widehat{x}(\omega + ir + t) \to \widehat{x}(\omega + ir)$, $\widehat{y}(\omega + ir + t) \to \widehat{y}(\omega + ir)$ and $\widehat{u}(\omega + ir + t) \to \widehat{u}(\omega + ir)$, as $t \to 0+$, for all $r \in \mathbf{R} \setminus N$. Let $r \in \mathbf{R} \setminus N$. Then

$$A\widehat{x}(\omega + ir + t) + B\widehat{u}(\omega + ir + t) = (\omega + ir + t)\widehat{x}(\omega + ir + t) \tag{6.91}$$
$$\to (\omega + ir)\widehat{x}(\omega + ir) = A\widehat{x}(\omega + ir) + B\widehat{u}(\omega + ir), \tag{6.92}$$

as $t \to 0+$. Therefore, $\widehat{x}(\omega + ir + t) \to \widehat{x}(\omega + ir)$ in H_B, by Lemma 6.3.18. Analogously, we see that $\widehat{x} \in \mathcal{C}(\mathbf{C}^+_\omega; H_B)$.

But $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\mathbf{C}^+_{\omega_A}$, by Lemma 6.3.10(a), hence $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\mathbf{C}^+_\omega$, by continuity. Consequently, $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\omega + i(\mathbf{R} \setminus N)$.

3° We get (6.90) from (3.34). □

In the setting of the above lemma, the following estimate is often useful:

Lemma 6.3.21 ($\|C_c x_0\| \le M(\|x_0\| + \|u_0\|)$) *Let (C_c, D_c) be a compatible pair for $\Sigma \in$ WPLS(U, H, Y). For each $\omega \in \mathbf{R}$, there is $M_\omega = M_{\omega,\Sigma,D_c} < \infty$ s.t. for all $x_0 \in H$, $u_0 \in U$, $s \in \mathbf{C}^+_\omega$ we have*

$$\|C_c x_0\|_Y \le M_\omega(\|x_0\|_H + \|u_0\|_U + \|(s - A)x_0 - Bu_0\|_H), \quad \textit{in particular}, \tag{6.93}$$
$$r \in \mathbf{R} \ \&\ (ir - A)x_0 = Bu_0 \implies \|C_c x_0\|_Y \le M_0(\|x_0\|_H + \|u_0\|_U). \tag{6.94}$$

Proof: Choose $\alpha > \omega_A$. Choose $z \in \mathbf{C}^+_{\alpha-\omega}$, so that $s + z \in \mathbf{C}^+_\alpha$. Set $x_1 := sx_0 - Ax_0 - Bu_0$.

Then $(s + z - A)x_0 = zx_0 + x_1 + Bu_0$, i.e., $x_0 = (s + z - A)^{-1}(zx_0 + x_1) + (s + z - A)^{-1}Bu_0$. Thus, by Lemma 6.3.10(a),

$$C_c x_0 = C(s + z - A)^{-1}(zx_0 + x_1) + (\widehat{\mathbb{D}}(s + z) - D_c)u_0. \tag{6.95}$$

Consequently, we can take $M_\omega := \|\widehat{\mathbb{C}}\|_{H^\infty(\mathbf{C}^+_\alpha; \mathcal{B}(H,Y))}(|z| + 1) + \|\mathbb{D}\|_{\mathrm{TIC}_\alpha} + \|D_c\|_{\mathcal{B}(U,Y)} < \infty$, by Theorem 6.2.11(c2).

(Note that $M_\omega = M_{\Sigma,D_c,z}$, where we can fix, e.g., $z := \omega_A + 1 - \omega$ to obtain $M_\omega = M_{\Sigma,D_c,\omega}$.) □

If $\mathbb{D}$ is ULR (or SLR), then we can improve the above estimate:

Lemma 6.3.22 ($\|C_w x_0\| \le M\|x_0\| + \varepsilon\|u_0\|$) *Let $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS(U, H, Y). If $\mathbb{D}$ is ULR, then, for each $\omega \in \mathbf{R}$ and $\varepsilon > 0$, there is $M_{\omega,\varepsilon} < \infty$ s.t.*

$$\|C_w x_0\|_Y \le \varepsilon\|u_0\|_U + M_{\omega,\varepsilon}(\|x_0\|_H + \|(s - A)x_0 - Bu_0\|_H), \tag{6.96}$$

for all $x_0 \in H$, $u_0 \in U$ and $s \in \mathbf{C}^+_\omega$. If $\mathbb{D}$ is merely SLR, then we can still choose $M_{u_0,\omega,\varepsilon} < \infty$ satisfying (6.96) for any (fixed) $u_0 \in U$, $\omega \in \mathbf{R}$ and $\varepsilon > 0$.

Proof: Assume that $\mathbb{D}$ is ULR. Given $\varepsilon > 0$, choose $\alpha > \omega_A$ s.t. $\|\widehat{\mathbb{D}}(z) - D\|_{\mathcal{B}} < \varepsilon$ for all $z \in \mathbf{C}^+_\alpha$. Work otherwise as in the proof of Lemma 6.3.21 (with $D_c := D$, $C_c := C_w$). Then the norm of the last term in (6.95) is at most $\varepsilon\|u_0\|$, hence we obtain $M_{\omega,\varepsilon}$ as above.

If $\mathbb{D}$ is SLR, then the above proof applies except that we have to choose $R > \omega$ s.t. $\|\widehat{\mathbb{D}}(s)u_0 - Du_0\|_{\mathcal{B}} < \varepsilon$ for all $s \in \mathbf{C}^+_R$. □

From the above estimate, we obtain the following implications:

Lemma 6.3.23 ($\mathbb{B}\tau \in$ SHPR&$\mathbb{D} \in$ SLR $\Rightarrow \mathbb{D} \in$ SHPR) *If $\mathbb{B}\tau$ is UHPR and $\mathbb{D}$ is ULR, then $\mathbb{D}$ is UHPR. If $\mathbb{B}\tau$ is UVR and $\mathbb{D}$ is ULR, then $\mathbb{D}$ is UVR. This lemma also holds with S in place of U.*

Proof: 1° $\mathbb{B}\tau \in$ UHPR*:* Let $\mathbb{B}\tau$ be UHPR. Then $\mathbb{B}\tau \in \mathrm{TIC}_\omega(U,H)$ for all $\omega > 0$, hence $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ for all $\omega > 0$, by Lemma 6.1.10(b2). Given $\varepsilon > 0$, choose $M := M_{0,\varepsilon/2}$ as in Lemma 6.3.22. Choose $R > 0$ s.t. $\|(s-A)^{-1}B\| < \varepsilon/2M$ for $s \in \mathbf{C}^+$ s.t. $|s| > R$.

Let $u_0 \in U$ and $\|u_0\|_U = 1$. Set $u := u_0\chi_{\mathbf{R}_+}$, $x := \mathbb{B}\tau u$, $y := \mathbb{D}u$, so that $u, x, y \in \mathrm{L}^2_\omega$ for all $\omega > 0$. Obviously, $\widehat{u}(s) = u_0/s$ ($s \in \mathbf{C}^+$).

By Lemma 6.3.20, we have $(s-A)\widehat{x}(s) = B\widehat{u}(s)$ (hence $(s-A)s\widehat{x}(s) = Bu_0$) and $\widehat{y} = C_w\widehat{x} + D\widehat{u}$ on $\mathbf{C}^+$ (even if $(s-A)^{-1}$ does not exist for all $s \in \mathbf{C}^+$). Therefore, $\widehat{\mathbb{D}}(s)u_0 = s\widehat{y}(s) = C_w s\widehat{x}(s) + Bu_0$ for $s \in \mathbf{C}^+$, hence

$$\|(\widehat{\mathbb{D}}(s) - D)u_0\|_Y = \|C_w s\widehat{x}(s)\| \leq \frac{\varepsilon}{2}\|u_0\| + M\|s\widehat{x}(s)\| \quad (s \in \mathbf{C}^+). \tag{6.97}$$

But $s\widehat{x}(s) = \widehat{\mathbb{B}\tau}(s)u_0$ (since both sides are holomorphic on $\mathbf{C}^+$ and equal to $(s-A)^{-1}Bu_0$ on $\mathbf{C}^+_{\max\{\omega_A,0\}}$), hence $\|(\widehat{\mathbb{D}}(s) - D)u_0\| \leq \varepsilon/2 + M\varepsilon/2M = \varepsilon$ when $|s| > R$.

2° $\mathbb{B}\tau \in$ UVR*:* This goes as in 1°, except that now we choose $R > 0$ for given $\beta > \alpha_{\mathbb{D}} := \max\{\alpha_{\mathbb{B}\tau}, \omega_A\}$ (cf. Definition 6.2.3).

3° *Cases SHPR and SVR:* Work as in 1° or 2° but choose R and M for a fixed $u_0 \in U$ (alternatively, replace $\mathbb{B}$ by $\mathbb{B}P$ and $\mathbb{D}$ by $\mathbb{D}P$, where $P\alpha := \alpha u_0$ for all $\alpha \in \mathbf{C}$, so that $\Sigma \in \mathrm{WPLS}(\mathbf{C}, H, Y)$ (cf. Lemma 6.7.17) and strong becomes equivalent to uniform, so that we can apply 1° or 2°). □

Curtain and Weiss [CW89] have shown that even if both B and C "fit to A", they need not "fit" simultaneously (by Lemma 6.3.16, this cannot happen if B or C is bounded):

Example 6.3.24 (**$\left[\frac{A \mid B}{C \mid}\right] \notin$ WPLS**) Let $H := \ell^2(\mathbf{N}$ and $U := Y := \mathbf{C}$. Then the system defined by

$$A := \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & \ddots \end{bmatrix}, \quad B := \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} C \qquad := \begin{bmatrix} 1 & 1 & 1 & \ldots \end{bmatrix} \tag{6.98}$$

(and any $D \in \mathcal{B}(C)$) is not well posed, but if we replace C by

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \ldots \end{bmatrix}, \tag{6.99}$$

then the system becomes well-posed (and UR). (See Example 6.1 of [CW89] for proofs.) $\triangleleft$

Given any WPLSs $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ and $\left[\frac{\mathbb{A}}{\mathbb{C}}\right]$, the operators $\left[\frac{A \mid B}{C \mid}\right]$ are the generators of a WPLS iff

$$(\alpha - \cdot)C(\cdot - A)^{-1}(\alpha - A)^{-1}B \in \mathrm{H}^\infty_\infty \tag{6.100}$$

for some (hence all) $\alpha \in \mathbf{C}^+_{\omega_A}$ (see, e.g., Theorem 9.4.6(iv) of [Sbook]; in this case, the transfer function of the system is (6.100) plus an arbitrary constant in $\mathcal{B}$).

Reachability means that we can control any initial state approximately to zero and observability means that any initial state can be observed from the output:

Definition 6.3.25 (Reachability and observability) *The* reachability subspace $H_{\mathbb{B}}$ *and* observability subpace $H_{\mathbb{C}}$ *of* $\Sigma = \left[\frac{\mathbb{A} \mid \mathbb{B}}{\mathbb{C} \mid \mathbb{D}}\right] \in \mathrm{WPLS}(U,H,Y)$ *are defined by*

$$H_{\mathbb{B}} := \overline{\mathbb{B}[\mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)]} \subset H, \quad H_{\mathbb{C}} := \mathrm{Ker}(\mathbb{C})^\perp \subset H. \tag{6.101}$$

We call Σ (approximately) reachable *if* $H_{\mathbb{B}} = H$, *and* (approximately) observable *if* $H_{\mathbb{C}} = H$. *We call* Σ exactly ω-reachable *(in infinite time) if* $\mathbb{B}$ *is* ω*-stable and* $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)] = H$*; we call* Σ exactly ω-observable *(in infinite time) if* $\mathbb{C}$ *is* ω*-stable and* $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$ *is coercive (we may drop* ω *for* $\omega = 0$*). If* Σ *is both reachable and observable, then* Σ *is called* minimal.

By exact reachability one sometimes means exact reachability in finite time (i.e., that $\mathrm{Ran}(\mathbb{B}^T) = H$ for some $T > 0$, see Definition 4.6 of [WR00]).

By (d) below, reachability and observability are extensions of the corresponding classical (finite-dimensional) concepts:

Lemma 6.3.26 *Let* $\Sigma = \left[\frac{\mathbb{A} \mid \mathbb{B}}{\mathbb{C} \mid \mathbb{D}}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$, $\alpha \in \mathbf{R}$. *Then the following hold:*

(a1) Σ *is [exactly* α*-]reachable iff* Σ^{d} *is [exactly* α*-]observable.*

(a2) $H_{\mathbb{B}} = H_{\mathbb{B}^{\mathrm{d}}} = \overline{\mathrm{Ran}(\mathbb{B})}$*, and* $H_{\mathbb{C}} = H_{\mathbb{C}^{\mathrm{d}}} = \overline{\mathrm{Ran}(\mathbb{C}^*)}$.

(b1) $H_{\mathbb{B}}$ *is the closure in* H *of any of* $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)]$, $\mathbb{B}[C^\infty_{\mathrm{c}}(\mathbf{R}_-;U)]$, $\cup_{t>0}\mathrm{Ran}(\mathbb{B}^t)$.

(b2) $H_{\mathbb{C}}^\perp = \cap_{t>0}\mathrm{Ker}(\mathbb{C}^t)$.

(b3) Let $\Sigma' \in \mathrm{WPLS}$. *If* $\mathbb{B}' = \mathbb{B}\mathbb{M}$ *for some* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*, then* $H_{\mathbb{B}'} = H_{\mathbb{B}}$. *If* $\mathbb{C}' = \mathbb{M}\mathbb{C}$ *for some* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*, then* $H_{\mathbb{C}'} = H_{\mathbb{C}}$.

(c1) Let Σ be stable. Then Σ is [exactly] reachable if $\mathbb{B}\mathbb{B}^* > 0$ [$\mathbb{B}\mathbb{B}^* \gg 0$] on L^2; Σ is [exactly] observable if $\mathbb{C}^*\mathbb{C} > 0$ [$\mathbb{C}^*\mathbb{C} \gg 0$] on L^2.

(c2) If Σ is exactly ω-observable (resp. exactly ω-reachable), then Σ is observable (resp. reachable).

(d) Let $A \in \mathcal{B}(H)$. Then, for any $t > 0$, we have

$$H_{\mathbb{B}} = \overline{\mathrm{Ran}(\mathbb{B}^t)} = \overline{\mathrm{span}(\cup_{n\in\mathbf{N}}\mathrm{Ran}(A^nB))}, \tag{6.102}$$

$$\mathrm{Ker}(\mathbb{C}) = \mathrm{Ker}(\mathbb{C}^t) = \cap_{n\in\mathbf{N}}\mathrm{Ker}(CA^n). \tag{6.103}$$

(e) If we replace H by $H_{\mathbb{B}}$ (with the topology inherited from H), we get a reachable realization

$$\left[\begin{array}{c|c} P\mathbb{A}P^* & P\mathbb{B} \\ \hline \mathbb{C}P^* & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U, H_{\mathbb{B}}, Y) \tag{6.104}$$

of $\mathbb{D}$, where $P \in \mathcal{B}(H, H_{\mathbb{B}})$ is the orthogonal projection $H \to H_{\mathbb{B}}$. If $B \in \mathcal{B}(U,H)$, then the input operator of (6.104) is $PB \in \mathcal{B}(U, H_{\mathbb{B}})$.

By taking, instead, P to be the orthogonal projection $H \to H_{\mathbb{C}}$, we get an observable realization of $\mathbb{D}$; if C is bounded, then $CP^* \in \mathcal{B}(H_{\mathbb{C}}, Y)$ is the (bounded) output operator of (6.104).

(f) If C is bounded and injective, then Σ is observable.

(g) $\mathbb{A}^t[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$ and $(\mathbb{A}^*)^t[H_{\mathbb{C}}] \subset H_{\mathbb{C}}$ for all $t \geq 0$.

By (b3), state feedback preserves reachability and output injection preserves observability; see Section 6.6 for details. Relations to stabilizability and detectability are explained on p. 241.

Part (d) does not hold for unbounded A in general, e.g., if Σ is the exactly reachable realization of $\mathbb{D} = \tau^{-2}$ (see p. 159), then $\chi_{[-1,0)}u_0 \in \mathrm{Ker}(\pi_{[0,1)}\mathbb{C}) \setminus \mathrm{Ker}(\mathbb{C})$, (and $CA^n\chi_{[-1,0)}u_0$ is not defined).

Proof: (b1) By definition, $H_{\mathbb{B}}$ is the closure of

$$\cup_{t>0}\mathrm{Ran}(\mathbb{B}^t) = \cup_{t<0}\mathrm{Ran}(\mathbb{B}\pi_{[-t,0)}) = \mathrm{Ran}(\mathbb{B}\cup_{t<0}\mathrm{L}^2([-t,0);U)). \tag{6.105}$$

But $\cup_{t<0}\mathrm{L}^2([-t,0);U)) = \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$ and $\mathcal{C}^\infty_{\mathrm{c}}(\mathbf{R}_-;U)$ are dense in $\mathrm{L}^2_\omega(\mathbf{R}_-;U)$, by Theorem B.3.11, hence (b1) holds (note that $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)] = \mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R}_-;U)]$).

(b2) Trivially, $\mathrm{Ker}(\mathbb{C}) \subset \mathrm{Ker}(\mathbb{C}^t)$ for any $t > 0$. Conversely, if $\mathbb{C}x \neq 0$, then $\mathbb{C}^t x \neq 0$ for some $t > 0$.

(a2) Now $\mathrm{Ran}(\mathbb{C}^{\mathrm{d}}) = \mathrm{Ran}(\mathbb{C}^*) = \mathrm{Ran}(\mathbb{C}^{\mathrm{H}})$ (see (6.2)), and $\mathrm{Ran}(\mathbb{C}^{\mathrm{H}})^\perp = \mathrm{Ker}(\mathbb{C})$, by Lemma A.3.1(c7). Therefore, $H_{\mathbb{C}} = \overline{\mathrm{Ran}(\mathbb{C}^{\mathrm{d}})} = H_{\mathbb{C}^{\mathrm{d}}}$, by (b1). Exchange Σ^{d} and $\Sigma = (\Sigma^{\mathrm{d}})^{\mathrm{d}}$ to obtain that $H_{\mathbb{B}^{\mathrm{d}}} = H_{\mathbb{B}}$.

(a1) The reachability claim follows from (a2). The exact reachability claim follows from the identity $\mathrm{Ran}(\mathbb{C}^{\mathrm{d}}) = \mathrm{Ran}(\mathbb{C}^{\mathrm{H}})$ (from (a2)).

(b3) We have $\mathbb{M}^t := \pi_{[0,t)}\mathbb{M}\pi_{[0,t)} \in \mathcal{GB}(\pi_{[0,t)}\mathrm{L}^2)$, by Lemma 2.2.8, and $(\mathbb{B}')^t := \mathbb{B}'\tau^t\pi_+ = \mathbb{B}^t\mathbb{M}^t$, hence $H_{\mathbb{B}'} = H_{\mathbb{B}}$. Apply this for $(\Sigma')^{\mathrm{d}}$ to obtain the $H_{\mathbb{C}'} = H_{\mathbb{C}}$ claim.

(c1) This follows from (c1) and (c9) of Lemma A.3.1 and (a2) above.

(c2) This follows from (b1) and (a1).

(d) Set $X := \cap_{n\in\mathbf{N}} \mathrm{Ker}(CA^n)$. Now $\mathbb{A}^t = e^{At} := \sum_{n\in\mathbf{N}} A^n t^n/n!$, hence $x \in X \Rightarrow (\mathbb{C}x)(t) = C\mathbb{A}^t x = 0$ for all $t \geq 0$, hence $X \subset \mathrm{Ker}(\mathbb{C})$. Let $t > 0$. By (b2), we have $\mathrm{Ker}(\mathbb{C}) \subset \mathrm{Ker}(\mathbb{C}^t)$. Let $x \in \mathrm{Ker}(\mathbb{C}^t)$, so that $Ce^{As}x = 0$ for $s \in [0,t)$. Differentiate this n times and set $s = 0$ to obtain that $CA^n x = 0$; because $n \in \mathbf{N}$ was arbitrary, we have $x \in X$. Thus, $\mathrm{Ker}(\mathbb{C}^t) \subset X$, and hence (6.103) holds.

Therefore, $H_{\mathbb{B}}^{\perp} = H_{\mathbb{B}^{\mathrm{d}}}^{\perp} = \mathrm{Ker}(\mathbb{B}^{\mathrm{d}}) = \cap_{n\in\mathbf{N}} \mathrm{Ker}(B^*(A^*)^n)$, hence

$$H_{\mathbb{B}} = (\cap_{n\in\mathbf{N}} \mathrm{Ran}(A^n B)^{\perp})^{\perp} = ((\cup_{n\in\mathbf{N}} \mathrm{Ran}(A^n B))^{\perp})^{\perp} = \overline{\mathrm{span}(\cup_{n\in\mathbf{N}} \mathrm{Ran}(A^n B))}. \tag{6.106}$$

Set $\widetilde{\mathbb{C}} := \mathbb{B}^{\mathrm{d}}$. Then $\mathrm{Ran}(\mathbb{B}^t)^{\perp} = \mathrm{Ker}((\mathbb{B}^t)^*) = \mathrm{Ker}(\pi_-\tau^{-t}\mathbb{B}) = \mathrm{Ker}(\widetilde{\mathbb{C}}^t) = \mathrm{Ker}(\widetilde{\mathbb{C}}) = \mathrm{Ran}(\mathbb{B})^{\perp} = H_{\mathbb{B}}^{\perp}$, hence also (6.102) holds.

(e) By (g), we have $P^*P\mathbb{A}P^*P = \mathbb{A}P^*P$ (note that $P^* \in \mathcal{B}(H_{\mathbb{B}}, H)$ is the embedding $H_{\mathbb{B}} \to H$, $P^*P \in \mathcal{B}(H)$ is the orthogonal projection $H \to H_{\mathbb{B}}$ with range space H, and $PP^* = I_{H_{\mathbb{B}}}$).

Because $P^*P\mathbb{B} = \mathbb{B}$, also the new system is an ω-stable WPLS, by Lemma 6.7.17.

An analogous claim holds for $\mathrm{H}_{\mathbb{C}}$. If $C \in \mathcal{B}(H,Y)$, then $Cx_0 = (\mathbb{C}x_0)(0) = 0$ for all $x_0 \in H_{\mathbb{C}}^{\perp}$, hence then (here $P \in \mathcal{B}(H, H_{\mathbb{C}})$ is the orthogonal projection) $CP\mathbb{A}P^* = C\mathbb{A}P^* = \mathbb{C}P^*$, i.e., C is the (unique) generator of $\mathbb{C}P^*$. By duality, we get the claim on $B \in \mathcal{B}(U,H)$.

(f) Obviously, now $\mathrm{Ker}(C\mathbb{A}) = \{0\}$.

(g) Let $t \geq 0$. We have $\mathbb{A}^t x_0 \in H_{\mathbb{B}}$ for all $x_0 \in \mathbb{B}\mathrm{L}^2_{\omega}$, by "2." of Definition 6.1.1, hence $\mathbb{A}^t x_0 \in H_{\mathbb{B}}$ for all $x_0 \in H_{\mathbb{B}}$, by continuity. Thus, $\mathbb{A}^t[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$.

Apply this to Σ^{d} to observe that $(\mathbb{A}^{\mathrm{d}})^t[H_{\mathbb{C}^{\mathrm{d}}}] \subset H_{\mathbb{C}^{\mathrm{d}}}$; which by (a2) means that $(\mathbb{A}^*)^t[H_{\mathbb{C}}] \subset H_{\mathbb{C}}$. $\square$

Notes

Parts of Proposition 6.3.1 and Lemma 6.3.2 are found in the literature, as stated in the proofs; the rest is rather obvious.

Theorem 6.3.9 is due to G. Weiss [SW01a] (with a somewhat different proof) and Lemma 6.3.13 is from [Sbook], which also contains methods similar to those in Lemma 6.3.11. Corollary 6.3.14 is well known.

An implicit form of Lemma 6.3.22 for bounded B and $x_0 \in H_1$ is contained in [Keu], p. 96. Example 6.3.24 is a simple conclusion of [PW]. Definition 6.3.25 is essentially from [Sbook], which contains further results on most subjects of this section (its final version probably overlaps more than explained above).

The regularity theory is usually more fruitful than compatibility theory, but the latter covers all WPLSs. In some applications, another approach that also covers all WPLSs, namely the use of a combined "*C&D*" operator, might be more practical, see [AN] and [Sbook] for details. Next we shall motivate the concept of a compatible pair and explain the history of this notion.

The Riccati equation theory of [WW] and of several articles by O. Staffans is based on the assumption that the system is regular and that the spectral factor of an optimation problem is SR and has an invertible feedthrough, so that also the corresponding (optimal) closed-loop system is regular (this corresponds to Proposition 6.6.18(d4)).

If the spectral factor is merely WR with invertible feedthrough, then the optimal closed-loop system may be irregular (i.e., not even WR). Nevertheless, we found that one can still find closed-loop "generators" that produce the output pointwise from the input and the state ("$y = C_c x + D_c u$"), and that by such methods one can extend most of Riccati equation theory for arbitrary control problems (regardless of regularity, we only need compatible output operators) as long as some "compatible feedthrough operator" of the spectral factor is invertible. It also appeared that by using these methods, the complete Riccati equation theory for optimal control can be extended to the case of a WR system and WR spectral factor with invertible feedthrough.

This fact lead us to define two weaker "regularity properties" for a WPLS in [Mik97a], the more general of which ("infraregular output operators") is equivalent to compatible output operators. We then developed a brief compatibility theory (including early versions of Lemmas 6.3.10 and 6.3.12 and Proposition 6.6.18) and used it to derive this extended Riccati equation theory. (We used the theory in the manuscript of [Mik97b] for the WR case, but in the final version of [Mik97b] the theory is used only implicitly, for briefness.)

After finding that William Helton used a similar concept for ULR systems in [Helton76a], O. Staffans developed the theory to a rather mature state in [Sbook]. Staffans and Weiss also presented this theory in [SWcompatible] and [SW01a], the former of which treats the relations between the three approaches mentioned above (compatibility, regularity and "*C&D*"; note that at that time not all WPLSs were known to have compatible pairs). We refer the reader to these works for further information on compatibility.

6.4 Spectral and coprime factorizations ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$)

Science is spectral analysis. Art is light synthesis.

— Karl Kraus (1874–1936)

In this section, we shall define spectral factorization, right, left and doubly coprime factorization, weak forms of coprimeness (quasi-, pseudo-), and inner and lossless factorization. We then explain the basic properties of these concepts to the degree required by the next two sections and Chapter 7.

The readers who wish to have a deeper understanding of the subject, may wish to read also Section 6.5, which is a further study on coprimeness, or Chapter 5 on spectral factorization. Other readers may skip the next section and visit it only when pointed by a reference.

The importance of the factorizations mentioned above is due to several reasons. For example, dynamic stabilization is intimately connected with coprime factorizations of the I/O map of the plant (see Chapter 7), and so is joint stabilizability and detectability (Theorem 6.6.28).

Stable control problems can be solved by using a spectral factorization of the corresponding Popov operator ($\mathbb{D}^*J\mathbb{D}$, where $\langle y, Jy\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)}$ is the cost function of the problem), and unstable problems by using a coprime, inner-right/left or lossless factorization (depending on the problem) of the I/O map ($\mathbb{D}$) of the system; this will be explained in Part III.

We shall start this section by defining three forms of coprimeness. In number theory, the word "coprime" means having no common divisors (except units). Thus, numbers $n, m \in \mathbf{Z}$ are coprime iff $n = n_0 k,\ m = m_0 k,\ n_0, m_0, k \in \mathbf{Z}$ implies that k is a unit (i.e., invertible, hence $k = \pm 1$). It is well known that an equivalent condition is that $xm + yn = 1$ for some $x, y \in \mathbf{Z}$, i.e., that $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ has a left inverse (e.g., $\begin{bmatrix} x & y \end{bmatrix} \in \mathbf{Z} \times \mathbf{Z}$).

If $\widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathcal{R} := \{$rational bounded scalar functions $\mathbf{C}^+ \to \mathbf{C}\}$, then $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{M}}$ have no common divisor (except units, i.e., elements of $\mathcal{GR}$) iff $\begin{bmatrix} \widehat{\mathbb{N}} \\ \widehat{\mathbb{M}} \end{bmatrix}$ has no left inverse in $\mathcal{R}$ (equivalently, in $\mathrm{H}^\infty(\mathbf{C}^+)$), by pp. 70 and 386 of [Vid] (an analogous claim holds for any other principal ideal domain in place of $\mathcal{R}$).

For $\mathrm{H}^\infty(\mathbf{C}^+)$ in place of $\mathcal{R}$, the latter condition (traditionally called coprimeness) becomes strictly stronger than the former (which is sometimes called weak coprimeness; we shall not need it). For matrix- or operator-valued functions, we must distinguish between right and left coprimeness (which imply having no common right or left divisors, respectively; see the comments below Lemma 6.5.2 for further information). Therefore, we shall use the traditional definition of coprimeness; we supplement it by two weaker concepts:

Definition 6.4.1 (Coprime)

(a) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* right coprime (r.c.)*, if* $\mathbb{N}$ *and* $\mathbb{M}$ *together with some* $\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}} \in \mathrm{TIC}$ *satisfy the (right) Bezout identity*

$$\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I_U. \tag{6.107}$$

(b) The operators $\widetilde{\mathbb{N}} \in \mathrm{TIC}(U,Y)$ *and* $\widetilde{\mathbb{M}} \in \mathrm{TIC}(Y)$ *are* left coprime (l.c.)*, if* $\widetilde{\mathbb{N}}$ *and* $\widetilde{\mathbb{M}}$ *together with some* $\mathbb{Y}, \mathbb{X} \in \mathrm{TIC}$ *satisfy the (left) Bezout identity*

$$\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I_Y. \tag{6.108}$$

(c) The operators $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \mathbb{M} \in \mathrm{TIC}$ *are* doubly coprime (d.c.)*, if they together with some* $\mathbb{Y}, \mathbb{X}, \widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}} \in \mathrm{TIC}$ *satisfy the double Bezout identity*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I_U & 0 \\ 0 & I_Y \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}. \tag{6.109}$$

*In (a)–(c), we add the words "*over $\mathcal{A}$*", if* $\mathcal{A} \subset \mathrm{TIC}$ *and the requirements are met with* $\mathcal{A}$ *in place of* TIC*. The word* exponential*, e.g., in "exponentially d.c." will refer to "over* $\mathrm{TIC}_{\mathrm{exp}}$*".*

(d) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* pseudo–right coprime (p.r.c.) *if* $\widehat{\mathbb{N}}^*\widehat{\mathbb{N}} + \widehat{\mathbb{M}}^*\widehat{\mathbb{M}} \geq \varepsilon I$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$*.*

We call $\mathbb{N}^{\mathrm{d}}, \mathbb{M}^{\mathrm{d}}$ *pseudo–left coprime (p.l.c.) if* $\mathbb{N}, \mathbb{M}$ *are p.r.c. (i.e., iff* $\widehat{\mathbb{N}}\widehat{\mathbb{N}}^* + \widehat{\mathbb{M}}\widehat{\mathbb{M}}^* \geq \varepsilon I$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$*).*

(e) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* quasi–right coprime (q.r.c.) *if* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right] u \notin \mathrm{L}^2$ *whenever* $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2$*.*

We call $\mathbb{N}^{\mathrm{d}}, \mathbb{M}^{\mathrm{d}}$ *quasi–left coprime (q.l.c.) if* $\mathbb{N}, \mathbb{M}$ *are q.r.c.*

(f) By the coprimeness of $\widehat{\mathbb{N}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ *we refer to the coprimeness of* $\mathbb{N}$ *and* $\mathbb{M}$ *(in any of the above senses). An analogous comment applies to Definition 6.4.4.*

(Recall that $\mathrm{L}^2_\infty := \cup_{\omega \in \mathbf{R}} \mathrm{L}^2_\omega$; see Theorem 6.2.1 for (f).) Before motivating the above definitions, we observe some basic facts:

Lemma 6.4.2 *D.c. implies r.c. and l.c., r.c. implies p.r.c., and p.r.c. implies q.r.c.*

The maps $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are [p.]r.c. iff* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ *is [pseudo-]left-invertible in* $\mathrm{TIC}(U)$*, or equivalently, iff* $\mathbb{N}^{\mathrm{d}}$ *and* $\mathbb{M}^{\mathrm{d}}$ *are [p.]l.c. The maps* $\mathbb{N}$ *and* $\mathbb{M}$ *are q.r.c. iff* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ *is quasi-left-invertible in* $\mathrm{TIC}(U)$*.*

(See pp. 128 and 131 for pseudo/quasi-left-invertibility.)

The above facts will be used in the sequel without further mention; the same applies to the obvious fact that if $\mathbb{M} \in \mathcal{G}\mathrm{TIC}(U)$ and $\mathbb{N} \in \mathrm{TIC}(U,*)$ (resp. $\mathbb{N} \in \mathrm{TIC}(*,U)$), then $\mathbb{M}$ and $\mathbb{N}$ are r.c. (resp. l.c.). (Analogous claims hold for pseudo- or quasi-left-invertible $\mathbb{M} \in \mathrm{TIC}(U)$; e.g., τ^{-1} and $\mathbb{N}$ are q.r.c. (resp. q.l.c.).)

The use of any "left" results is minimal in this monograph, we prefer using "right"[3] results and the duality stated in the lemma. Explicit forms of many such "left" results can be found in [Sbook].

Proof of Lemma 6.4.2: Trivially, d.c. implies r.c. and l.c.; the other two implications follows from Lemma 6.5.2(ii)&(b1). The equivalence follows directly from the definitions. □

It is instructive to observe the meaning of coprimeness in the case of scalar transfer functions:

Lemma 6.4.3 *Let* $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}(\mathbf{C})$. *Then the following are equivalent:*

(i) $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]r.c.*

(ii) $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]l.c.*

(iii) $|\widehat{\mathbb{N}}| + |\widehat{\mathbb{M}}| \geq \varepsilon$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$.

If $\widehat{\mathbb{N}}$ *and* $\widehat{\mathbb{M}}$ *are continuous on* $\overline{\mathbf{C}^+} \cup \{\infty\}$ *(e.g., they are rational or in* $\widehat{\mathrm{MTIC}^{\mathrm{L}^1}}$*), then (iii) holds iff* $\widehat{\mathbb{N}}$ *and* $\widehat{\mathbb{M}}$ *have no common zeros on* $\overline{\mathbf{C}^+} \cup \{\infty\}$. □

(This follows from Lemma 6.5.3(a)&(c) and the compactness of $\overline{\mathbf{C}^+} \cup \{\infty\}$.)

Thus, if $\mathbb{N}$ and $\mathbb{M}$ are scalar and coprime, and $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, then "$\mathbb{N}$ cancels no poles of $\mathbb{M}^{-1}$", i.e., "$\mathbb{M}^{-1}$ and $\mathbb{N}\mathbb{M}^{-1}$ have the same poles". See Lemma 6.5.4 for the general case.

Classical coprimeness has its advantages, especially in dynamic feedback (see Chapter 7) including the H^∞ four-block problem. However, the most useful properties of coprimeness are the ones given in (b1) and (c1) of Lemma 6.5.1, hence for most results using coprimeness, also quasi-coprimeness is a sufficient assumption.

Furthermore, quasi-coprimeness has two important advantages to coprimeness: it can often be more easily verified and it is preserved in inverse discretization (see Theorem 13.4.4(e1); we do not know if this is the case for pseudo-coprimeness), thus allowing us to prove several important results in discrete time. Indeed, the verification of pseudo-coprimeness is a simple, nonconstructive process, and "p.r.c." implies "q.r.c.". Moreover, the I/O map of an exponentially stabilizable and detectable system has a q.r.c. factorization, by Theorem 6.7.15(c2) (see Corollary 6.7.16 for similar implications).

For the above reasons, we usually use coprimeness in connection with dynamic stabilization and quasi-coprimeness for other occasions, including state feedback. Pseudo-coprimeness seldom implies anything useful that quasi-coprimeness would not imply, hence we mostly neglect it.

Next we define several notions that are used in connection with feedback and optimal control:

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[3] This is not an ideological statement.

Definition 6.4.4 (Factorizations) *Let $J = J^* \in \mathcal{B}(Y)$, $S \in \mathcal{GB}(U)$, and $\mathcal{A} \subset \mathrm{TIC}$.*

(a) An operator $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *is* (J,S)-inner, *if* $\mathbb{N}^* J\mathbb{N} = S$.

(b) An operator $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *is* (J,S)-lossless, *if* $\mathbb{N}^* J\mathbb{N} = S$ *and* $\mathbb{N}^*\pi_- J\mathbb{N} \leq \pi_- S$.

(c) **(SpF)** *A factorization* $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$ *is a* spectral factorization [over $\mathcal{A}$] *of* $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$, *if* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *[$\mathbb{X} \in \mathcal{GA}(U)$]. In this case we call* $\mathbb{X}$ *an* S-spectral factor *of* $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$.

(d1) **(r.c.f.)** *A factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* [quasi–]right coprime factorization ([q.]r.c.f.) *of* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, *if* $\mathbb{N},\mathbb{M} \in \mathrm{TIC}$ *are [q.]r.c. and* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

If, in addition, $\mathbb{N}$ *is (J,S)-inner (resp. (J,S)-lossless), then* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* (J,S)-inner [q.]r.c.f. *(resp.* (J,S)-lossless [q.]r.c.f.*).*

(d2) A factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* right factorization *of* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, *if* $\mathbb{N},\mathbb{M} \in \mathrm{TIC}$ *and* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

If, in addition, $\mathbb{N}^* J\mathbb{N} = S$, *then* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* (J,S)-inner-right factorization

(e) A factorization $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ *is a* [quasi–]left coprime factorization ([q.]l.c.f.) *of* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, *if* $\widetilde{\mathbb{N}},\widetilde{\mathbb{M}} \in \mathrm{TIC}$ *are [q.]l.c. and* $\widetilde{\mathbb{M}} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$.

(f) **(d.c.f.)** *A factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ *is a* doubly coprime factorization (d.c.f.) [over $\mathcal{A}$] *of* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, *if* $\mathbb{N},\mathbb{M},\widetilde{\mathbb{N}},\widetilde{\mathbb{M}}$ *are d.c. [over $\mathcal{A}$] and* $\mathbb{M},\widetilde{\mathbb{M}} \in \mathcal{G}\mathrm{TIC}_\infty$.

If (6.109) is the corresponding double Bezout identity, we say that $\mathbb{D}$ *and* $\mathbb{Y}\mathbb{X}^{-1}$ *(equivalently,* $\mathbb{D}$ *and* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*) have a* joint d.c.f. [over $\mathcal{A}$].[4]

*As in (f), we add the words "*over $\mathcal{A}$*" in (d) and (e) too, if the factors are coprime over $\mathcal{A}$ instead of* TIC. *For example, the factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* r.c.f. over $\mathcal{A}$, *if* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$ *and* $\mathbb{M},\mathbb{N},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathcal{A}$ *satisfy (6.107).*

We also use the above definitions with "pseudo" in place of "quasi" and "p." in place of "q.".

The words exponentially stable*, e.g., in "an exponentially stable d.c.f.", will refer to "over* $\mathrm{TIC}_{\mathrm{exp}}$*".*

In parts (a)–(d2), we call S a signature operator *(note that necessarily $S = S^*$). By "$(J,*)$" we mean "(J,S) for some $S \in \mathcal{GB}$".*

Thus, a d.c.f. contains a r.c.f. and a l.c.f.; the converse is given in Lemma 6.5.8; moreover, a r.c.f. is a right factorization. These factorizations are unique up to a unit, see Lemmas 6.4.5 and 6.5.9. In (d)–(f), the operators $\mathbb{N}$ and $\widetilde{\mathbb{N}}$ are called *numerators* and $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ *denominators*.

Inner-right factorizations are often called inner-outer factorizations if the right factor has a stable inverse. Losslessness is further treated in Section 2.5.

[4] If $\mathbb{X}$ is not invertible in a reasonable sense (e.g., in TIC_∞), this will be considered merely as a formal definition until Section 7.3. Note that $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$.

Note that we have required $\mathbb{X}$ and S to be invertible unlike some authors do. This way (c) coincides with Definition 5.1.1, by Lemma 5.2.1(d), and these strong definitions are more useful for the results presented in this monograph.

We remark that, by Lemma 6.5.9(a2), any operators satisfying (6.109) constitute a d.c.f. of $\mathbb{N}\mathbb{M}^{-1}$ if(f) $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$. In the finite-dimensional case, on often constructs a d.c.f. of a map by choosing a minimal, hence (exponentially) jointly stabilizable and detectable realization. The same approach can be used for the I/O map for any WPLS having SR jointly stabilizing state feedback and output injection operators, as noted below Theorem 6.6.28.

When $\dim U < \infty$, one sometimes does not require that $\mathbb{M}^{-1} \in \mathrm{TIC}_\infty$, just that $\det\widehat{\mathbb{M}} \not\equiv 0$, and thus (d1) becomes the definition of a r.c. "$\mathrm{H}^\infty/\mathrm{H}^\infty$ factorization". By Lemma 6.5.4(d2), this is equivalent to our definition when $\mathbb{D} \in \mathrm{TIC}_\infty$.

Observe from Lemma 6.4.3 that a scalar right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is coprime iff $\mathbb{N}$ does not cancel any poles of $\mathbb{M}^{-1}$, i.e., iff $\mathbb{M}^{-1}$ and $\mathbb{D}$ have the same poles. In a sense, the same holds also in the infinite-dimensional case, by the comments below Lemma 6.5.4.

Coprime and spectral factorizations are unique up to unit:

Lemma 6.4.5 (Uniqueness of factorizations)

(a) *Let $\mathbb{X}_0^* S_0 \mathbb{X}_0$ be a spectral factorization of $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$. Then all spectral factorizations of $\mathbb{E}$ are given by $\mathbb{X} = E\mathbb{X}_0$, $S = (E^*)^{-1} S_0 E^{-1}$ with $E \in \mathcal{GB}(U)$.*

(b) *Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a right factorization of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all right (not necessarily coprime) factorizations of $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}$ are given by $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ ($\mathbb{U} \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ if $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ is a q.r.c.f.).*

(c) *Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a q.r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all q.r.c.f.'s of $\mathbb{D}$ are given by $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$.*

Moreover, if one of them is a [p.]r.c.f., then all of them are [p.]r.c.f.'s.

(d) *Let $(\widetilde{\mathbb{N}}_0, \widetilde{\mathbb{M}}_0)$ be a [q.]l.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all [q.]l.c.f.'s of $\mathbb{D}$ are given by $\widetilde{\mathbb{N}} = \mathbb{U}\widetilde{\mathbb{N}}_0$, $\widetilde{\mathbb{M}} = \mathbb{U}\widetilde{\mathbb{M}}_0$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$.*

(e) *Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a (J, S_0)-inner [q.]r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all $(J,*)$-inner [q.]r.c.f.'s of $\mathbb{D}$ are given by $\mathbb{N} = \mathbb{N}_0 E$, $\mathbb{M} = \mathbb{M}_0 E$ (and $S = E^* S_0 E$) with $E \in \mathcal{GB}(U)$.*

Moreover, all $(J,*)$-inner-right factorizations of $\mathbb{D}$ are [q.]r.c.f.'s, hence of the above form. (See also (c).)

(f) *Parts (a) and (e) hold even if the signature operators (S and S_0) are required to be merely one-to-one (not necessarily invertible).*

Note that for $\mathbb{E} \gg 0$ one can take $E := \sqrt{S_0}$ to get $\mathbb{E}^* = \mathbb{X}^*\mathbb{X}$ in (i), as is often done (e.g., in [WW]). For WR $\mathbb{X}_0$ with an invertible feedthrough operator $X_0 := \widehat{\mathbb{X}}_0(+\infty)$, another common normalization is to take $E := X_0^{-1}$ (i.e., $X = I$;

this corresponds to zero feedthrough when $\mathbb{F} := I - \mathbb{X}$ is used to construct a state feedback pair, as in Theorem 9.9.10(g1) and Corollary 9.9.11).

Proof: (a) This follows from Lemma 5.2.1(d)&(f).

(b) Set $\mathbb{U} := \mathbb{M}_0^{-1}\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$ to obtain $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$. If $\mathbb{N}_0$ and $\mathbb{M}_0$ are q.r.c., then $\mathbb{U} \in \mathrm{TIC}$, by Lemma 6.5.1(c1).

(c) The parametrization follows from (b) (interchange the roles of $(\mathbb{N},\mathbb{M})$ and $(\mathbb{N}_0,\mathbb{M}_0)$ to see that $\mathbb{U}^{-1} \in \mathrm{TIC}$).

If $\mathbb{N}_0,\mathbb{M}_0$ are r.c., i.e., $\widetilde{\mathbb{X}}\mathbb{N}_0 - \widetilde{\mathbb{Y}}\mathbb{M}_0 = I$ for some $\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathrm{TIC}$, then $\widetilde{\mathbb{U}}^{-1}\widetilde{\mathbb{X}}\mathbb{N} - \widetilde{\mathbb{U}}^{-1}\widetilde{\mathbb{Y}}\mathbb{M} = I$, hence then $\mathbb{N},\mathbb{M}$ are r.c. The p.r.c. claim follows analogously from, e.g., Lemma 6.5.2(ii).

(d) Apply (c) to $\mathbb{D}^{\mathrm{d}}$. (Of course, the other claims in (c) also have their duals for (d).)

(e) It is obvious that $\mathbb{D} = (\mathbb{N}_0E)(\mathbb{M}_0E)^{-1}$ is a (J,E^*S_0E)-inner [q.]r.c.f. for each $E \in \mathcal{G}\mathcal{B}(U)$ (see (c)), hence we only have to study the (extended) converse.

Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a (J,S)-inner-right factorization of $\mathbb{D}$. Set $E := \mathbb{M}_0^{-1}\mathbb{M} \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ as in (b), so that $\mathbb{N} = \mathbb{N}_0E$. Then

$$E^*S_0E = E^*\mathbb{N}_0^*J\mathbb{N}_0E = \mathbb{N}^*J\mathbb{N} = S, \tag{6.110}$$

hence $E \in \mathcal{G}\mathcal{B}$, by Lemma 6.5.5(a); in particular, $\mathbb{N}\mathbb{M}^{-1}$ is a [q.]r.c.f., by (c).

(f) For (a) we note that $\mathbb{X}^*S\mathbb{X} = \mathbb{X}_0^*S_0\mathbb{X}_0$ implies that $E^*SE = S_0$, where $E = \mathbb{X}\mathbb{X}_0^{-1} \in \mathcal{G}\mathrm{TIC}$, hence $E \in \mathcal{G}\mathcal{B}$, by Lemma 6.5.5(a). The converse is trivial.

The proof of (e) above applies for noninvertible S too. $\square$

Next we recall two important facts from Lemma 2.2.2(a1)&(d):

Lemma 6.4.6 *Let $\mathbb{E} \in \mathrm{TI}(U)$. We have $\mathbb{E} \gg 0$ iff $\pi_+\mathbb{E}\pi_+ \gg 0$. Moreover, if $\pi_+\mathbb{E}\pi_+$ is invertible (on $\pi_+\mathrm{L}^2$), then so is $\mathbb{E}$ (on L^2, i.e., in $\mathrm{TI}(U)$).* $\square$

The converse to the latter claim is not true (e.g., take $\mathbb{E} = \tau(1)$).

We repeat here some results from Lemma 5.2.1:

Lemma 6.4.7 (SpF) *Let $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$. Then we have the following:*

(a) $\mathbb{E} \gg 0$ iff $\mathbb{E}$ has the spectral factorization $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$.

Assume now that $\mathbb{E} \in \mathrm{TI}(U)$ has a spectral factorization $\mathbb{E} = \mathbb{X}^*S\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$, $S = S^* \in \mathcal{G}\mathcal{B}(U)$. Then we have the following:

(b) The Toeplitz operator $\pi_+\mathbb{E}\pi_+$ is invertible on $\pi_+\mathrm{L}^2$, and $\pi_+\mathbb{X}^{-1}\pi_+S^{-1}\mathbb{X}^{-*}\pi_+$ is its inverse.

(See Theorem 8.4.12 for the converse for $\mathbb{E} \in \mathrm{MTI}$.)

(c) If, in addition, $\mathbb{E} \in \mathrm{TI}_\omega(U)$ for some $\omega \neq 0$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\exp}(U)$.

(d) The map $\mathbb{E}^{\mathrm{d}} := \boldsymbol{\text{Я}}\mathbb{E}\boldsymbol{\text{Я}} \in \mathrm{TI}(U)$ has the co-spectral factorization *$\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}}S(\mathbb{X}^{\mathrm{d}})^*$ (where $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(U)$).* $\square$

(This is a direct consequence of Lemma 5.2.1.) Clearly $\mathbb{E}^{\mathrm{d}}$ has a co-spectral factorization iff $\mathbb{E}$ has a spectral factorization, i.e., the converse to (d) is also true.

Section 8.4 contains a study on the equivalence between spectral or coprime factorizations and the coercivity of the cost function of a control problem; see especially Theorem 8.4.12 for MTIC classes. In Section 9.1, we establish a third equivalent condition in terms of Riccati equations. For very regular systems, we give more neat results in Theorem 9.2.14 and Corollary 9.2.15. See also Chapter 5 on spectral factorization.

We finish this section by noting that the search for a (J,S)-inner r.c.f. can be reduced to a spectral factorization problem:

Lemma 6.4.8 ((J,S)-inner r.c.f. vs. SpF) *The following hold:*

(a) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$. If $\mathbb{X}^* S \mathbb{X}$ is a spectral factorization of $\mathbb{D}^* J \mathbb{D}$, then $(\mathbb{D}\mathbb{X}^{-1}, \mathbb{X}^{-1})$ is a (J,S)-inner r.c.f. of $\mathbb{D}$. Conversely, if $(\mathbb{N},\mathbb{M})$ is a (J,S)-inner q.r.c.f. of $\mathbb{D}$, then $\mathbb{M}^{-1} S \mathbb{M}^{-1}$ is a spectral factorization of $\mathbb{D}^* J \mathbb{D}$.

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a [q.]r.c.f., let $J = J^* \in \mathcal{B}(Y)$, and let $S = S^* \in \mathcal{G}\mathcal{B}(U)$. Then $\mathbb{D}$ has a (J,S)-inner [q.]r.c.f. iff $\mathbb{N}^* J \mathbb{N}$ has an S-spectral factor.

(c) Let $\mathbb{E} \in \mathcal{G}\mathrm{TI}(U \times W)$. Then $\mathbb{E}^* J_1 \mathbb{E} = J_1 \Leftrightarrow \mathbb{E} J_1 \mathbb{E}^* = J_1 \Leftrightarrow \mathbb{E}^{-*} J_1 \mathbb{E}^{-1} = J_1$.

Proof: (a) The first claim follows from (denote $\mathbb{M} := \mathbb{X}^{-1}$) $\mathbb{X}^* S \mathbb{X} = \mathbb{D}^* J \mathbb{D} \Leftrightarrow S = (\mathbb{D}\mathbb{M})^* J (\mathbb{D}\mathbb{M})$ and from $0 + \mathbb{X}\mathbb{M} = I$. The converse follows from the fact that $\mathbb{X} := \mathbb{M}^{-1} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.5.6(b).

(b) If $\mathbb{N}'\mathbb{M}'^{-1}$ is a (J,S)-inner [q.]r.c.f., then $\mathbb{N}' = \mathbb{N}\mathbb{X}^{-1}$, $\mathbb{M}' = \mathbb{M}\mathbb{X}^{-1}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c), and $(\mathbb{N}\mathbb{X}^{-1})^* J \mathbb{N}\mathbb{X}^{-1} = S$ implies that $\mathbb{X}S\mathbb{X} = \mathbb{N}^* J \mathbb{N}$. By going backwards we get the converse implication.

(c) The first equivalence follows from Lemma A.1.1(h1); the second is obvious (recall that $J_1 := \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$). □

Notes

Definition 6.4.1(a)–(c), Lemma 6.4.6, Lemma 6.4.7(a)&(b), and most of Definition 6.4.4 and Lemmas 6.4.5 and 6.4.8 are from [S98a] and [S98c] (see also [Sbook]); most of these are classical results/definitions (except possibly for signs). Also Lemma 6.4.2 and 6.4.3 are well known.

Naturally, from each result of this section on coprimeness, one can obtain a corresponding result on "ω-coprimeness" or on exponential coprimeness, by shifting (see Remark 6.1.9); an analogous claim applies to most other results concerning stability.

The class $\mathcal{F} := \{\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1} \mid \widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathbf{C}^{*\times m}),\ \det \widehat{\mathbb{M}} \not\equiv 0\}$ of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions"has been studied extensively in the literature. This class is not contained in, nor does it contain the class of matrix-valued well-posed transfer functions. We can apply our theory to any $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ having a right factorization (such a factorization is implied by any kind of stabilizability), due to Lemma 6.5.4(d2). Conversely, most proofs in this and the next section also apply (mutatis mutandis) to the $\mathcal{F}$ setting or to its generalization to infinite-dimensional input and output spaces with $\widehat{\mathbb{M}}$ required to be invertible at least at one point of $\mathbf{C}^+$.

The monograph [Vid] is an excellent reference to coprime factorization. Its emphasis is on rational matrix-valued transfer functions (equivalently, on the I/O maps having a finite-dimensional realization), but it also contains some results on several more general classes. Also [Logemann93] treats coprime factorizations in various subclasses of $\mathcal{F}$ (cf. Remark 6.5.11), and [Smith] is instructive in the general $\mathcal{F}$ setting.

During the preparation of this book, we have received several articles on coprime factorization and dynamic feedback in a WPLS setting, these include [CWW96], [WC] and [CWW01]. Also [Sbook] contains further results.

The notions of quasi- and pseudo-coprimeness may be new. Ruth Curtain [Curtain02] has lately studied the latter using so called *reci-procal systems*; one of her results is stated below Lemma 6.5.10.

6.5 Further coprimeness and factorizations

I tell them to turn to the study of mathematics, for it is only there that they might escape the lusts of the flesh.
— Thomas Mann (1875–1955), "The Magic Mountain"

In this section we give deeper explanations of the nature of classical, pseudo- and quasi-coprimeness and their relations to each other. We also give certain further results on coprime factorizations and inner maps.

We first deduce the basic facts on quasi–right coprime maps from those of quasi-left-invertible ones:

Lemma 6.5.1 (q.r.c.) *Assume that* $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are q.r.c. Then, for any* $\omega \geq 0$*, we have*

(a) $\mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M} \gg 0$.

(b1) $\mathbb{N}u, \mathbb{M}u \in \mathrm{L}^2 \Leftrightarrow u \in \mathrm{L}^2$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b3) There is $\varepsilon > 0$ *s.t. s.t.* $\varepsilon\|u\|_{\mathrm{L}^2} \leq \|\mathbb{N}u\|_{\mathrm{L}^2} + \|\mathbb{M}u\|_{\mathrm{L}^2} \leq \varepsilon^{-1}\|u\|_{\mathrm{L}^2}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(c1) We have $\mathbb{N}\mathbb{D}, \mathbb{M}\mathbb{D} \in \mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathrm{TIC}$ *(and* $\|\mathbb{D}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}(\|\mathbb{N}\mathbb{D}\|_{\mathrm{TIC}} + \|\mathbb{M}\mathbb{D}\|_{\mathrm{TIC}}$*) when* $\mathbb{D} \in \mathrm{TIC}_\infty(H,U)$.

(c2) We have $\mathbb{N}\mathbb{C}, \mathbb{M}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2)$ *(and* $\|\mathbb{C}\| \leq \varepsilon^{-1}(\|\mathbb{N}\mathbb{C}\| + \|\mathbb{M}\mathbb{C}\|)$*) when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega(\mathbf{R}_+;U))$.

(d) If $\mathbb{N}$ *and* $\mathbb{M}$ *are [q.]r.c. and* $\begin{bmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}(Y \times U)$ *is [quasi–]left invertible (over* TIC*), then* $\mathbb{D}_1 \begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix}$ *and* $\mathbb{D}_2 \begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix}$ *are [q.]r.c.; in particular,* $\mathbb{N}$ *and* $\mathbb{M} + \mathbb{U}\mathbb{N}$ *are [q.]r.c. for any* $\mathbb{U} \in \mathrm{TIC}$.

(e) For each $u_0 \in U \setminus \{0\}$*, we have* $\|\widehat{\mathbb{N}}u_0\|_Y + \|\widehat{\mathbb{M}}u_0\|_U \neq 0$ *on* $\mathbf{C}^+$ *and* $\|\widehat{\mathbb{N}}(ir)u_0\|_Y + \|\widehat{\mathbb{M}}(ir)u_0\|_U \geq \varepsilon\|u_0\|_U$ *for a.e.* $r \in \mathbf{R}$ *and some* $\varepsilon > 0$.

(f) If $\begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix} = \begin{bmatrix} \mathbb{N}_0 \\ \mathbb{M}_0 \end{bmatrix}\mathbb{U}$*, and* $\mathbb{U}, \mathbb{N}_0, \mathbb{M}_0 \in \mathrm{TIC}$*, then* $\mathbb{U}$ *is quasi–left-invertible (left-invertible over* TIC *if* $\mathbb{N}, \mathbb{M}$ *are r.c.).*

Proof: This follows from corresponding claims in Lemma 4.1.8. (Set $\mathbb{D} = \begin{bmatrix} I & 0 \\ \mathbb{U} & I \end{bmatrix}$ in (d). The "non-quasi" claims in (d) and (f) are obvious.) □

We go on to list some properties of p.r.c. maps (hence of r.c. maps too):

Lemma 6.5.2 (p.r.c.) *Let* $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$*. The following are equivalent:*

(i) $\mathbb{N}$ *and* $\mathbb{M}$ *are p.r.c.;*

(ii) $\widehat{\widehat{\widehat{\mathbb{X}}}}\widehat{\mathbb{M}} - \widehat{\widehat{\widehat{\mathbb{Y}}}}\widehat{\mathbb{N}} \equiv I$ *on* $\mathbf{C}^+$ *for some bounded* $\widehat{\widehat{\widehat{\mathbb{X}}}}, \widehat{\widehat{\widehat{\mathbb{Y}}}} : \mathbf{C}^+ \to \mathcal{B}$*;*

(iii) There is $M < \infty$ *s.t. for each* $\omega \geq 0$ *there are* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TI}_\omega(Y,U)$ *satisfying* $\|\widetilde{\mathbb{X}}\|_{\mathrm{TI}_\omega} \leq M$*,* $\|\widetilde{\mathbb{Y}}\|_{\mathrm{TI}_\omega} \leq M$*, and* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I_{\mathrm{TI}_\omega}$*;*

(iv) There is $\varepsilon > 0$ *s.t.* $\|\mathbb{M}u\|_{\mathrm{L}^2_\omega} + \|\mathbb{N}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*) for all* $\omega \geq 0$.

Moreover, if $\mathbb{N}$ *and* $\mathbb{M}$ *are p.r.c.,* $\alpha \geq \omega \geq 0$*, and* $T, \beta \in \mathbf{R}$*, then*

(a) Condition (ii) holds with some $\widehat{\widehat{\mathbb{X}}}, \widehat{\widehat{\mathbb{Y}}} \in \mathcal{C}_b(\mathbf{C}^+; \mathcal{B})$.

(b1) $\mathbb{N}u, \mathbb{M}u \in \mathrm{L}^2_\omega \Leftrightarrow u \in \mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R}; U)$.

(b2) $\mathbb{N}u, \mathbb{M}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega \Leftrightarrow u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R}; U)$.

(b3) There is $\varepsilon > 0$ *s.t.*

$$\varepsilon\|u\|_{\mathrm{L}^2_\omega} \leq \|\mathbb{M}u\|_{\mathrm{L}^2_\omega} + \|\mathbb{N}u\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|u\|_{\mathrm{L}^2_\omega}, \quad (u \in \mathrm{L}^2_\alpha(\mathbf{R}; U)). \tag{6.111}$$

(c1) Let $\mathbb{D} \in \mathrm{TIC}_\infty(H, U)$*. Then* $\mathbb{N}\mathbb{D}, \mathbb{M}\mathbb{D} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{D} \in \mathrm{TIC}_\omega$.

(c2) Let $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\beta(\mathbf{R}_+; U))$*. Then* $\mathbb{N}\mathbb{C}, \mathbb{M}\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega) \Leftrightarrow \mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega)$.

(d) If $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]r.c., then so are* $\mathbb{N}$ *and* $\mathbb{M} + \mathbb{U}\mathbb{N}$ *for any* $\mathbb{U} \in \mathrm{TIC}$.

(e) We have $\mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M} \gg 0$.

(f) If $\begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix} = \begin{bmatrix} \mathbb{N}_0 \\ \mathbb{M}_0 \end{bmatrix} \mathbb{U}$*, and* $\mathbb{U}, \mathbb{N}_0, \mathbb{M}_0 \in \mathrm{TIC}$*, then* $\mathbb{U}$ *is pseudo–left-invertible on* TIC *(left-invertible if* $\mathbb{N}, \mathbb{M}$ *are r.c.).*

Note that if $\dim U < \infty$, then $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$ in (f), by Proposition 2.2.5(3). Thus, when $\dim U < \infty$, all common right divisors of [p.]r.c. maps are invertible. We conclude that "p.r.c.", is a stronger property than "weakly r.c."; in fact, it is strictly stronger (e.g., take $f(s) = se^{-s}/(s+1)$, $g = 1/(s+1)$, so that $f, g \in \mathrm{H}^\infty(\mathbf{C}^+; \mathbf{C})$ are "weakly r.c." but not [p.]r.c., because $f(+\infty) = 0 = g(+\infty)$; this is Example of [Smith]).

On the other hand, "weakly r.c." is not implied by "q.r.c.": τ^{-1} and τ^{-1} are q.r.c. but not "weakly r.c.", because τ^{-1} (or τ^{-r} for any $r \in (0,1]$) is their common divisor and not a unit (in TIC). We note that also the definition of "weakly r.c." used in part III of [Smith] is weaker than r.c. (=p.r.c., by Lemma 6.5.3), by Lemma 4 of [Smith]. However, we shall not use the concept "weakly r.c.".

Proof of Lemma 6.5.2: The equivalence and (a)–(c2) follow by setting $\mathbb{D} := \begin{bmatrix} \mathbb{M} \\ \mathbb{N} \end{bmatrix}$, $[\widehat{\widehat{\mathbb{X}}} \ -\widehat{\widehat{\mathbb{Y}}}] := \widehat{\mathbb{V}}$ in Proposition 4.1.7 (for (iv) and (b3) we use $(|r| + |s|)/2 \| \begin{bmatrix} r \\ s \end{bmatrix} \| \leq (|r| + |s|)$) which also gives additional equivalent conditions.

(d) If $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$, then $\widetilde{\mathbb{X}}(\mathbb{M} + \mathbb{U}\mathbb{N}) - (\widetilde{\mathbb{X}}\mathbb{U} + \widetilde{\mathbb{Y}})\mathbb{N} = I$; use (ii) for the p.r.c. case.

(e) By (v) (with $\omega = 0$), we have $\mathbb{M}^*\mathbb{M} + \mathbb{N}^*\mathbb{N} \geq \varepsilon I$.

(f) Now $(\widehat{\widehat{\mathbb{X}}}\widehat{\mathbb{M}}_0 - \widehat{\widehat{\mathbb{Y}}}\widehat{\mathbb{N}}_0)\mathbb{U} = I$, by (ii). □

Due to (a) and (c) below, one does not meet the notions "p.r.c." and "q.r.c." in the finite-dimensional theory:

Lemma 6.5.3 *Let* $\mathbb{M} \in \mathrm{TIC}(U)$ *and* $\mathbb{N} \in \mathrm{TIC}(U, Y)$*. Then*

(a) If $\dim U < \infty$*, then* $\mathbb{M}$ *and* $\mathbb{N}$ *are p.r.c. iff they are r.c.*

(b) If $\dim U = \infty$*, then* $\mathbb{M}$ *and* $\mathbb{N}$ *may be p.r.c. even if they are not r.c.*

(c) If $\dim U < \infty$ *and* $\widehat{\mathbb{M}}, \widehat{\mathbb{N}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U, *))$ *are rational and* $M \in \mathcal{G}\mathcal{B}(U)$*, then* $\widehat{\mathbb{M}}$*,* $\widehat{\mathbb{N}}$ *are q.r.c. iff they are r.c. (iff* $\|\widehat{\mathbb{M}}u_0\|_U + \|\widehat{\mathbb{N}}u_0\|_Y \neq 0$ *on* $\overline{\mathbf{C}^+}$ *for all* $u_0 \in U \setminus \{0\}$*).*

(d) A rational q.r.c.f. is a r.c.f.

See (the Corona) Theorem 4.1.6 for further results in case $\dim U < \infty$.

Even when $U = \mathbf{C} = Y$, we may have $\widehat{\mathbb{M}} = e^{-s}$, $\widehat{\mathbb{N}} = 0$ so that $\mathbb{M}$ and $\mathbb{N}$ are q.r.c. but not p.r.c. We do not know whether $\mathbb{M}$ and $\mathbb{N}$ can be q.r.c. without being p.r.c. if we require that $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

Proof: (a) This holds by Theorem 4.1.6(b).

(b) Let $\mathbb{M}$ be the map $\mathbb{D}$ of Lemma 4.1.10, and set $\mathbb{N} = 0$. Then $\widehat{\mathbb{M}}^*\widehat{\mathbb{M}} + 0^*0 \geq I$ on $\mathbf{C}^+$, hence $\mathbb{M}$ and $\mathbb{N}$ are p.r.c., but $\left[\begin{smallmatrix}\mathbb{M}\\ \mathbb{N}\end{smallmatrix}\right]$ is not left-invertible on TIC, i.e., $\mathbb{M}$ and $\mathbb{N}$ are not r.c.

(c) (In fact, $\widehat{\mathbb{M}}, \widehat{\mathbb{N}}$ need not be rational, it suffices that $\dim U < \infty$, $\widehat{\mathbb{M}}, \widehat{\mathbb{N}} \in \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\}; \mathcal{B}(U,*)) \cap \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,*))$ and $M \in \mathcal{G}\mathcal{B}(U)$.)

Set $\varepsilon := \min_{\|u_0\|_U = 1,\ s \in \overline{\mathbf{C}^+} \cup \{\infty\}} f(s, u_0)$ (this exists, since $\overline{\mathbf{C}^+} \cup \{\infty\}$ is compact), where $f(s,u_0) := \|\widehat{\mathbb{M}}(s)u_0\|^2 + \|\widehat{\mathbb{N}}(s)u_0\|^2$. If $\varepsilon > 0$, then $\mathbb{M}$ and $\mathbb{N}$ are p.r.c., hence r.c. (by (a)). If $\mathbb{M}$ and $\mathbb{N}$ are r.c., then they are q.r.c. It remains to show that $\varepsilon = 0$ implies that $\mathbb{N}$ and $\mathbb{M}$ are not q.r.c.

Assume that $\varepsilon = 0$. Then there are $\{s_n\}$, $\{u_n\}$ as above s.t. $f(s_n, u_n) \to 0$ as $n \to +\infty$. Replaces the above sequences by subsequences so that $s_n \to s$ and $u_n \to u$ for some s and u (since $M \in \mathcal{G}\mathcal{B}(U)$ we have $s \neq \infty$). Use the uniform continuity of f on $\overline{\mathbf{C}^+} \cup \{\infty\} \times \{\|v\|_U = 1\}$ to obtain that $f(s_n, u) \to 0$, hence $f(s,u) = 0$, hence $\mathbb{M}$ and $\mathbb{N}$ are not q.r.c., by Lemma 6.5.1(e).

(d) This follows from (c) (since $M := \widehat{\mathbb{M}}(+\infty) \in \mathcal{G}\mathcal{B}(U)$, by Proposition 6.3.1(c)). □

By Lemma 6.4.3, a scalar right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is coprime iff $\mathbb{N}$ does not cancel any poles of $\mathbb{M}^{-1}$, i.e., iff $\mathbb{M}^{-1}$ and $\mathbb{D}$ have the same poles. Claim (c) above is an extension of this, below we give similar claims in the general case:

Lemma 6.5.4 (R.c. maps do not have common zeros) *Let $\mathbb{M} \in \mathrm{TIC}(U)$ and $\mathbb{N} \in \mathrm{TIC}(U,Y)$ be p.r.c. Let $\Omega := \{s \in \mathbf{C}^+ \,|\, \widehat{\mathbb{M}}(s) \in \mathcal{G}\mathcal{B}(U)\} \neq \emptyset$ (this set is open, by Lemma A.3.3(A2)). Let $\Omega' \subset \mathbf{C}^+$ be open and connected, let $\Omega_1 \subset \Omega$ be bounded, and let $\Omega_2 \subset \Omega$ satisfy $0 \notin \overline{\Omega_2}$. Then the following hold:*

(a) $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega; \mathcal{B}(U))$.

(b) Let s_0 be a boundary point of Ω in $\mathbf{C}^+$. Then $\|\big(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}\big)(s)\| \to \infty$, as $s \to s_0$ and $s \in \Omega$.

(c1) Let $s_0 \in \overline{\Omega}$ and $N \in \mathbf{N}$. Then $(s - s_0)^N \big(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}\big)(s)$ is bounded on Ω_1 iff $(s - s_0)^N \widehat{\mathbb{M}}^{-1}(s)$ is bounded on Ω_1.

(c2) Let $N \in \mathbf{N}$. Then $s^{-N}\big(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}\big)(s)$ is bounded on Ω_2 iff $s^{-N}\widehat{\mathbb{M}}^{-1}(s)$ is bounded on Ω_2.

(c3) Let $\Omega_3 \subset \Omega$. Then $\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is bounded on Ω_3 iff $\widehat{\mathbb{M}}^{-1}$ is bounded on Ω_3.

(d1) If $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega'; \mathcal{B}(U,Y))$ (resp. $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U,Y))$) and $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on some open, nonempty subset of $\Omega \cap \Omega'$, then $\Omega' \subset \Omega$, $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega'; \mathcal{B}(U))$ (resp. $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U))$) and $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on Ω'.

(d2) If some $\mathbb{D} \in \mathrm{TIC}_\omega$ *satisfies* $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ *on some open subset of* $\mathbf{C}^+_\omega$*, then* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega$ *and* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

In the finite-dimensional setting, a r.c.f. is often defined so that $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is required only on the set where $\widehat{\mathbb{M}}^{-1}$ exists. By (d2), this is equivalent to our definition whenever $\mathbb{D}$ is well-posed (i.e., in TIC_∞).

If a map $\mathbb{D} \in \mathrm{TIC}_\infty$ can be written as $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\mathbb{N}$ and $\mathbb{M}$ are p.r.c. and $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, then the poles of $\mathbb{M}^{-1}$ on $\overline{\mathbf{C}^+} \cup \{\infty\}$ are exactly those of $\mathbb{D}$, and these poles have same multiplicities, by (c1) and (c2) (the latter treats the point ∞).

In a sense, also the converse holds (if $\mathbb{N}$ and $\mathbb{M}$ are not p.r.c., then there are $\{s_n\} \subset \mathbf{C}^+$, $s_\infty \in \overline{\mathbf{C}^+} \cup \{\infty\}$, $\{u_n\} \subset U$ s.t. $\|u_n\| = 1$ $(n \in \mathbf{N})$, $s_n \to s_\infty$, and $\widehat{\mathbb{N}}(s_n)u_n, \widehat{\mathbb{M}}(s_n)u_n \to 0$, so that $\widehat{\mathbb{N}}$ has a zero and $\widehat{\mathbb{M}}^{-1}$ has a nonremovable singularity at s_∞.

Note that, by Lemma 3.3.9, for a map $\mathbb{M} \in \mathrm{TIC}(U)$ with U infinite-dimensional, the set of singularities ("poles") of $\widehat{\mathbb{M}}^{-1}$ can be any closed subset of $\mathbf{C}^+$ (excluding $\mathbf{C}^+_\omega$ if we wish that $\mathbb{M} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TIC}_\omega$; take, e.g., $\mathbb{N} = I$ to obtain a r.c.f.).

Proof: (a) This follows from Lemma D.1.2(b2).

(b) This follows from Lemma A.3.3(A3)) and the proof of (c1) with $N = 0$.

(c1) Let $\widehat{\widehat{\mathbb{X}}}\widehat{\mathbb{M}} - \widehat{\widehat{\mathbb{Y}}}\widehat{\mathbb{N}} = I$ on $\mathbf{C}^+$ and $M := \max(\|\widehat{\widehat{\mathbb{X}}}\|, \|\widehat{\widehat{\mathbb{Y}}}\|, d(\Omega_1)) < \infty$, as in Lemma 6.5.2(ii). Assume that $(s - s_0)^N\widehat{\mathbb{M}}^{-1}(s)$ is unbounded on Ω_1, i.e., that there is $\{s_n\} \subset \Omega$ s.t. $\|\widehat{\mathbb{M}}(s_n)^{-1}\| > n$ $(n \in \mathbf{N})$. Choose $\{u_n\} \subset U$ s.t. $\|v_n\| > n$ and $\|u_n\| < 1$ for $n \in \mathbf{N}$, where $v_n := (s_n - s_0)^N\widehat{\mathbb{M}}(s_n)^{-1}u_n$. Note that $\|\widehat{\mathbb{M}}(s_n)v_n\| < |s_n - s_0|^N \leq M^N$. Now

$$M\|(s_n - s_0)^N\widehat{\mathbb{N}}(s_n)\widehat{\mathbb{M}}(s_n)^{-1}u_n\| = M\|\widehat{\mathbb{N}}(s_n)v_n\| \geq \|\widehat{\widehat{\mathbb{Y}}}(s_n)\widehat{\mathbb{N}}(s_n)v_n\| \qquad (6.112)$$

$$\geq \|Iv_n\| - \|\widehat{\widehat{\mathbb{X}}}(s_n)\widehat{\mathbb{M}}(s_n)v_n\| > n - M|s_n - s_0|^N \to \infty, \qquad (6.113)$$

as $n \to \infty$, as desired.

(c2) Replace $(s - s_0)$ by $1/s$ in the proof of (c1) (and let $\infty > M \geq \sup_{s\in\Omega_2} |1/s|$).

(c3) This follows from the proof of (c1) with $N = 0$, $|s_n - s_0|^N = 1$.

(d1) By Lemma D.1.2(e), we have $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on $\Omega \cap \Omega'$, If there were some $s_0 \in \Omega' \cap \partial\Omega$, then we would have $\|\widehat{\mathbb{D}}(s)\| \to \infty$ as $s \to s_0$ in $\Omega' \cap \Omega$, by (a), hence $\Omega' = \Omega \cup (\Omega' \cap \overline{\Omega}^c)$. Because Ω' is connected, this implies that $\Omega' \cap \overline{\Omega}^c = \emptyset$, hence $\Omega' \subset \Omega$; consequently, $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega'; \mathcal{B}(U))$. If $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is bounded on Ω', then so is $\widehat{\mathbb{M}}^{-1}$, by (c3).

(d2) If some $\mathbb{D} \in \mathrm{TIC}_\omega$ satisfies $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on some open subset of $\mathbf{C}^+_\omega$, then $\widehat{\mathbb{M}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U))$, by (d1), hence $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega$ and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$. Conversely, if $\mathbb{M}^{-1} \in \mathrm{TIC}_\omega$, then $\mathbb{N}\mathbb{M}^{-1} \in \mathrm{TIC}_\omega$. □

We take now a look at the approach of Frank Callier, Charles Desoer and others to obtain a d.c.f. Assume that $\mathbb{N} \in \mathrm{TIC}_{-\varepsilon}(\mathbf{C}^m, Y)$ and $\mathbb{M} \in \mathrm{TIC}_{-\varepsilon}(\mathbf{C}^m) \cap \mathcal{G}\mathrm{TIC}_\infty$ for some $m \in \mathbf{N} + 1$, $\varepsilon > 0$. Then $\widehat{\mathbb{D}} := \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is meromorphic on $\mathbf{C}^+_{-\varepsilon}$.

By Lemma D.1.2(e), the set $Z := \{s \in \mathbf{C}^+_{-\varepsilon} \mid \det\widehat{\mathbb{M}}(s) = 0\}$ has no limit points on $\mathbf{C}^+_{-\varepsilon}$. Assume now that $|\det\widehat{\mathbb{M}}| \ge \varepsilon$ when $s \in \mathbf{C}^+$ and $|s| > R$, for some $R, \varepsilon > 0$ (e.g., $\mathbb{M} \in$ UHPR). Then ∞ is not a limit point of $Z \cap \overline{\mathbf{C}^+}$ (see also Lemma 6.3.6(a2)), hence then $Z \cap \overline{\mathbf{C}^+}$ is finite.

It has been shown for several subclasses of such transfer functions that from these zeros one can construct a rational (exponentially stable) denominator $\widehat{\mathbb{M}}_0 \in \mathrm{H}^\infty(\mathbf{C}^+; \mathbf{C}^n)$ s.t. $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ is a r.c.f., where $\mathbb{N}_0 := \mathbb{D}\mathbb{M}_0 \in \mathrm{TIC}_{\exp}$, and that $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2$, where $\mathbb{D}_1 \in \mathrm{TIC}_{\exp}$ and $\widehat{\mathbb{D}}_2 \in \mathrm{H}^\infty_\infty$ is rational. See, e.g., [CD78], [CD80], [Logemann93] or [Logemann87] for further details and information.

The following is a generalization of a standard result:

Lemma 6.5.5 (Inner–outer is constant)

(a) If $J, S \in \mathcal{B}$, S is one-to-one, $\mathbb{N}^* J \mathbb{N} = S$ and $\mathbb{N} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{N} \in \mathcal{G}\mathcal{B}$.

(b) Let $J = J^* \in \mathcal{B}$ and $S = S^* \in \mathcal{G}\mathcal{B}$. If $\mathbb{N} \in$ is (J,S)-inner and $\mathbb{N}$ is outer *(i.e., $\mathbb{N}\pi_+\mathrm{L}^2$ is dense in $\pi_+\mathrm{L}^2$), then $\mathbb{N} \in \mathcal{G}\mathcal{B}$.*

Proof: (a) Now $\mathrm{TIC} \ni S\mathbb{N}^{-1} = \mathbb{N}^* J \in \mathrm{TIC}^*$, hence $L := S\mathbb{N}^{-1} \in \mathcal{B}$, by Lemma 2.1.7, Because $S\widehat{\mathbb{N}}^{-1} \equiv L$ on $\mathbf{C}^+$ (see Theorem 6.2.1), $\widehat{\mathbb{N}}^{-1}$ is a constant$\in \mathcal{B}$, hence so is $\mathbb{N}$.

(b) Because $\pi_+(S^{-1}\mathbb{N}^* J)\pi_+\mathbb{N}\pi_+ = \pi_+$ (recall that $\pi_+\mathbb{N}\pi_+ = \mathbb{N}\pi_+$), the range $\mathbb{N}\pi_+\mathrm{L}^2$ is closed in $\pi_+\mathrm{L}^2$, by Lemma A.3.1(v)&(iv), hence $\pi_+\mathbb{N}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$. Being coercive and onto on $\pi_+\mathrm{L}^2$, $\mathbb{N}$ is invertible on $\pi_+\mathrm{L}^2$, by Lemma A.3.1(c3)(ii)&(i), hence $\mathbb{N} \in \mathcal{G}\mathrm{TIC}$, by Lemma 2.2.3. Consequently, $\mathbb{N} \in \mathcal{G}\mathcal{B}$, by (a). $\square$

As noted above, "$\mathbb{D}$ is as stable as $\mathbb{M}^{-1}$" if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. This implies several useful facts:

Lemma 6.5.6 ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\alpha \ge \omega \ge 0$. Then the following hold:*

(a1) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a p.r.c.f. and $u \in \mathrm{L}^2_\alpha$. Then $u, \mathbb{D}u \in \mathrm{L}^2_\omega \Leftrightarrow \mathbb{M}^{-1}u \in \mathrm{L}^2_\omega$.

(a2) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a p.r.c.f. Then there is $\varepsilon > 0$ s.t.

$$\varepsilon\|\mathbb{M}^{-1}u\|_{\mathrm{L}^2_\omega} \le \|u\|_{\mathrm{L}^2_\omega} + \|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le \varepsilon^{-1}\|\mathbb{M}^{-1}u\|_{\mathrm{L}^2_\omega} \quad (u \in \mathrm{L}^2_\alpha(\mathbf{R};U)). \tag{6.114}$$

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a [p.]r.c.f. and $\mathbb{T} \in \mathrm{TIC}(H,U)$. Then $\mathbb{D} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{M}^{-1} \in \mathrm{TIC}_\omega$, and $\mathbb{D}\mathbb{T} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{M}^{-1}\mathbb{T} \in \mathrm{TIC}_\omega$.

(c) $\mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. of $\mathbb{D}$ iff $\mathbb{M}^{-\mathrm{d}}\mathbb{N}^{\mathrm{d}}$ is a [p.]l.c.f. of $\mathbb{D}^{\mathrm{d}}$.

(d) If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f., $U = U_1 \times U_2$, and $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(U_2)$, then $\mathbb{D}$ has a [p.]r.c.f. of the form

$$\mathbb{D} = \mathbb{N}' \begin{bmatrix} \mathbb{M}'_{11} & \mathbb{M}'_{12} \\ 0 & I \end{bmatrix}^{-1}. \tag{6.115}$$

If $\mathcal{A} \subset \mathrm{TIC}$ and $\mathbb{N}, \mathbb{M}, \mathbb{M}_{22}^{-1} \in \mathcal{A}$, then we can take $\mathbb{N}', \mathbb{M}' \in \mathcal{A}$; if $\mathbb{M}_{22}^{-1} \in \mathcal{A}$ and $\mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. over $\mathcal{A}$, then also $\mathbb{N}'(\mathbb{M}')^{-1}$ is a [p.]r.c.f. over $\mathcal{A}$,.

(e) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *be a [p.]r.c.f. Then* $\mathbb{D}$ *has a (normalized) [p.]r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ *s.t.* $\mathbb{N}'^*\mathbb{N}' + \mathbb{M}'^*\mathbb{M}' = I$.

(f) For $\omega = 0$*, we can replace "p." by "q." in (a1)–(e).*

Proof: (a1)&(a2) Apply Lemma 6.5.2(b1)&(b3) to $u_{\circlearrowleft} := \mathbb{M}^{-1}u$.

(b) Apply Lemma 6.5.2(c) to $\mathbb{M}^{-1}\mathbb{T} \in \mathrm{TIC}_\infty$.

(c) This is trivial (recall that $\mathbb{D}^{\mathrm{d}} := \mathbf{Я}\mathbb{D}^*\mathbf{Я} \in \mathrm{TIC}_\infty$).

(d) Now $\mathbb{U} := \begin{bmatrix} I & 0 \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U_1 \times U_2)$, by Lemma A.1.1(b1), hence $\mathbb{D} = (\mathbb{N}\mathbb{U})(\mathbb{M}\mathbb{U})^{-1}$ is obviously an [p.]r.c.f. too. But $\mathbb{M}\mathbb{U} = \begin{bmatrix} \mathbb{M}_{11} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ 0 & I \end{bmatrix}$, as desired.

The $\mathcal{A}$ case is obvious (if $\mathbb{U}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$, then so are $\mathbb{U}^{-1}\widetilde{\mathbb{X}}$ and $\mathbb{U}^{-1}\widetilde{\mathbb{Y}}$).

(e) By Lemma 6.5.2(e) and Lemma 6.4.7(a), we have $\mathbb{X}^*\mathbb{X} = \mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Set $\mathbb{N}' := \mathbb{N}\mathbb{X}^{-1}$, $\mathbb{M}' := \mathbb{M}\mathbb{X}^{-1}$.

(f) Use Lemma 6.5.1 instead of Lemma 6.5.2 in the proofs of (a1)–(e). □

See Theorem 4.1.6(d) on the connection between right and left coprime factorizations.

A coprime factorization of a perturbed map is obtained as follows:

Lemma 6.5.7 (D.c.f.+TIC) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *and* $\widetilde{\mathbb{D}} \in \mathrm{TIC}(U,Y)$*. Then the following hold:*

(a) If $\mathbb{D}$ *has the d.c.f. (6.109), then*

$$\begin{bmatrix} I & 0 \\ \widetilde{\mathbb{D}} & I \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = \left(\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\widetilde{\mathbb{D}} & I \end{bmatrix} \right)^{-1} \tag{6.116}$$

is a d.c.f. of $\mathbb{D} + \widetilde{\mathbb{D}}$.

(b) Similarly, $\widetilde{\mathbb{D}} + \mathbb{N}\mathbb{M}^{-1} = (\mathbb{N} + \widetilde{\mathbb{D}}\mathbb{M})\mathbb{M}^{-1}$ *is a [q.]r.c.f., whenever* $\mathbb{N}\mathbb{M}^{-1}$ *is.*

(c) $\begin{bmatrix} \mathbb{D} \\ I \end{bmatrix} = \widetilde{\mathbb{N}}\mathbb{M}^{-1}$ *is a [q.]r.c.f. iff* $\widetilde{\mathbb{N}} = \begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix}$ *for a [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

Thus, when factorizing of a regular transfer function $\widehat{\mathbb{D}}$, we may assume that $D := \widehat{\mathbb{D}}(+\infty) = 0$.

Proof: (a) This is obvious.

(b) This follows from Lemma 6.5.1(e).

(c) Let $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. We have $\begin{bmatrix} \mathbb{D} \\ I \end{bmatrix} = \widetilde{\mathbb{N}}\mathbb{M}^{-1}$ iff $\widetilde{\mathbb{N}} = \begin{bmatrix} \mathbb{D} \\ I \end{bmatrix}\mathbb{M} =: \begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix}$. Obviously, $\begin{bmatrix} \mathbb{N} \\ \mathbb{M} \end{bmatrix}$ and $\mathbb{M}$ are [q.]r.c. iff $\mathbb{N}$ and $\mathbb{M}$ are [q.]r.c. □

A map has a d.c.f. iff it has a r.c.f. and a l.c.f.:

Lemma 6.5.8 (D.c.f.⇔r.c.f. & l.c.f.) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *have the r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *[over* $\mathcal{A}$*] and the l.c.f.* $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ *[over* $\mathcal{A}$*]. Then these factorizations can be extended to a d.c.f.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}, \tag{6.117}$$

[over $\mathcal{A}$] of $\mathbb{D}$. Moreover, if $\mathbb{X},\mathbb{Y} \in \mathrm{TIC}$ satisfying (6.108) are given, there are $\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathrm{TIC}$ satisfying (6.117) (the dual claim with left and right interchanged holds as well).

See Theorem 6.6.28 for the equivalence of joint stability and detectability and the existence of a d.c.f.

By Lemma 6.5.6(e) (and its dual), we can above require the normalizations $\mathbb{N}^*\mathbb{N}+\mathbb{M}^*\mathbb{M} = I$ and $\widetilde{\mathbb{M}}\widetilde{\mathbb{M}}^* + \widetilde{\mathbb{N}}\widetilde{\mathbb{N}}^* = I$.

Proof of Lemma 6.5.8 Lemma A.1.1(e1) shows that if $\mathbb{X}_0,\mathbb{Y}_0,\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathrm{TIC}$ satisfying (6.108) and (6.107), respectively, are given, then $\mathbb{X} := \mathbb{X}_0 + \mathbb{N}(\widetilde{\mathbb{Y}}\mathbb{X}_0 - \widetilde{\mathbb{X}}\mathbb{Y}_0)$ and $\mathbb{Y} := \mathbb{Y}_0 + \mathbb{M}(\widetilde{\mathbb{Y}}\mathbb{X}_0 - \widetilde{\mathbb{X}}\mathbb{Y}_0)$ satisfy the second equality in (6.117) [note that also $\mathbb{X},\mathbb{Y} \in \mathcal{A}$]. The equation $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{smallmatrix}\right]\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right] = I$ holds in TIC_∞, by Lemma A.1.1(e5), hence in TIC (by analytic extension on $\mathbf{C}^+$). By taking (causal) adjoints, one gets the dual claim. □

(The invertibility of $\mathbb{M}$ was used in the above proof; the corresponding assumption in Lemma A.1.1(e5) is not superfluous.)

Given a d.c.f. of a TIC_∞ map, all d.c.f.'s of that map are obtained from (d) below:

Lemma 6.5.9 (All d.c.f.'s) *Let*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix} = I = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} \tag{6.118}$$

in $\mathrm{TIC}(U\times Y)$. *Then we have the following:*

(a1) $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{M}} \in \mathcal{G}\mathrm{TIC}_\infty$.

(a2) If $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*, then (6.118) is a d.c.f. of* $\mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$.

(b) All possible choices $\mathbb{X},\mathbb{Y},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathrm{TIC}$ *in (6.118) (for fixed* $\mathbb{M},\mathbb{N},\widetilde{\mathbb{M}},\widetilde{\mathbb{N}}$*) are parametrized by*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}+\mathbb{M}\mathbb{U}\\ \mathbb{N} & \mathbb{X}+\mathbb{N}\mathbb{U}\end{bmatrix}^{-1} = \begin{bmatrix}\widetilde{\mathbb{X}}+\mathbb{U}\widetilde{\mathbb{N}} & -(\widetilde{\mathbb{Y}}+\mathbb{U}\widetilde{\mathbb{M}})\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix} \quad (\mathbb{U} \in \mathrm{TIC}). \tag{6.119}$$

(c) All completions of $\left[\begin{smallmatrix}\mathbb{M}\\ \mathbb{N}\end{smallmatrix}\right]$ *to an invertible operator in* $\mathrm{TIC}(U\times Y)$ *are parametrized by*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}I & \mathbb{U}\\ 0 & \mathbb{V}\end{bmatrix} = \begin{bmatrix}\mathbb{M} & \mathbb{Y}\mathbb{V}+\mathbb{M}\mathbb{U}\\ \mathbb{N} & \mathbb{X}\mathbb{V}+\mathbb{N}\mathbb{U}\end{bmatrix} \quad (\mathbb{U} \in \mathrm{TIC},\ \mathbb{V} \in \mathcal{G}\mathrm{TIC}). \tag{6.120}$$

(d) If (6.118) is a d.c.f. (i.e., if $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*), then all d.c.f.'s of* $\mathbb{N}\mathbb{M}^{-1}$ *are given by*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}\mathbb{W} & \mathbb{U}\\ 0 & \mathbb{V}\end{bmatrix} = \left(\begin{bmatrix}\mathbb{W} & \mathbb{U}\\ 0 & \mathbb{V}\end{bmatrix}^{-1}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\right)^{-1} \quad (\mathbb{U} \in \mathrm{TIC},\ \mathbb{V},\mathbb{W} \in \mathcal{G}\mathrm{TIC}). \tag{6.121}$$

(e) Let (6.118) be a d.c.f., and let $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$, $\mathbb{X} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$, $\widetilde{\mathbb{X}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. *Then all d.c.f.'s of* $\mathbb{N}\mathbb{M}^{-1}$ *with such* $\mathbb{M}, \widetilde{\mathbb{M}}, \mathbb{X}, \widetilde{\mathbb{X}}$ *are given by (6.121) with extra requirements* $\mathbb{W} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\mathbb{U} = \left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$, $\mathbb{V} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$.

Proof: (a) Part (a1) follows from Lemma A.1.1(c1) and (a2) is trivial.

(b) This follows from (c), because fixing $\widetilde{\mathbb{M}}$ and $\widetilde{\mathbb{N}}$ forces $\mathbb{V}$ to be I.

(c) Clearly $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ whenever $\mathbb{U} \in \mathrm{TIC}$ and $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$. For the converse, assume that $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}(U \times Y)$. Set

$$\begin{bmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}.$$

We must have $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$, hence we get (6.120) by multiplying the above equation by $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right]$ to the left.

(d) Clearly $\left[\begin{smallmatrix} \mathbb{M}\mathbb{W} & \mathbb{Y} \\ \mathbb{N}\mathbb{W} & \mathbb{X} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{W} & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$. By (b), all d.c.f.'s corresponding to the pair $(\mathbb{N}\mathbb{W}, \mathbb{M}\mathbb{W})$ are given by $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{W} & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{smallmatrix}\right]$, and these are of the form (6.121) (with $\mathbb{W}\mathbb{U}$ in place of $\mathbb{U}$). On the other hand, all r.c.f.'s of $\mathbb{N}\mathbb{M}^{-1}$ are of the form $(\mathbb{N}\mathbb{W}, \mathbb{M}\mathbb{W})$ for some $\mathbb{W} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5.

(e) This follows by writing (6.121) out (the equation $\mathbb{U}_{11} = 0$ is obtained from the right-hand-side and the equation $\mathbb{W}\left[\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right]\mathbb{V} = \left[\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right]$, or, alternatively, from the fact that $\mathbb{M}_{11}$ is injective). □

An operator not having a d.c.f. is somewhat pathological, as shown below:

Lemma 6.5.10 *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U, Y)$. *The existence of a d.c.f. is guaranteed in the following cases:*

(a) If $\mathbb{D}$ *is stable, then it has the d.c.f.* $\left[\begin{smallmatrix} I & 0 \\ \mathbb{D} & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & 0 \\ -\mathbb{D} & I \end{smallmatrix}\right]$.

(b1) If $\widehat{\mathbb{D}}$ *is rational, i.e.,* $\mathbb{D}$ *has a finite-dimensional realization, then* $\mathbb{D}$ *has a d.c.f.*

(c) If $\widehat{\mathbb{D}}$ *belongs to the Callier–Desoer class "*$\mathbb{B}(0)^{n \times m}$*", then* $\mathbb{D}$ *has a d.c.f. with* $\widehat{\mathbb{M}}$ *and* $\widehat{\widetilde{\mathbb{M}}}$ *rational.*

(d) If $\mathbb{D}$ *has a jointly stabilizable and detectable realization, then it has a d.c.f.*

(e1) Assume that $\mathbb{D}$ *is DF-stabilizable (stabilizable by dynamic (output) feedback; see Section 7.1). If* $\dim U, \dim Y < \infty$ *or some DF-stabilizing controller of* $\mathbb{D}$ *has a d.c.f., then* $\mathbb{D}$ *has a d.c.f.*

(e2) If $\mathbb{D}$ *DF-stabilizable by a stable controller, then* $\mathbb{D}$ *has a d.c.f. In particular, this is the case if* $\mathbb{D}$ *is stabilizable by static output feedback.*

(e3) If $\mathbb{D}$ *has an exponentially DF-stabilizable realization with bounded input and output operators (or as a WPLS of the form of Lemma 6.8.5), then* $\mathbb{D}$ *has an exponential d.c.f. in* $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$.

By (e1), the maps not having a d.c.f. are not interesting from the point of view of DF-stabilization and optimization, at least not in the case with finite-dimensional input and output spaces.

In cases (b) and (c), the d.c.f. can, in fact, be chosen to be exponentially coprime (and with rational $\mathbb{M}$ and $\widetilde{\mathbb{M}}$).

According to [Curtain02], a d.c.f. also exists if Σ and Σ^d are SOS-stabilizable, or Σ and Σ^d are state-and-output-stabilizable, provided that $\dim U, \dim Y < \infty$ (or that we accept pseudo-coprimeness) and that $\sigma(A)$ satisfies certain assumptions.

Proof: (a) This is clear.

(b) By [Vid, p. 387], the set $\mathbf{C}[s]$ of (complex) rational functions over the field $\mathbf{C}$ (or any other field) is a proper Euclidean domain, hence a Bezout domain [Vid, Fact A.4.6], so every $\widehat{\mathbb{D}} \in \mathbf{C}[s]^{n\times m}$ has a r.c.f. and a l.c.f., by [Vid, Corollary 8.1.8]. (Take a minimal realization (see, e.g., Section 6.4 of [LR]) and use Theorem 6.6.28 to obtain an alternative proof.)

(c) This is contained in Theorem 2.1 of [CD80]. The class "$\mathbb{B}(0)^{n\times m}$" refers to $\mathbf{C}^{n\times m}$-valued (matrix) functions with elements of form $\widehat{f}/\widehat{g}$ s.t. $f, g \in \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}(\mathbf{C})$ and $g \in \mathcal{G}\mathrm{TIC}_\infty(\mathbf{C})$.

(d) This is proved in Theorem 6.6.28.

(e1) The first claim is Lemma 7.1.4 and the second one is Proposition 7.1.6(d) and contains (e2) as a special case, by (a).

(e3) This follows from Theorem 7.2.4(a) and Corollary 9.2.13(c). □

The factorization results over TIC (as well as most results of Chapter 7 among others) could as well have been stated over MTIC or over any other structures where $\mathrm{TIC}(U)$ is replaced with a ring with identity etc. (cf. Lemma A.1.1) we state this notion in a form that will be applied in Section 7.1:

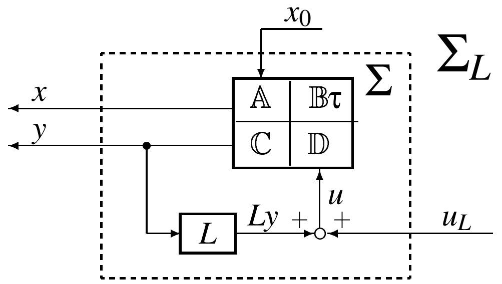
Remark 6.5.11 *Let $\mathcal{X}$, $\mathcal{A}'$ and $\mathcal{A} \subset \mathcal{A}'$ be as in Definition 6.2.4. Define r.c.f.'s, l.c.f.'s and d.c.f.'s as above, with $\mathcal{A}$ in place of* TIC *and $\mathcal{A}'$ in place of* TIC_∞ *(i.e., consider $\mathcal{A}'$ as the class of all admissible I/O maps and $\mathcal{A}$ as the class of "stable" I/O maps).*

Then Lemma 6.4.5(b)–(d), Lemmas 6.5.9 and 6.5.7, and most of Lemmas 6.5.8 and 6.5.6 (and almost all I/O results of Section 6.6 and most of Chapter 7; cf. Remark 7.0.1) hold with $\mathcal{A}$ in place of TIC*, $\mathcal{A}'$ in place of* TIC_∞ *and "q." and "p." removed.*

This is particularly useful when we let $\mathcal{A}$ be MTIC *or some of its subclasses, and take $\mathcal{X} :=$ {all Hilbert spaces} and $\mathcal{A}' := \mathrm{TIC}_\infty$, or when $\mathcal{A} = \mathrm{tic}$ and $\mathcal{A}' = \mathrm{tic}_\infty$.*

Notes

Lemma 6.5.8 is essentially from [S98a]. Lemmas 6.5.5, 6.5.7 and 6.5.9 are known at least to some extent. Probably also many of the other results are known at least in the case where "pseudo-" or "quasi-" is dropped from the assumptions and $\dim U, \dim Y < \infty$. See also the references in the text and the notes to Section 6.4.



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$
$$u = Ly + u_L$$

Figure 6.2: Static output feedback

6.6 Feedback and Stabilization (Σ_L, $\Sigma_\flat$, $\Sigma_\sharp$)

The Universe is populated by stable things.

— Richard Dawkins

One often wants to stabilize and possibly also regulate or optimize a system by feeding its state or output back into the system through some kind of controller, as in, e.g., Figure 6.2. All control problems of Chapters 8–12 are of this form.

In this section, we shall introduce static output feedback (Figure 6.2), state feedback (Figure 6.3) and output injection (Figure 6.5), hence also the concepts "stabilizability" and "detectability" in their various forms.

In finite-dimensional control theory, optimization is usually restricted by the requirement that the (controlled) closed-loop system should be exponentially stable. However, sometimes strong stability or some other form of stability has been allowed, and for infinite-dimensional systems this has become increasingly popular during recent years. Therefore, we have decided to study all forms of feedback w.r.t. all forms of stability in this section.

Most of our definitions and some of the results follow [S97b], [S98a] and [Sbook]; in particular, we reduce all forms of feedback to static output feedback. We note that by shifting (see Remark 6.1.9) any result on stabilization, one obtains a result on exponential stabilization, but the converse is not true, and results on (nonexponential) stabilization are often weaker or harder to prove.

In Section 6.7, we shall present further results and related concepts. Different forms of dynamic [partial] output feedback are the subject of Chapter 7. We have collected the definition of all forms of feedback to Summary 6.7.1.

Now we shall introduce static output feedback, where we feed a part Ly of the output y of Σ back into the input, as Figure 6.2 shows; here $L \in \mathcal{B}(Y,U)$ and $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. Under an external output $u_L : \mathbf{R}_+ \to U$ (external input, control, disturbance or analogous), the effective input u becomes equal to $Ly + u_L$. In the initial value setting (6.3) with initial value $x_0 \in H$, the signals in the resulting dashed *closed-loop system* Σ_L of Figure 6.2 clearly satisfy the following equations:

$$x(t) = \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \qquad (t \geq 0), \tag{6.122}$$

$$y = \mathbb{C}x_0 + \mathbb{D}u, \tag{6.123}$$

$$u = Ly + u_L. \tag{6.124}$$

Obviously, this can be uniquely solved in terms of x_0 and u_L iff $(I - L\mathbb{D})$ is invertible (the corresponding solution is given by (6.125)). We call such an L admissible:

Definition 6.6.1 (Admissible static output feedback) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. *An operator* $L \in \mathcal{B}(Y,U)$ *is called an* admissible (static) output feedback operator *for* Σ *(or for* $\mathbb{D}$*) if* $I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*, or equivalently, if* $I - \mathbb{D}L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$.

This is the case iff $I - \mathbb{D}L$ is locally invertible, by Lemma 2.2.8. For $\mathbb{D} \in \mathrm{ULR}$, this is equivalent to $I - DL \in \mathcal{GB}(Y)$, by Proposition 6.3.1(c). A more thorough motivation of admissibility is given in [W94b, Proposition 3.6].

As shown in [W94b, Section 6], the signals x and y in (6.122)–(6.124) can be interpreted as the state and output of another well-posed linear system:

Proposition 6.6.2 (Σ_L) *Let* $L \in \mathcal{B}(Y,U)$ *be an admissible output feedback operator for* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

Then $\Sigma_L \in \mathrm{WPLS}(U,H,Y)$*, where*

$$\Sigma_L := \left[\begin{array}{c|c} \mathbb{A}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau L(I - \mathbb{D}L)^{-1}\mathbb{C} & \mathbb{B}(I - L\mathbb{D})^{-1} \\ \hline (I - \mathbb{D}L)^{-1}\mathbb{C} & \mathbb{D}(I - L\mathbb{D})^{-1} \end{array}\right] \tag{6.125}$$

$$= \Sigma \begin{bmatrix} I & 0 \\ -L\mathbb{C} & I - L\mathbb{D} \end{bmatrix}^{-1} = \Sigma \begin{bmatrix} I & 0 \\ (I - L\mathbb{D})^{-1}L\mathbb{C} & (I - L\mathbb{D})^{-1} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_L \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6.126}$$

We call Σ_L *the* closed-loop system *with output feedback operator L. In the initial value setting (6.3) with initial value* x_0 *and control* u_L*, the controlled state* $x(t)$ *at time t and the output y of* Σ_L *form the unique solution of equations (6.122)–(6.124).* □

Note that (6.126) follows easily from

$$\begin{bmatrix} x_0 \\ u_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ -L\mathbb{C} & I - L\mathbb{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I - L\mathbb{D})^{-1}L\mathbb{C} & (I - L\mathbb{D})^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ u_L \end{bmatrix} \tag{6.127}$$

(here $u, u_L : \mathbf{R}_+ \to U$ and Σ refers to $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$; cf. (6.123)–(6.124)).

Repeated feedback behaves in the expected way:

Lemma 6.6.3 ($\boldsymbol{\Sigma_{L+K} = (\Sigma_L)_K}$) *Let L be admissible for* Σ *as above. Then* $K \in \mathcal{B}(Y,U)$ *is admissible for* Σ_L *iff* $K + L$ *is admissible for* Σ*; and in that case* $\Sigma_{L+K} = (\Sigma_L)_K$ *(in particular,* $(\Sigma_L)_{-L} = \Sigma$*).* □

(See [W94b, Remark 6.5] for the proof.)

Definition 6.6.4 (Stabilizing *L*) *Let $L \in \mathcal{B}(Y,U)$ be an admissible output feedback operator for $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. The operator L is* stabilizing *for Σ if the corresponding closed loop system Σ_L is stable.*

The same applies to all stability concepts (prefices for "stabilizing") defined in Definition 6.1.3. E.g., L is strongly internally ω-stabilizing if Σ_L is strongly internally ω-stable, i.e., if $\mathbb{A}_L$ is strongly ω-stable.

Analogously, we call L $\mathbb{B}$-stabilizing *if $\mathbb{B}_L$ is stable; the same applies to other components of Σ_L.*

We call L [exponentially] stabilizing for $\mathbb{D}$ *if L is admissible and makes $\mathbb{D}_L = \mathbb{D}(I - L\mathbb{D})^{-1}$ [exponentially] stable.*

Stabilizes *means* is stabilizing for.

Thus, an admissible output feedback operator L I/O-stabilizes $\Sigma \in \mathrm{WPLS}(U,H,Y)$ iff $y \in \mathrm{L}^2$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$, and L stabilizes Σ [strongly] iff x is bounded [and $x(t) \to 0$ as $t \to \infty$] and $y \in \mathrm{L}^2(\mathbf{R}_+;Y)$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$. The operator L is exponentially stabilizing for Σ iff $x \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$ (and $u_L = 0$); or equivalently, iff $x, y \in \mathrm{L}^2$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$, by Lemma 6.1.10(a1) and Lemma A.4.5. (Here $x := \mathbb{A}_L x_0 + \mathbb{B}_L \tau u_L$ and $y := \mathbb{C}_L x_0 + \mathbb{D}_L u_L$ are the state and output of the closed-loop system Σ_L with initial state x_0 and input u_L.)

Usually we shall only need the definitions of two first paragraphs of Definition 6.6.4. Terms like "$\mathbb{B}$-stabilizing" are useful only when referring to a part of a component of Σ (above we could say "input-stabilizing"). The last definition will be needed in connection with dynamic output feedback (when only I/O maps are specified).

Note that L is admissible for $\mathbb{D}$ iff L is admissible for Σ, but L may stabilize $\mathbb{D}$ even if L does not stabilize Σ. The same, of course, holds for the admissibility and stability concepts derived from this later.

The maps $u_L, y_L \mapsto u, y$ in a stabilizing feedback induce a d.c.f.:

Lemma 6.6.5 *The operator $L \in \mathcal{B}(Y,U)$ is a stabilizing output feedback operator for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} := \begin{bmatrix}(I - L\mathbb{D})^{-1} & L\\ \mathbb{D}_L & I\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}^{-1} := \begin{bmatrix}I & -L\\ -\mathbb{D}_L & (I - \mathbb{D}L)^{-1}\end{bmatrix}^{-1} \tag{6.128}$$

defines a d.c.f. of $\mathbb{D}$ (still $\mathbb{D}_L := \mathbb{D}(I - L\mathbb{D})^{-1}$), i.e., iff the operators in (6.128) are (well-posed and) stable.

If we input an extra signal y_L to the output y (as y is output from Σ in Figure 6.2), then the closed loop map $\left[\begin{smallmatrix}u_L\\ y_L\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ is the stable mapping

$$\begin{bmatrix}u\\ y\end{bmatrix} = \begin{bmatrix}\mathbb{M} & \widetilde{\mathbb{N}}\\ \mathbb{N} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}u_L\\ y_L\end{bmatrix}. \tag{6.129}$$

This is based on the useful formula $(I-L\mathbb{D})^{-1} = I + L\mathbb{D}_L$.

Proof: Obviously, (6.128) is sufficient for L to stabilize $\mathbb{D}$ (because then $\mathbb{D}_L$ is well-posed and stable), so we will assume that L stabilizes $\mathbb{D}$ and prove (6.128).

We have $\mathbb{M} := (I-L\mathbb{D})^{-1} = I + L\mathbb{D}(I-L\mathbb{D})^{-1} = I + L\mathbb{D}_L \in \mathrm{TIC}$, and $\widetilde{\mathbb{M}} := (I-\mathbb{D}L)^{-1} = I + (I-\mathbb{D}L)^{-1}\mathbb{D}L = I + \mathbb{D}_L L \in \mathrm{TIC}$ (because $\mathbb{D}_L := \mathbb{D}(I-L\mathbb{D})^{-1} = (I-\mathbb{D}L)^{-1}\mathbb{D}$, by Lemma A.1.1(f6)); clearly (6.128) follows from this.

Equation (6.129) is obviously the solution of equations

$$y = y_L + \mathbb{D}u \tag{6.130}$$

$$u = u_L + Ly. \tag{6.131}$$

□

If $\mathbb{D}$ has a r.c.f., then it is enough to stabilize $u_L \mapsto u$:

Lemma 6.6.6 *Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then $L \in \mathcal{B}(U,Y)$ is an admissible output feedback operator for $\mathbb{D}$ iff $\mathbb{M} - L\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.*

An admissible L is $\mathbb{D}$-stabilizing (i.e., $\mathbb{D}_L = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$ is stable) iff $(\mathbb{M}-L\mathbb{N})^{-1} \in \mathrm{TIC}$ is stable. Note that then $(I-L\mathbb{D})^{-1} = \mathbb{M}(\mathbb{M}-L\mathbb{N})^{-1} : u_L \mapsto u$.

Proof: Now $I - L\mathbb{D} = (\mathbb{M}-L\mathbb{N})\mathbb{M}^{-1}$, so admissibility is equivalent to $\mathbb{M} - L\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, and we have $(I-L\mathbb{D})^{-1} = \mathbb{M}(\mathbb{M}-L\mathbb{N})^{-1} =: \mathbb{M}_L$ and $\mathbb{D}_L := \mathbb{D}(I-L\mathbb{D})^{-1} = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$.

If L is stabilizing, then $\mathbb{M}_L$ and $\mathbb{D}_L$ are (stable and) r.c., by Lemma 6.6.5, and this in turn implies that $(\mathbb{M}-L\mathbb{N})^{-1} = \mathbb{X}\mathbb{M}_L - \mathbb{Y}\mathbb{D}_L$ is stable, if $\mathbb{X},\mathbb{Y} \in \mathrm{TIC}$ are s.t. $\mathbb{X}\mathbb{M} - \mathbb{Y}\mathbb{N} = I$. On the other hand, if $(\mathbb{M}-L\mathbb{N})^{-1}$ is stable, then so is $\mathbb{D}_L = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$. □

Lemma 6.6.7 ($\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}}$ [strongly] stable) *Assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$ and $\widetilde{\mathbb{C}} \in \mathcal{B}(H,\mathrm{L}^2(\mathbf{R};U))$, and that $\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}}$ is a C_0-semigroup.*

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is [[exponentially] strongly] stable, then $\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}}$ is [[exponentially] strongly] stable.

Proof: Since $\mathbb{B}\tau\widetilde{\mathbb{C}}$ is bounded, the stable case is obvious. If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable, then so is $\mathbb{B}$, by Lemma 6.1.13, hence then $\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}}$ is strongly stable.

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is exponentially stable, then $\mathbb{B}\tau$ is stable, by Lemma 6.1.10(a2), hence then $(\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}})x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, so that $\mathbb{A}+\mathbb{B}\tau\widetilde{\mathbb{C}}$ is exponentially stable, by Lemma A.4.5. □

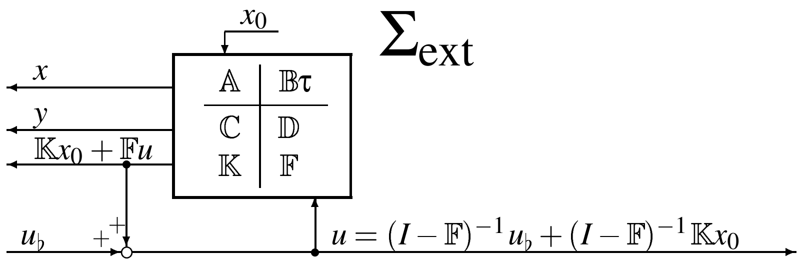


Figure 6.3: State feedback connection

From the above lemma we conclude the following:

Lemma 6.6.8 ($\mathbb{A}$ iff $\mathbb{A}_L$ stable) *Let L, Σ, and Σ_L be as in Proposition 6.6.2.*

(a) Assume that $\mathbb{B}$ and $L\mathbb{C}_L$ are stable. If $\mathbb{A}$ is [strongly] stable, then $\mathbb{A}_L$ is [strongly] stable.

(b) Assume that $\mathbb{B}_L L$ and $\mathbb{C}$ are stable. Then $\mathbb{A}$ is stable iff $\mathbb{A}_L$ is stable.

(c) Assume that $\mathbb{B}_L L$ or $L\mathbb{C}_L$ is stable. If $\mathbb{A}$ is exponentially stable, then $\mathbb{A}_L$ is exponentially stable.

Proof: (a) By (6.125), $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau L\mathbb{C}_{\circlearrowleft}$, hence this follows from Lemma 6.6.7.

(b) By (6.125), $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau(I - L\mathbb{D})^{-1}L\mathbb{C} = \mathbb{B}_L L\tau\mathbb{C}$, which is bounded, hence (b) holds.

(c) If $L\mathbb{C}_L$ is stable, then this follows from Lemma 6.6.7 as in (a). If $\mathbb{B}_L L$ is stable, then we apply the same for Σ^{d} and L^{d}. □

When Σ and Σ_L are I/O-stable, their stabilities are equivalent:

Corollary 6.6.9 (Σ iff Σ_L stable) *Let the assumptions of Proposition 6.6.2 hold. Let $I - L\mathbb{D} \in \mathcal{G}$TIC. Then Σ is [SOS-/strongly/exponentially] stable iff Σ_L is.*

Note that $\mathbb{D}, \mathbb{D}_L \in \text{TIC} \Rightarrow I - L\mathbb{D} \in \mathcal{G}\text{TIC}$, because $I + L\mathbb{D}_L = (I - L\mathbb{D})^{-1}$.

Proof: "Only if": The stable and SOS-stable case follow from (6.125); the strongly [exponentially] stable case follows from the stable case and Lemma 6.6.8(a)[(c)]. "If": Exchange the roles of Σ and Σ_L. □

Next we shall formulate and study stabilizability and detectability. A reader more familiar with matrix presentations of systems than (abstract) integral operators might wish to still think the systems as groups of generating operators ($\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$), which can be done in the regular case (and the formulae look the same, just "the font is changed", from $\mathbb{A}$ to A etc., as in Proposition 6.6.18).

A state feedback can be reduced to an output feedback as follows. The appropriate connection has been drawn in Figure 6.3.

Definition 6.6.10 (Admissible state feedback) *1. A pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is called an* admissible state feedback pair *for* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *if the extended system*

$$\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right] \tag{6.132}$$

is a WPLS *and* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*, i.e.,* $L := \begin{bmatrix} 0 & I \end{bmatrix}$ *is an admissible output feedback operator for* Σ_{ext}.

We often denote the corresponding closed loop system by (here $\mathbb{M} := (I - \mathbb{F})^{-1}$*)*

$$\Sigma_\flat = \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat\tau \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}+\mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M}\tau \\ \hline \mathbb{C}+\mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M}-I \end{array}\right] \tag{6.133}$$

$$= \Sigma_{\mathrm{ext}} \begin{bmatrix} I & 0 \\ -\mathbb{K} & I-\mathbb{F} \end{bmatrix}^{-1} = \Sigma_{\mathrm{ext}} \begin{bmatrix} I & 0 \\ \mathbb{M}\mathbb{K} & \mathbb{M} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_\flat \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ u-u_\flat \end{bmatrix}. \tag{6.134}$$

2. The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is a* stabilizing state feedback pair *if, in addition, L is a stabilizing output feedback operator for* Σ_{ext}*; we use prefices as in Definition 6.6.4 above, and add, in addition, the prefix "*[q.]r.c.*" (resp. suffix "*in $\mathcal{A}$*"), if* $\mathbb{N} := \mathbb{D}\mathbb{M}$ *and* $\mathbb{M}$ *are [q.]r.c. (resp.* $\mathbb{N},\mathbb{M} \in \mathcal{A}$*; cf. Definition 6.2.4). The pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is* exponentially [q.]r.c.-stabilizing *for* Σ *if* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is exponentially stabilizing and* $\mathbb{N}$ *and* $\mathbb{M}$ *are exponentially [q.]r.c.; equivalently, if there is* $\omega > 0$ *s.t.* $\mathcal{T}_\omega \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-stabilizing for* $\mathcal{T}_\omega\Sigma$ *(see Remark 6.1.9).*

3. The system $\Sigma \in$ WPLS *is* stabilizable *if it has a stabilizing feedback pair; we use here prefices and suffices as above (e.g.,* strongly r.c.-stabilizable*).*

4. We call $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *stable iff* $\mathbb{K}$ *and* $\mathbb{F}$ *are stable. We call* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *WR (resp. SR, UR, ...) iff* $\mathbb{F}$ *is WR (resp. SR, UR, ...).*

5. We call K_{c} *an* admissible compatible state feedback operator *for* Σ *if* $(K_{\mathrm{c}}, 0)$ *is a compatible pair for* $\left[\begin{smallmatrix} A & B \\ K & F \end{smallmatrix}\right]$*. We call* K_{w} *an* admissible WR state feedback operator *for* Σ *if* $\mathbb{F}$ *is WR and* $\left[\begin{smallmatrix} A & B \\ K_{\mathrm{w}} & 0 \end{smallmatrix}\right]$ *generate* $\left[\begin{smallmatrix} A & B \\ K & F \end{smallmatrix}\right]$*. We use here prefices and suffices as above (see 2. and 4.).*

Thus, e.g., a stabilizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is *stabilizing in* MTIC for Σ iff $\mathbb{M}, \mathbb{D}\mathbb{M} \in$ MTIC, equivalently, iff $\left[\begin{smallmatrix} \mathbb{D}_\flat \\ \mathbb{F}_\flat \end{smallmatrix}\right] \in$ MTIC; an admissible $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right]$-stabilizing for Σ iff $\mathbb{C}_\flat$ and $\mathbb{D}_\flat$ are stable. Of course, $\Sigma_{\mathrm{ext}} \in$ WPLS iff $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right] \in$ WPLS. Thus, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$.

In the literature, notion "K is admissible for A (resp. Σ)" often means that $\left[\begin{smallmatrix} A \\ K \end{smallmatrix}\right]$ (resp. $\left[\begin{smallmatrix} A & B \\ C & * \\ K & * \end{smallmatrix}\right]$) generate a WPLS, i.e., that K "fits" to A (resp. to Σ). To avoid misinterpretation, we never use this convention, hence admissibility always contains the requirement that the corresponding feedback is well-posed, e.g., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible for Σ iff it "fits" to Σ *and* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, we follow the convention of [S95]–[S98d].

If $\dim H < \infty$, then a reachable system is exponentially stabilizable, by p. 91 of [LR]. This is not the case in general: if $\mathbb{D} \in \mathrm{TIC}(U,Y)$, then the dual of (6.11) is stable and exactly 0-stabilizable (in infinite time) but not even optimizable (see Definition 6.7.3), hence not exponentially stabilizable.

Any bounded K is an admissible ULR state feedback operator:

Lemma 6.6.11 (Bounded *K*) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\begin{bmatrix} K & F \end{bmatrix} \in \mathcal{B}(H \times U, U)$.

Then $\begin{bmatrix} K & F \end{bmatrix}$ *generate an admissible state feedback pair for* Σ *iff* $I - F \in \mathcal{GB}(U)$. *If* $I - F \in \mathcal{GB}(U)$, *then* $(I-F)^{-1}K \in \mathcal{B}(H,U)$ *is an ULR admissible state feedback operator for* Σ, *and corresponding closed-loop systems are identical modulo* $E := (I-F)^{-1}$ *(see (6.136)).*

By Proposition 6.6.18(d), for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$, the closed-loop system corresponding to $K \in \mathcal{B}(H,U)$ is generated by $\left[\begin{array}{c|c} A+BK & B \\ \hline K & 0 \end{array}\right]$.

Proof: By Lemma 6.3.16(c), $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}(U,H,U)$ and $\mathbb{F}$ is ULR. By Proposition 6.3.1(c), $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible iff $I - F \in \mathcal{GB}(U)$.

Assume that $I - F \in \mathcal{GB}(U)$. Set $K' := EK$, where $E := (I-F)^{-1}$. Then K' is admissible and ULR, by the above. If $\left[\begin{array}{c|c} qK' & \mathbb{F}' \end{array}\right]$ is the corresponding state feedback pair, then

$$\mathbb{F}' = K'\mathbb{B}\tau = E(K\mathbb{B}\tau + F) - EF = E\mathbb{F} + I - E, \tag{6.135}$$

because $EF = E - I$, hence then Lemma 6.6.12 applies. □

A coordinate transform in the feedback signal does not essentially affect the system:

Lemma 6.6.12 (Making *F* zero) *Let* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, Σ *and* $\Sigma_\flat$ *be as in Definition 6.6.10. Let* $E \in \mathcal{GB}$. *Then also* $\left[\begin{array}{c|c} E\mathbb{K} & E\mathbb{F} + I - E \end{array}\right]$ *is an admissible state feedback pair for* Σ *(having same prefices and suffices as* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *does), and the corresponding closed-loop system is given by*

$$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat E^{-1}\tau \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat E^{-1} \\ \mathbb{K}_\flat & \mathbb{F}_\flat E^{-1} + E^{-1} - I \end{array}\right] \tag{6.136}$$

□

(We leave the easy verification of this and the following two lemmas to the reader.)

Thus, if $\mathbb{F}$ is WR and $E := I - F \in \mathcal{GB}(U)$ (this is necessary if $\mathbb{F}$ is UR or $\mathbb{F}, \mathbb{F}^* \in \mathrm{SR}$, by Proposition 6.3.1(a1)&(b1)), then we can take $F = 0$ w.l.o.g. In particular, any UR state feedback is equivalent (in the sense of the lemma) to a UR state feedback operator. We shall often use this fact.

The following is obvious:

Lemma 6.6.13 (Stable is stabilizable) *The pair* $\left[\begin{array}{c|c} 0 & 0 \end{array}\right]$ *is [IO-/SOS-/strongly/exponentially] r.c.-stabilizing for* Σ *iff* Σ *is [IO-/SOS-/strongly/exponentially] stable.*

Thus, an [IO-/SOS-/strongly/exponentially] stable system is [IO-/SOS-/strongly/exponentially] r.c.-stabilizable (but not vice versa). □

Naturally, analogous claims also hold for detectability and for joint stabilizability and detectability (Definition 6.6.21).

We can always use negative feedback to recover the original system:

Lemma 6.6.14 ($\begin{bmatrix} -\mathbb{K}_\flat & \big| & -\mathbb{F}_\flat \end{bmatrix}$ for $\Sigma_\flat$) *Let $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$, Σ and $\Sigma_\flat$ be as in Definition 6.6.10. Then $\begin{bmatrix} -\mathbb{K}_\flat & \big| & -\mathbb{F}_\flat \end{bmatrix} = \begin{bmatrix} -\mathbb{M}\mathbb{K} & \big| & I-\mathbb{M} \end{bmatrix}$ is an admissible state feedback pair for $\Sigma_\flat$, and the corresponding closed-loop system is Σ_{ext} with the added row $\begin{bmatrix} -\mathbb{K} & \big| & -\mathbb{F} \end{bmatrix}$.* □

Remark 6.6.15 (Historical definitions of exponential stabilizability) *In most infinite-dimensional classes, one traditionally defines only exponential stabilizability (not stabilizability, strong stabilizability etc.) and uses a stronger definition: one requires the existence of a* bounded *($K \in \mathcal{B}(H,U)$) exponentially stabilizing state-feedback operator (see, e.g., [CZ]).*

For Pritchard–Salamon-systems (see Section 6.9), the definition is even stronger: $\Sigma_\flat$ must be exponentially stable also in the larger state space "$\mathcal{V}$", not merely in $H := \mathcal{W}$.

Our definition was adopted from [S98a]. However, in some articles on regular WPLSs (e.g., in [WR00]), one requires the existence of a SR exponentially stabilizing state-feedback operator; also this definition is stronger than our definition of exponential stabilizability (we do not know whether it is strictly stronger).

In the works of I. Lasiecka, R. Triggiani and others (e.g., in [FLT]), sometimes also "non-admissible" feedback is accepted, i.e., the map $u \mapsto u_L$ need not be well-posed; this is a special case of the setting of Definition 8.3.15 and Section 9.7 (see also (8.34)), Therefore, under that convention one usually treats only "the left column of $\Sigma_\flat$", which is well posed; any external input ("$u_\flat$"; e.g., disturbance or modeling error) might "explode" the system. This convention makes optimizability equivalent to exponential stabilizability. (Optimizability is the weakest reasonable extension of the classical concept "exponential stabilizability"; see Definition 6.7.3.)

By Theorem 9.2.12, all these definitions of exponential stability coincide for systems having B bounded (or $\mathbb{A}Bu_0 \in \mathrm{L}^1_{\text{loc}}(\mathbf{R}_+;H)$ for each $u_0 \in U$). Note also that by "stabilizability" one often means exponential stabilizability. □

All this applies also to (exponential) detectability (Definition 6.6.21) and estimatability, mutatis mutandis.

A state feedback pair induces a factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ of $\mathbb{D}$. However, most such factorizations do not correspond to state feedback.

It is not enough to find a right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ to obtain an I/O-stabilizing feedback pair, there must also exist a suitable $\mathbb{K}$ for $\mathbb{F} := I - \mathbb{M}^{-1}$. Therefore, $\widehat{\mathbb{F}}$ must be analytic outside the spectrum of A (this is most obvious when $\widehat{\mathbb{F}} = K(s-A)^{-1}B + F$ is rational; in the general case one should use (6.40) with $\widehat{\mathbb{D}} \mapsto \widehat{\mathbb{F}}$ and $C \mapsto K$ on some right half-plane and (possibly) analytic extension elsewhere). Hence, in the finite-dimensional case, the McMillan degree of $\mathbb{F}$ must not exceed the dimension of the state space H. Thus, only a small fraction of all right factorizations of $\mathbb{D}$ correspond to I/O-stabilizing state feedback pairs.

The concept "[q.]r.c.-stabilizability" is equivalent to stabilizability by such a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ that the right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ mentioned above is a [q.]r.c.f. Since this concept is new, we shall look at it more closely below.

In Chapters 8.3–12.3, the importance of this concept will become obvious; e.g., the Popov operator of a sufficiently regular strongly stable system has a spectral factorization iff the corresponding Riccati equation has a q.r.c.-stabilizing solution, and under suitable assumptions such solutions lead to solutions of several control problems also in the unstable case. Also Remark 9.9.9 explains the advantages of q.r.c.-stabilizability in optimal control; some sufficient conditions for q.r.c.-stabilizability are given in Corollary 6.7.16.

By definition, [q.]r.c.-stabilization means such a stabilization that the map $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right] : u_\flat \mapsto \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ (see (6.133)–(6.134) or Figure 6.3) becomes [quasi–]left invertible over TIC. In particular, for such a pair the map $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \mapsto u_\flat$ is well-defined and stable.

Therefore, to construct a system that is stabilizable but not q.r.c.-stabilizable, we let $\widehat{\mathbb{B}}$ have a pole (at 1) that is not shared by $\widehat{\mathbb{D}}$, so that we must have $\widehat{\mathbb{M}}(1) = 0$ to make $\mathbb{B}_\flat := \mathbb{B}\mathbb{M}$ stable, in which case $\widehat{\mathbb{N}} := \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ also has a zero at 1, i.e., $\mathbb{N}$ and $\mathbb{M}$ are not q.r.c.:

Example 6.6.16 (Exponentially stabilizable but not q.r.c.-stabilizable) For the system $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ generated by $\left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \end{array}\right]$, a state feedback operator $K \in \mathbf{C}$ results in $\widehat{\mathbb{M}}(s) = (s-1)/(s-1-K) = \widehat{\mathbb{N}}(s)$, $A + BK = 1 + K$, hence only $K = 0$ is q.r.c.-I/O-stabilizing (even q.r.c.-SOS-stabilizing), whereas $K < -1$ is exponentially stabilizing, $K = -1$ is $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right]$-stabilizing (but not I/O-stabilizing), and $K > -1$, $K \neq 0$ is only output-stabilizing. In particular, this system is not q.r.c.-stabilizable (see Example 8.4.4 of [Sbook] for a less trivial example). This follows from the fact that $\mathbb{A}$ has an unobservable pole. $\triangleleft$

As above, lack of q.r.c.-stabilizability typically corresponds to situations where the semigroup has unobservable poles. Conversely, one can often make a stabilizing pair r.c.-stabilizing by canceling the common zeros of $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{M}}$. Indeed, from Lemma 6.5.4 one observes that in the finite-dimensional case, $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (q.)r.c.f. iff the zeros of $\widehat{\mathbb{M}}$ coincide with the poles of $\widehat{\mathbb{D}}$, up to the multiplicity. Thus, then (q.)r.c.-I/O-stabilization means I/O-stabilization with a minimal number of zeros. (See also Lemma 6.5.3(d).) The infinite-dimensional situation is similar.

For an exponentially stabilizable and detectable system (e.g., a minimal finite-dimensional system), any I/O-stabilizing state feedback pair is exponentially q.r.c.-stabilizing, by Theorem 6.7.15(c1). This is one of the reasons why [q.]r.c.-stabilizability is not as important in the finite-dimensional case.

We do not know whether there is a SOS-stabilizable system that is not q.r.c.-SOS-stabilizable (or a (well-posed) right factorization that cannot be reduced to a q.r.c.f.).

For a stable system, any stable, stabilizing state feedback pair is [q.]r.c.-stabilizing (and conversely, see (c)). This and related results are given below:

Lemma 6.6.17 (Stable case: $\mathbb{F} \in$ TIC $\Leftrightarrow$ r.c.-stab.) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS.

(a) Let $\mathbb{D}$ be stable, and let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ I/O-stabilize Σ. Then the following are equivalent:

(i) $\mathbb{F}$ is stable;
(ii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is r.c.-I/O-stabilizing for Σ;
(iii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-I/O-stabilizing for Σ;
(iv) $I - \mathbb{F} \in \mathcal{G}$TIC.

Moreover, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [q.]r.c.-SOS-stabilizing iff it is stable (i.e., iff $\mathbb{K}$ and $\mathbb{F}$ are stable) and $\Sigma \in$ SOS.

(b) If $\Sigma \in$ SOS, then a SOS-stabilizing feedback pair for Σ is stable iff it is [q.]r.c.-SOS-stabilizing.

(c) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be a stable I/O-stabilizing feedback pair for Σ, and let $\mathbb{D}$ be stable.

Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is r.c.-I/O-stabilizing. Moreover, it is r.c.-SOS-stabilizing iff $\Sigma \in$ SOS, and it is [strongly/exponentially] r.c.-stabilizing iff Σ is [strongly/exponentially] stable.

(d) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be admissible for $\Sigma \in$ SOS. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is r.c.-SOS-stabilizing iff it is stable and $(I - \mathbb{F})^{-1} \in$ TIC.

Proof: (a) If $\mathbb{F} \in$ TIC, then $\mathbb{M} := (I - \mathbb{F})^{-1} \in \mathcal{G}$TIC, which makes $\mathbb{N} := \mathbb{D}(I - \mathbb{F})^{-1} \in$ TIC and $\mathbb{M} := (I - \mathbb{F})^{-1} \in$ TIC r.c. (see Definition 6.6.10). Conversely, if $\mathbb{M}$ and $\mathbb{N}$ are q.r.c., then $\mathbb{M}^{-1} \in$ TIC, by Lemma 6.5.6(b), hence then $\mathbb{F} \in$ TIC.

Let $\mathbb{F}$ be stable. The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [q.]r.c.-SOS-stabilizing iff $\mathbb{K}_\flat := \mathbb{M}\mathbb{K}$ and $\mathbb{C}_\flat = \mathbb{C} + \mathbb{N}\mathbb{K}$ are stable. But $\mathbb{M}\mathbb{K}$ is stable iff $\mathbb{K} := \mathbb{M}^{-1}\mathbb{M}\mathbb{K}$ is (because $\mathbb{M} \in \mathcal{G}$TIC). If $\mathbb{K}$ is stable, then $\mathbb{C}$ is stable iff $\mathbb{C}_\flat$ is.

(b) This follows from (a).

(c) The first two claims are from (a), the last one follows from Corollary 6.6.9.

(d) This follows from (a) and (b). □

If B and C are bounded, then it easy to verify that the generators of (6.125) are given by $\left[\begin{array}{c|c} A+BL(I-DL)^{-1}C & B(I-LD)^{-1} \\ (I-DL)^{-1}C & D(I-LD)^{-1} \end{array}\right]$, i.e., formally we just (remove τ and) replace each operator by its feedthrough operator; an analogous comment applies to other forms of feedback too. For unbounded B and C, the situation is similar but much trickier, as obvious from the following important results:

Proposition 6.6.18 (Generators $\left[\begin{array}{c|c} A_L & B_L \\ C_L & D_L \end{array}\right]$ of Σ_L) *Make the assumptions and use the notation of Proposition 6.6.2. Then (a1)–(c4) hold:*

(a1) $\mathrm{Dom}(A_L) \subset H_{B_L} = H_B$ *and* $H^*_{C_L} = H^*_C$.

(a2) For $\mathrm{Re}\, s > \max\{\omega_A, \omega_{A_L}\}$ *we have*

$$(s-A_L)^{-1} - (s-A)^{-1} = (s-A)^{-1}BLC_L(s-A_L)^{-1} \tag{6.137}$$

$$= (s-A_L)^{-1}B_L LC(s-A)^{-1} \in \mathcal{B}(H, H_B). \tag{6.138}$$

(a3) The generators A *and* A_L *are given by*

$$A_L = A + BLC_L : \mathrm{Dom}(A_L) \to H, \tag{6.139}$$

$$A = A_L - B_L LC : \mathrm{Dom}(A) \to H. \tag{6.140}$$

(b1) **(Σ_L)** *Let* Σ *have compatible output operator pair* (C_c, D) *s.t.* $I - LD$ *or* $I - DL$ *is left-invertible.*

Then both $I - LD$ *and* $I - DL$ *are left-invertible,* Σ_L *has compatible generators (here* $(I-DL)^{-1}_{\mathrm{left}}$ *may be any left inverse of* $I - DL$*)*

$$\left[\begin{array}{c|c} A_L & B_L \\ \hline (C_L)_\mathrm{c} & (D_L)_\mathrm{c} \end{array}\right] = \left[\begin{array}{c|c} A+BLC_L & B_L \\ \hline (I-DL)^{-1}_{\mathrm{left}}C_\mathrm{c} & (I-DL)^{-1}_{\mathrm{left}}D \end{array}\right] \text{ on } \mathrm{Dom}(A_L) \times U, \tag{6.141}$$

$A = A_L - B_L LC_\mathrm{c} \in \mathcal{B}(H_B, V)$ $(C_L)_\mathrm{c} = (I-DL)^{-1}_{\mathrm{left}}C_\mathrm{c} \in \mathcal{B}(H_B, Y)$ *and* $B = B_L(I - LD) \in \mathcal{B}(U, V)$ *(see the proof for the identification of* $B_L U \subset H^L_{-1}$ *and* $BU \subset H_{-1}$*; here* V *is a Banach space s.t.* $H \underset{c}{\subset} V \underset{c}{\subset} H_{-1}$ *and* $V \underset{c}{\subset} H^L_{-1}$*).*

If $I - LD \in \mathcal{GB}$*, then* $B_L = B(I-LD)^{-1} \in \mathcal{B}(U, V)$ *and* $A_L = A + BL(C_L)_\mathrm{c} \in \mathcal{B}(H_B, V)$.

(b2) Let Σ *be WR and let* $I - LD$ *or* $I - DL$ *be left-invertible. Then (b1) applies with* $(C_\mathrm{c}, D) = (C_\mathrm{w}, D)$.

If, in addition, $I - LD \in \mathcal{GB}$*, then* $B^*_L = (I - D^*L^*)^{-1}B^*_\mathrm{w}$ *on* $\mathrm{Dom}(A_L)$.

(b3) Let Σ *be SR. Then* $I - LD$ *and* $I - DL$ *are left-invertible, and (b1) and (b2) apply with* $(C_\mathrm{c}, D) = (C_\mathrm{s}, D)$*. Moreover, then* Σ_L *is* SR *iff* $I - LD \in \mathcal{GB}$.

If $I - LD \in \mathcal{GB}$*, then*

$$\left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] = \left[\begin{array}{c|c} A+BLC_L & B(I-LD)^{-1} \\ \hline (I-DL)^{-1}C_\mathrm{s} & (I-DL)^{-1}D \end{array}\right] \in \mathcal{B}(H_B \times U, V \times Y). \tag{6.142}$$

(see the proof for the identification of $B_L[U]$ *and* $B[U]$*),* $A_L = A + B(I - LD)^{-1}LC_\mathrm{s} \in \mathcal{B}(H_B, V)$*,* $B, B_L \in \mathcal{B}(U, V)$*, where* V *is a Banach space s.t.* $H \underset{c}{\subset} V$ *densely,* $V \underset{c}{\subset} H_{-1}$*,* $V \underset{c}{\subset} H^L_{-1}$*. See also (c4).*

(b4) Let Σ *be UR. Then* $I - LD \in \mathcal{GB}$*,* Σ_L *is UR and (6.142) holds.*

(c1) Let $\mathbb{D}$ *be SR. Then* $\mathbb{D}_L$ *is SR iff* $I - DL \in \mathcal{GB}(Y)$.

(c2) Let $\mathbb{D}L$ *be WR and let* $I-DL$ *have a left inverse* $(I-DL)^{-1}_{\text{left}}$*. Then* $(C_L)_{\text{s}} \subset (I-DL)^{-1}_{\text{left}} C_{\text{w}}$.

(c3) Let $\mathbb{D}L$ *be SR. Then* $I-DL$ *has a left inverse* $(I-DL)^{-1}_{\text{left}}$*, and* $(C_L)_{\text{s}} \subset (I-DL)^{-1}_{\text{left}} C_{\text{s}}$.

(c4) Let $\mathbb{D}$ *be SR and* $I-DL \in \mathcal{GB}(Y)$*. Then* $(C_L)_{\text{s}} = (I-DL)^{-1}C_{\text{s}}$*, and* $(B_L^*)_{\text{s}} = (I-D^*L^*)^{-1}B_{\text{s}}^*$.

In particular, $\mathrm{Dom}((C_L)_{\text{s}}) = \mathrm{Dom}(C_{\text{s}})$ *and* $\mathrm{Dom}((B_L^*)_{\text{s}}) = \mathrm{Dom}(B_{\text{s}}^*)$ *(with equivalent norms).*

(c5) Let $(\mathbb{D}L)^{\text{d}}$ *be SR and let* $I-DL$ *have a left inverse. Then* $I-DL \in \mathcal{GB}(Y)$ *and* $(C_L)_{\text{w}} = (I-DL)^{-1}C_{\text{w}}$.

(c6) Let $\mathbb{D}$ *be SR. Assume that* $\mathbb{D}L \in \mathrm{MTIC}_\infty$*. Then* $I-DL \in \mathcal{GB}(Y)$*,* $(C_L)_{\text{L,s}} = (I-DL)^{-1}C_{\text{L,s}}$*, and* $(B_L^*)_{\text{L,s}} = (I-D^*L^*)^{-1}B_{\text{L,s}}^*$.

In particular, $\mathrm{Dom}((C_L)_{\text{L,s}}) = \mathrm{Dom}(C_{\text{L,s}})$ *and* $\mathrm{Dom}((B_L^*)_{\text{L,s}}) = \mathrm{Dom}(B_{\text{L,s}}^*)$ *(with equivalent norms).*

Make the assumptions and use the notation of Definition 6.6.10. Then

(d1) **($\Sigma_\flat$)** *Let* $(\left[\begin{smallmatrix} C_{\text{c}} \\ K_{\text{c}} \end{smallmatrix}\right], \left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right])$ *be a compatible pair for* Σ_{ext}*. Assume that* $X := I - F$ *has a left inverse* M*. Then* $\Sigma_\flat$ *is compatible with*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A + B_\flat K_{\text{c}} & B_\flat \\ \hline C_{\text{c}} + DMK_{\text{c}} & DM \\ MK_{\text{c}} & M - I \end{array}\right] \text{ on } H_B \times U, \tag{6.143}$$

$A_\flat = A + BMK_{\text{c}}$ *on* $\mathrm{Dom}(A)$*, and* $B = B_\flat X$ *on* U *(the remarks of (b1) apply). By (a),* $\mathrm{Dom}(A_\flat) \subset H_{B_\flat} = H_B$.

Moreover, $\widehat{\mathbb{X}}(s) = X - K_{\text{c}}(s-A)^{-1}B$*, and* $\widehat{\mathbb{M}}(s) = M + MK_{\text{c}}(s-A_\flat)^{-1}B_\flat$*. If* $\mathbb{X}$ *is WR, then* $\mathbb{K}_\circlearrowleft x_0 = MK_{\text{w}}\mathbb{A}_\circlearrowleft(\cdot)x_0$ *a.e.; if* $\mathbb{X}$ *is SR, then* $\mathbb{K}_\circlearrowleft x_0 = MK_{\text{s}}\mathbb{A}_\circlearrowleft(\cdot)x_0$ *a.e., for any* $x_0 \in H$.

If $X \in \mathcal{GB}$*, then* $B_\flat = BM$ *on* U*, and* $A_\flat = A + BMK_{\text{c}}$ *on* H_B.

(d2) $\boldsymbol{((K_{\text{c}}, 0))}$ *Let* $(\left[\begin{smallmatrix} C_{\text{c}} \\ K_{\text{c}} \end{smallmatrix}\right], \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right])$ *be a compatible pair for* Σ_{ext}*. Then* $\Sigma_\flat$ *is compatible with*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A + BK_{\text{c}} & B \\ \hline C_{\text{c}} + DK_{\text{c}} & D \\ K_{\text{c}} & 0 \end{array}\right] \text{ on } H_B \times U, \tag{6.144}$$

(the remarks of (b1) apply). Moreover, (d1) applies (with $M = I = X$*).*

In particular, $-K_{\text{c}}$ *is admissible for* $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$*, and the resulting closed-loop system is* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ -\mathbb{K} & -\mathbb{F} \end{array}\right]$.

$$\begin{tikzcd}
 & \Sigma \arrow[dl, "K"'] \arrow[dr, "K' = K + K_{\natural\mathrm{w}}"] & \\
\Sigma_\flat \arrow[rr, "K_\natural"'] & & \Sigma_\circlearrowleft, \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}'_\circlearrowleft & \mathbb{F}'_\circlearrowleft \end{array}\right]
\end{tikzcd}$$

Figure 6.4: The setting of Proposition 6.6.18(f)

(d3) Assume that $\mathbb{D}, \mathbb{F} \in \mathrm{WR}$ *and* $F = 0$*. Then* $\Sigma_\flat$ *has compatible generators*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A + BK_\mathrm{w} & B \\ \hline C_\mathrm{w} + DK_\mathrm{w} & D \\ K_\mathrm{w} & 0 \end{array}\right] \text{ on } H_B \times U. \tag{6.145}$$

Moreover, then $(C_\flat)_\mathrm{s} \subset C_\mathrm{w} + DK_\mathrm{w}$*,* $(K_\flat)_\mathrm{s} \subset K_\mathrm{w}$*,* $\widehat{\mathbb{X}}(s) = I - K_\mathrm{w}(s-A)^{-1}B$*,* $\widehat{\mathbb{M}}(s) = I + K_\mathrm{w}(s - A - BK_\mathrm{w})^{-1}B$*, and* $\mathbb{K}_\circlearrowleft x_0 = K_\mathrm{w}\mathbb{A}_\circlearrowleft(\cdot)x_0$ *a.e.*

(d4) $\boldsymbol{((K_\mathrm{s}, 0))}$ *Assume that* $\mathbb{D} \in \mathrm{WR}$*,* $\mathbb{F} \in \mathrm{SR}$ *and* $F = 0$*. Then* $\mathbb{F}_\flat$ *is SR,* $\Sigma_\flat$ *is WR with generators (6.145) (which holds also with* K_s *in place of* K_w*). Moreover, then* $(C_\flat)_\mathrm{w} = C_\mathrm{w} + DK_\mathrm{s}$ *on* H_B*, and* $(K_\flat)_\mathrm{s} = K_\mathrm{s}$*.*

If $D = 0$*, then* $(C_\flat)_\mathrm{w} = C_\mathrm{w}$*. If* $\mathbb{D}$ *is SR, then* $\Sigma_\flat$ *is SR and* $(C_\flat)_\mathrm{s} = C_\mathrm{s} + DK_\mathrm{s}$*.*

(e) Assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is SR and that* $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{S} & \mathbb{T} \end{array}\right] \in \mathrm{WPLS}(U, H, Z)$*. If* $\mathbb{T}$ *is WR (resp. SR), then the weak (resp. strong) Weiss extensions of* $S : \mathrm{Dom}(A_\flat) \to Z$ *and* $S_\mathrm{w}|_{\mathrm{Dom}(A)} : \mathrm{Dom}(A) \to Z$ *are identical on* H_B*.*

(f) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be SR and* $F = 0$*. Let* $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *be WR and admissible for* $\Sigma_\flat$ *with closed-loop system* $\Sigma_\circlearrowleft$ *and* $F_\natural = 0$*.*

Then $K' := K + K_{\natural\mathrm{w}}|_{\mathrm{Dom}(A)}$ *is WR and admissible for* Σ*, and* $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$ *forms the top two rows of the corresponding closed-loop system (and* $K'_\mathrm{w} = K_\mathrm{s} + K_{\natural\mathrm{w}}$ *on* H_B*, by (e)). The pair generated by* K' *and corresponding closed-loop maps are given by*

$$\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural + \mathbb{X}_\natural\mathbb{K} & \mathbb{F} + \mathbb{F}_\natural - \mathbb{F}_\natural\mathbb{F} \end{array}\right], \tag{6.146}$$

$$\left[\begin{array}{c|c} \mathbb{K}'_\circlearrowleft & \mathbb{F}'_\circlearrowleft \end{array}\right] = \left[\begin{array}{c|c} \mathbb{M}\mathbb{K}_\circlearrowleft + \mathbb{K}_\flat & \mathbb{M}\mathbb{M}_\natural - I \end{array}\right]. \tag{6.147}$$

Thus, if K *and* $K_\natural$ *have some strong or uniform regularity property, then so do* K' *and* $K'_\circlearrowleft$*.*

(g) We have $K' = K''$ *(on* H_B*)* $\Leftrightarrow \mathbb{K}'_\flat = \mathbb{K}''_\flat \Leftrightarrow \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}'' & \mathbb{F}'' \end{array}\right]$ *for any compatible admissible state feedback operators* K' *and* K''*.*

Recall that $(C_L)_\mathrm{s} \subset (I - DL)^{-1}_{\mathrm{left}}C_\mathrm{w}$ means that the latter is an extension of the former, i.e., that $\mathrm{Dom}((C_L)_\mathrm{s}) \subset \mathrm{Dom}((I - DL)^{-1}_{\mathrm{left}}C_\mathrm{w})$ and $(C_L)_\mathrm{s}x_0 = (I - DL)^{-1}_{\mathrm{left}}C_\mathrm{w}x_0$ for all $x_0 \in \mathrm{Dom}((C_L)_\mathrm{s})$. In (c2)–(c5), "$DL$" denotes the feedthrough operator of $\mathbb{D}L \in \mathrm{WR}$; obviously, "$DL$"$= D \cdot L$ whenever $\mathbb{D} \in \mathrm{WR}$ (so that "D" exists).

By (c4), we have $\mathrm{Dom}((C_L)_{\mathrm{s}}) = (I - DL)^{-1}\,\mathrm{Dom}(C_{\mathrm{s}})$ despite the fact that the strong extensions are computed w.r.t. different semigroup generators (A_L and A, respectively); therefore, $(K_\flat)_{\mathrm{s}} = K_{\mathrm{s}}$ in (d4) (see the proofs of (c3) and (c2) for details; also (e) relates to this).

It is not known, whether, e.g., (c4) holds with $C_{\mathrm{L,s}}$ in place of C_{s} etc. (with the exception of (c6); do not mix the generator C_L of $\mathbb{C}_L$ with the strong Lebesgue extension $C_{\mathrm{L,s}}$ of C). However, when $\mathbb{D}$ is SR, we have $C_{\mathrm{s}} = C_{\mathrm{L,s}}$ on H_B, by Proposition 6.2.8(c1)&(c4)&(d1). Analogously, most of the above proposition can be written with Lebesgue extensions.

Recall also that compatible output operators (e.g., $(C_L)_{\mathrm{c}}$ and $(D_L)_{\mathrm{c}}$) are not unique in general and may differ from $(C_L)_{\mathrm{w}}$ and $(D_L)_{\mathrm{w}}$ even if Σ_L is WR (but $(C_L)_{\mathrm{w}}$ and $(D_L)_{\mathrm{w}}$ are unique).

The closed-loop system Σ_L need not be WR in (b2):

Example 6.6.19 If we set $\widetilde{\mathbb{D}} := \begin{bmatrix} 0 & \mathbb{D} \\ \mathbb{D}^{\mathrm{d}} & 0 \end{bmatrix} \in \mathrm{TIC} \cap \mathrm{WHPR}$ and $L := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, where $\mathbb{D} \in \mathrm{TIC} \cap \mathrm{SHPR}$ is as in Example 6.2.6, then $\widetilde{\mathbb{D}}_L = \begin{bmatrix} \mathbb{D}\mathbb{D}^{\mathrm{d}} & \mathbb{D} \\ \mathbb{D}^{\mathrm{d}} & 0 \end{bmatrix} \notin \mathrm{WR}$, although $I - L\widetilde{D} = I \in \mathcal{GB}$. Thus, if Σ is any realization of $\widetilde{\mathbb{D}}$, then (b1) and (b2) apply but Σ_L is not regular. $\triangleleft$

This fact and the need for closed-loop generators for Riccati equation theory was our main motivation to introduce compatible generators in [Mik97a]. Cf. also Lemma 6.3.11.

Proof of Proposition 6.6.18: (a1) By [S97b, Proposition 37], we have $H_B = H_{B_L} := (s - A_L)^{-1}[H + BU] \supset \mathrm{Dom}(A_L)$, and $H_{C_L}^* = H_C^*$ (for the stable case; the proofs apply in the general case too; alternatively, use shifting (see Remark 6.1.9) or see [Sbook] for the general case).

(a2) This is Proposition 6.6 of [W94b] (just take the Laplace transform of the identity $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau L\mathbb{C}_L = \mathbb{B}_L\tau L\mathbb{C}$, which follows from (6.125)).

(b1) The (left-)invertibility of $I - DL$ is equivalent to that of $I - LD$, by Lemma A.1.1(f6). The other claims are given in Theorem 6.5.1 of [Sbook] (originally mostly from [Mik97a] except for B_L). We sketch the proof below.

1° *A_L, $(C_L)_{\mathrm{c}}$ and $(D_L)_{\mathrm{c}}$:* The formula for A was given in (a3). Set $\omega := \max\{\omega_A, \omega_{A_L}\}$. By Theorem 6.2.11(c1), Lemma 6.3.10(a) and (6.125), we have

$$\widehat{\mathbb{C}}(s) = C(s-A),\ \ \widehat{\mathbb{C}}_L(s) = C_L(s-A_L)^{-1}, \tag{6.148}$$

$$\widehat{\mathbb{D}}(s) = D + C_{\mathrm{c}}(s - A^{-1})B \ \text{ and } \ (I - \widehat{\mathbb{D}}L)\widehat{\mathbb{C}}_L = \widehat{\mathbb{C}} \ \ (s \in \mathbf{C}^+_\omega). \tag{6.149}$$

Thus, for any $s \in \mathbf{C}^+_\omega$, we have

$$(I-DL)C_L(s-A_L)^{-1} = C_c\left((s-A)^{-1} + (s-A)^{-1}BLC_L(s-A_L)^{-1}\right) \quad (6.150)$$

$$= C_c(s-A_L)^{-1} \in \mathcal{B}(H,Y), \quad (6.151)$$

by (a3), hence $C_L = (I-DL)^{-1}_{\text{left}}C_c$ on $\text{Ran}(s-A_L) = \text{Dom}(A_L)$. Since $(s-A_L)^{-1}B_L = (s-A)^{-1}B(I-L\widehat{\mathbb{D}}(s))^{-1}$, by (6.125) and Theorem 6.2.11, we obtain that

$$D + C_c(s-A_L)^{-1}B_L(I-L\widehat{\mathbb{D}}(s)) = (I-DL)\widehat{\mathbb{D}}(s), \quad \text{hence} \quad (6.152)$$

$$(I-DL)^{-1}(D+C_c(s-A_L)^{-1}B_L) = \widehat{\mathbb{D}}(s)(I-L\widehat{\mathbb{D}}(s))^{-1} = \widehat{\mathbb{D}}_L(s), \quad (6.153)$$

by (6.125), so that $(I-DL)^{-1}(C_c, D_c)$ is a compatible pair for Σ_L (we can use for $(C_L)_c := (I-DL)^{-1}C_c$ the same W as for C_c), by Definition 6.3.10. (An alternative proof would have used Lemma 6.3.12.)

2° B_L: Choose $\alpha \in \sigma(A)^c \cap \sigma(A_L)^c$, so that $(\alpha - A) \in \mathcal{GB}(H, H_{-1})$ and $(\alpha - A_L)\mathcal{GB}(H, H^L_{-1})$, where $H_{-1} := \text{Dom}(A^*)^*$, $H^L_{-1} := \text{Dom}(A_L^*)^*$, by Lemma A.4.6. Set

$$V := (\alpha - A)W, \quad \|(\alpha - A)x_0\|_V := \|x_0\|_W, \quad (6.154)$$

$$V_L := (\alpha - A_L)W, \quad \|(\alpha - A_L)x_0\|_{V_L} := \|x_0\|_W, \quad (x_0 \in W); \quad (6.155)$$

$$J := (\alpha - A_L + B_L LC_c)(\alpha - A)^{-1} \in \mathcal{B}(V, V_L) \quad (6.156)$$

$$J_L := (\alpha - A - BL(C_L)_c)(\alpha - A_L)^{-1} \in \mathcal{B}(V_L, V). \quad (6.157)$$

It follows that $H \underset{c}{\subset} V \underset{c}{\subset} H_{-1}$, $H \underset{c}{\subset} V_L \underset{c}{\subset} H^L_{-1}$, and that V and V_L are independent of α (obviously, $H = (\alpha - A)H_1 \subset V$ and $V \underset{c}{\subset} H_{-1}$; by the resolvent equation, V is independent of α as a set; by Lemma A.3.6, also the topology of V is independent of α, by Lemma A.4.6).

Assuming that $(I-DL)$ is left-invertible, a direct computation (see [Sbook]) shows that $J_LJ = I \in \mathcal{B}(V)$, i.e., that $(\alpha - A)^{-1}J_LJ(\alpha - A) = I \in \mathcal{B}(W)$. Since J is boundedly left-invertible, we have $J \in \mathcal{GB}(V, V')$, were $V' := J[V] \subset V_L$. Thus, J defines an embedding $V \to V_L$ with range V'. By (a3), $J_{|H} = I = J_{L|H}$, hence we can and will identify V to V' (note that this may alter the norm of V from the one defined above, but the new norm is equivalent, i.e., the topology of V is unchanged).

Then J becomes the identity on V, hence $\alpha - A = \alpha - (A_L - B_LLC_c) \in \mathcal{B}(W, V)$ (use the definition of J). By Lemma 6.3.10(g), it follows that $B \in$

$\mathcal{B}(U,V)$. From equation

$$(s-A_L)^{-1}B_L = (s-A)^{-1}B(I-L\mathbb{D}(\alpha))^{-1} \in \mathcal{B}(U,W), \tag{6.158}$$

one can compute that $B_L(I-LD) = B$ (and $B_L \in \mathcal{B}(U,V_L)$, by Lemma 6.3.10(g)).

If $I-LD \in \mathcal{GB}(U)$, then we get the final two claims by exchanging the roles of Σ_L and Σ (note that then $JJ_L = I_{V_L}$, hence then $V' = V_L$, i.e., $V = V_L$).

3° *Remark:* We do not know whether necessarily $(C_L)_{\mathrm{w}} = (C_L)_{\mathrm{c}} := (I-DL)^{-1}_{\mathrm{left}}C_{\mathrm{c}}$, or equivalently (by Lemma 6.3.10(d3)), $D_L = (D_L)_{\mathrm{c}}$, when Σ and Σ_L are WR and $I-DL \in \mathcal{GB}(Y)$. Of course, this is not the case in (6.142), where "WR" is replaced by "SR".

(b2) This follows from (b1), except for $B_L^* = (I-D^*L^*)^{-1}B_{\mathrm{w}}^*$, which is from (c2) applied to $\Sigma^{\mathrm{d}} \in \mathrm{WR}(Y,H,U)$ and L^*, giving $(B_L^*)_{\mathrm{s}} = (I-D^*L^*)^{-1}B_{\mathrm{w}}^*$ on $\mathrm{Dom}((B_L^*)_{\mathrm{s}})$.

(As noted below the proposition, Σ_L need not be WR even if Σ is WR and $I-LD \in \mathcal{GB}(U)$.)

(b3) (Note that $\mathrm{Dom}(A) =: H_1$ and $\mathrm{Dom}(A_L)$ are in general proper subsets of H_B; the notion "$A \in \mathcal{B}(H_B,V)$" in (b3) refers to $A|_{H_B}$, where A stands for, e.g., its extension to H (see Lemma 6.1.16(a)). The same applies for A_L.)

We prove the case $I-DL \in \mathcal{GB}(Y)$ below; the rest follows from (b2) and Proposition 6.3.1(a1).

1° *The proof:* We fix $\alpha \in \sigma(A)^c \cap \sigma(A_L)^c$, so that $(\alpha - A) \in \mathcal{GB}(H_k,H_{k-1})$ ($k \in \mathbf{Z}$). Let W be the closure of H_1 in $\mathrm{Dom}(C_{\mathrm{s}})$ (equivalently, in $\mathrm{Dom}((C_L)_{\mathrm{s}})$, by (c4)). Since H_1 is dense in $\mathrm{Dom}(C_{\mathrm{L,s}})$, and $\mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{s}})$, we have

$$H_1 \underset{c}{\subset} H_B \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} W \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{s}}) \underset{c}{\subset} H \underset{c}{\subset} V := (\alpha - A)W \underset{c}{\subset} H_{-1}. \tag{6.159}$$

Analogously, we see that W is the closure of $\mathrm{Dom}(A)$ in $\mathrm{Dom}((C_L)_{\mathrm{s}}) = \mathrm{Dom}(C_{\mathrm{s}})$. Now H is dense in V, because H_1 is dense in W. Therefore, J in (b1) becomes the unique continuous extension of $I : H \to H$ to $V \to V_L$.

This way, we have $C_{\mathrm{c}} := C_{\mathrm{s}}|_W$, $(I-DL)^{-1}C_{\mathrm{c}} = (C_L)_{\mathrm{s}}|_W$, $(I-DL)^{-1}D$ becomes the feedthrough operator of $\mathbb{D}_L$, and we obtain (b3) from the formulae of (b1) and (c4) (note from the proof of (b1) that now $V = V_L$ and $B, B_L \in \mathcal{B}(U,V)$).

2° *An equivalent construction from [W94b], pp. 52–56:* G. Weiss shows that, for each $x_0 \in V$, the element $Jx_0 := V_L\text{-}\lim_{r\to+\infty} r(r-A)^{-1}x_0 \in V_L$ exists. Conversely $J^Lx_0 := V\text{-}\lim_{r\to+\infty} r(r-A_L)^{-1}x_0 \in V$ exists for each $x_0 \in V_L$ and $J = (J^L)^{-1} \in \mathcal{GB}(V,V_L)$. He then shows that J and J^L are (unique) continuous extensions of $I : H \to H$. By density, they must be equal to the extensions of 1° (i.e., of (b1)). It follows that

$$x_0 = \lim_{r\to+\infty} r(r-A)^{-1}x_0 = \lim_{r\to+\infty} r(r-A_L)^{-1}x_0 \quad (x_0 \in V = V_L) \tag{6.160}$$

(both limits converge in both H_{-1} and H_{-1}^L).

By Proposition 6.2.8(e), we also have $B \in \mathcal{B}(U, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$. We do not know whether $\mathrm{Dom}(B_{\mathrm{L,s}}^*)^*$ (and $\mathrm{Dom}((B_L^*)_{\mathrm{L,s}})^*$) is always contained in V.

(b4) By Proposition 6.3.1(b1), the admissibility of L implies that Σ_L is uniformly regular and $I - DL \in \mathcal{GB}$; the rest follows from (b3).

(c1) This follows from Proposition 6.3.1(a3) (and Lemma 6.2.5).

(c2) (Here "DL" denotes the feedthrough operator of $\mathbb{D}L$; if $\mathbb{D} \in \mathrm{WR}$, then, obviously, $DL = D \cdot L$).

We use the argument of Proposition 7.1 of [W94b] (which contains (c3) and (c4)):

For $x_0 \in \mathrm{Dom}((C_L)_\mathrm{s})$, by Lemma 6.2.12, the limit

$$C_\mathrm{w} x_0 = \text{w-lim}\, Cs(s-A)^{-1}x_0 = \text{w-lim}\, s\widehat{\mathbb{C}x_0}(s) \tag{6.161}$$

$$= \text{w-lim}\, s\widehat{(I-\mathbb{D}L)\mathbb{C}_L x_0}(s) = \text{w-lim}\, s(I - \widehat{\mathbb{D}}(s)L)\widehat{\mathbb{C}_L x_0}(s) \tag{6.162}$$

$$= \text{w-lim}(I - \widehat{\mathbb{D}}(s)L)C_L s(s-A_L)^{-1}x_0 = (I-DL)(C_L)_\mathrm{s} x_0 \tag{6.163}$$

exists as $s \to +\infty$ (because $C_L s(s-A_\circlearrowleft)^{-1}x_0 \to (C_L)_\mathrm{s} x_0$ strongly and $(I - \widehat{\mathbb{D}}(s)L) \to I - DL$ weakly; see also Lemma A.3.1(j2)), hence C_w is an extension of $(I-DL)(C_L)_\mathrm{s}$. Because $I - DL$ is left-invertible, we obtain (c1).

(c3) By Proposition 6.3.1(a1), $I - DL$ has a left inverse. The second claim is obtained exactly as in (c2) (with strong limits).

(c4) The $(C_L)_\mathrm{s}$ formula follows from (c3) and (c1) (and Lemma 6.6.3); in particular, $\mathrm{Dom}((C_L)_\mathrm{s}) = \mathrm{Dom}(C_\mathrm{s})$. By Lemma A.3.6, the domains must have equal norms (since both $\underset{c}{\subset} H$). The $(B_L^*)_\mathrm{s}$ formula follows from this (as in the proof of (b2)).

(c5) By Proposition 6.3.1(a1), $(I-DL)^*$ has a left inverse, i.e., $I - DL$ has a right inverse, hence $I - DL \in \mathcal{GB}(Y)$. Now the proof of (c2) shows that $C_\mathrm{w} x_0 = (I-DL)(C_L)_\mathrm{w} x_0$ for all $x_0 \in \mathrm{Dom}((C_L)_\mathrm{w})$, hence $(C_L)_\mathrm{w} \subset C_\mathrm{w}$.

But $((I-\mathbb{D}L)^{-1})^\mathrm{d} = ((I-\mathbb{D}L)^\mathrm{d})^{-1} \in \mathrm{SR}$, by Proposition 6.3.1(a3), hence one can analogously verify that $C_\mathrm{w} \subset (C_L)_\mathrm{w}$.

(c6) By Theorem 2.6.4(c2) (shifted), $\mathbb{D}L \in \mathrm{MTIC}_\infty \Leftrightarrow \mathbb{E} := (I-\mathbb{D}L)^{-1} \in \mathrm{MTIC}_\infty$; assume that this holds and that $\mathbb{D}$ is SR.

It follows that $\mathbb{E} \in \mathcal{G}\mathrm{ULR}$, $E \in \mathcal{GB}(Y)$ and $I - \mathbb{D}L = \mathbb{E}^{-1} \in \mathrm{ULR}$, by Proposition 6.3.1(c)&(a3), so that $I - DL = E^{-1} \in \mathcal{GB}(Y)$ and $\mathbb{D}_L \in \mathrm{SR}$. Set $g* := \mathbb{E} - E$. Note that $\mathbb{C}_L = \mathbb{E}\mathbb{C} = E\mathbb{C} + g*\mathbb{C}$.

Choose $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$. Then $f := \mathbb{C}x_0 \in \mathrm{L}^2_\infty$ and $\frac{1}{t}\int_0^t f\,dm =: y_0 + \varepsilon(t) \to y_0$ as $t \to 0+$, where $y_0 := EC_{\mathrm{L,s}} x_0 \in Y$, by Proposition 6.2.8(c4)&(c3). If $g \in \mathrm{L}^1_\infty$, then, by the Fubini Theorem, we have

$$\frac{1}{t}\int_0^t g*\mathbb{C}x_0\,dm = \frac{1}{t}\int_0^t \int_0^s g(r)f(s-r)\,dr\,ds = \frac{1}{t}\int_0^t g(r)\int_r^t f(s-r)\,ds\,dr \tag{6.164}$$

$$= \int_0^t g(r)\frac{t-r}{t}(y_0 + \varepsilon(t-r))\,dr \to 0, \tag{6.165}$$

as $t \to 0+$. Analogously, if $\sum_k \|E_k\|_{\mathcal{B}(Y)} < \infty$, and $T_k > 0$ for all $k \in \mathbf{N}$, then

$$\|\frac{1}{t}\int_0^t \sum_k E_k \tau^{-T_k} f\|_Y \leq \sum_k \|E_k\| \|\frac{t-T_k}{t}(y_0 + \varepsilon(t-T_k))\|_Y \to 0, \tag{6.166}$$

as $t \to 0+$, where the sum runs over $\{k \in \mathbf{N} \mid T_k < t\}$. From (6.164) and (6.166) we obtain that $\frac{1}{t}\int_0^t (\mathbb{E}-E) f\, dm \to 0$, as $t \to 0+$. Because $\frac{1}{t}\int_0^t E f\, dm \to E y_0$, as $t \to 0+$, this implies that $(C_L)_{\mathrm{L,s}} x_0$ exists and is equal to $y_0 = E C_{\mathrm{L,s}} x_0$, by Proposition 6.2.8(c4)&(c3).

Since $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ was arbitrary, we have $C_{\mathrm{L,s}} \subset E(C_L)_{\mathrm{L,s}}$. Exchange the roles of Σ and Σ_L to obtain that $(C_L)_{\mathrm{L,s}} \subset E^{-1} C_{\mathrm{L,s}}$. The claim on B_L^* follows by duality.

(d1) This follows from (b1), except for the $\widehat{\mathbb{X}}$, $\widehat{\mathbb{M}}$ and $\mathbb{K}_{\circlearrowleft}$ formulae, which are from Lemma 6.3.10(a) and Lemma 6.2.12 (we have $(K_\flat)_{\mathrm{s}} \subset K_{\mathrm{w}}$ $[(K_\flat)_{\mathrm{s}} = K_{\mathrm{w}}]$, by (c2) [(c4)]). Note that the "$(I-DL)^{-1}_{\mathrm{left}}$" of (b1) is given by $(I - \left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & I \end{smallmatrix}\right])^{-1}_{\mathrm{left}} = \left[\begin{smallmatrix} I & DM \\ 0 & M \end{smallmatrix}\right]$.

(d2) This follows from (d1).

(d3) If $\mathbb{X} \in \mathrm{SR}$, then $\mathbb{M} \in \mathrm{SR}$, by Proposition 6.3.1(a3), hence then $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{WR}$. The rest follows from (b1), (c2) and (d1). (Note that (6.145)$\in \mathcal{B}(H_B \times U, V \times Y \times U)$, by (b1).)

(d4) Now $Ks(s-A)^{-1} = s\widehat{\mathbb{X}}(s) K_\flat (s-A_\flat)^{-1}$ and $C_\flat s(s-A_\flat)^{-1} = Cs(s-A)^{-1} + \widehat{\mathbb{D}}(s)\widehat{\mathbb{M}}(s) Ks(s-A)^{-1}$ (cf. (c2)); the claims follow from these equations and Proposition 6.3.1(a3) (because $H_B \subset \mathrm{Dom}(K_{\mathrm{s}})$ etc.).

(e) (In fact, without any regularity assumptions on $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, we have the equality on $\mathrm{Dom}((K_\flat)_{\mathrm{s}})$; i.e., for $x_0 \in \mathrm{Dom}((K_\flat)_{\mathrm{s}})$ we have $x_0 \in \mathrm{Dom}(S_{\mathrm{s}}) \Leftrightarrow x_0 \in \mathrm{Dom}(S_{\mathrm{s}}^A)$ with equal values on domains, as one easily observes from the proof.)

Set $S_{\mathrm{w}} := \text{w-lim}_{s \to +\infty} Ss(s-A_\flat)^{-1}$, $S_{\mathrm{w}}^A := \text{w-lim}_{s \to +\infty} S_{\mathrm{w}} s(s-A)^{-1}$.

Let $x_0 \in H_B$. Then, by (a2), we have

$$S_{\mathrm{w}} s(s-A)^{-1} x_0 = Ss(s-A_\flat)^{-1} x_0 + S_{\mathrm{w}}(s-A)^{-1} B K_\flat s(s-A_\flat)^{-1} x_0 \rightharpoonup S_{\mathrm{w}} x_0 + 0, \tag{6.167}$$

because $S_{\mathrm{w}}(s-A)^{-1} B \rightharpoonup 0$ and $K_\flat s(s-A_\flat)^{-1} x_0 \to (K_\flat)_{\mathrm{s}} x_0$ (see also Lemma A.3.1(j2)). Therefore, $x_0 \in \mathrm{Dom}(S_{\mathrm{w}}^A)$ and $S_{\mathrm{w}}^A x_0 = S_{\mathrm{w}} x_0$. The SR case is analogous.

(f) Recall from (a1) that "H_B" is the same for all four systems. From Lemma 6.7.12 we obtain (6.146) and (6.147).

Because $s\widehat{\mathbb{K}}'(s) x_0 \rightharpoonup (K_\natural)_{\mathrm{w}} x_0 + I K_{\mathrm{s}} x_0)$, as $s \to +\infty$, for all $x_0 \in H_B$ (even for all $x_0 \in \mathrm{Dom}(K_{\mathrm{s}}) \cap \mathrm{Dom}((K_\natural)_{\mathrm{w}})$), the operator $\mathbb{F}'$ is WR and $K_{\mathrm{w}}' = K_{\mathrm{s}} + (K_\natural)_{\mathrm{w}}$ on H_B (even on $\mathrm{Dom}(K_{\mathrm{s}}) \cap \mathrm{Dom}((K_\natural)_{\mathrm{w}})$). Thus, $K' = (K_{\mathrm{w}}')|_{\mathrm{Dom}(A)} = K + (K_\natural)_{\mathrm{w}}|_{\mathrm{Dom}(A)}$.

By (6.146) and Lemma 6.2.5, we have $F' = F + F_\natural - F_\natural F = 0$ (since $\mathbb{F}$ is SR). Recall from Lemma 6.2.5 that strong and uniform regularity properties are preserved under composition and vector operations.

(g) By Lemma 6.3.10(d3), $\left[\ \mathbb{K}' \ \middle|\ \mathbb{F}'\ \right] = \left[\ \mathbb{K}'' \ \middle|\ \mathbb{F}''\ \right]$ iff $K' = K''$ on H_B. Obviously, either implies that $\mathbb{K}'_\flat = \mathbb{K}''_\flat$.

Assume then that $\mathbb{K}'_\flat = \mathbb{K}''_\flat$. Then $\mathbb{A}'_\flat = \mathbb{A} + \mathbb{B}\tau\mathbb{K}'_\flat = \mathbb{A}''_\flat$, and $\mathbb{C}'_\flat = \mathbb{C} + \mathbb{D}\tau\mathbb{K}'_\flat = \mathbb{C}''_\flat$, hence $A'_\flat = A''_\flat$, $C'_\flat = C''_\flat$ and $K'_\flat = K''_\flat$. By (6.144), we have $\Sigma'_\flat = \Sigma''_\flat$. Since $-K'_\flat = -K''_\flat$ is admissible for $\Sigma'_\flat$, we obtain that $\left[\ \mathbb{K}' \ \middle|\ \mathbb{F}'\ \right] = \left[\ \mathbb{K}'' \ \middle|\ \mathbb{F}''\ \right]$, by Lemma 6.6.14. □

The formulae between different operators in Proposition 6.6.18 are straightforward as long as we stay in H (which obviously includes the domains and extended domains of C, K and B^*). However, when write formulae for B and B_L, or for A and A_L with input space such as H_B, larger than the domain of the original operator, we face the fact that $H_{-1} \neq H^L_{-1}$ in general.

Thus, these formulae depend on how we identify a part of H_{-1} (containing $H + B[U]$) to a part of H^L_{-1} (containing $H + B_L[U]$). We make the following remark on this:

Remark 6.6.20 ($B = B_\flat$) *(1.) The identification between B and $B_\flat$ in Proposition 6.6.18(d2) (the compatible case without feedthrough, due to O. Staffans [Sbook] (or [SW00])) is based on an isometric isomorphism $J : (\alpha - A)W \to (\alpha - A_\flat)W$ (see the proof of (b1)), which is a continuous extension of $I : H \to H$; here $W \supset H_B$ is the domain of $\begin{bmatrix} C_c \\ K_c \end{bmatrix}$.*

If H_1 is dense in W, then H is dense in $(\alpha - A)W$ and $(\alpha - A_\flat)W$, and thus this extension is unique. If this is not the case, then Σ has several compatible pairs for the same W, and the identification depends on the pair chosen. Thus, when applying the formula $B = B_\flat$, one should write out the spaces $(\alpha - A)W$ and $(\alpha - A_\flat)W$ for the system under study; in several applications it is completely natural to identify them.

If $I - F$ is merely left-invertible, then we can still embed $(\alpha - A)W$ into $(\alpha - A_\flat)W$, still so that the embedding is a continuous extension of $I : H \to H$, and we obtain a weaker connection between the operators, as shown in (d1) (or (b1)).

(2.) The identification in the SR case (e.g., Proposition 6.6.18(d4)), due to G. Weiss [W94b], is based on a rather natural identification; the space W has been chosen so that H_1 is dense in W. Of course, instead of V, one could choose to close H w.r.t. a different norm than that of V and try to obtain a space containing $H + BU$ and embeddable into H_{-1}.

Still, we recommend the reader to always check the meaning of V before applying the formula $B = B_\flat$ (or the formula for $A_\flat$ outside $\mathrm{Dom}(A_\flat)$).

Naturally, the case for static output feedback is analogous. See the proofs of (b1) and (b3) for details on the two identifications mentioned above.

As observed in the proof of (b3), the identification of Weiss is a special case of the identification of Staffans. □

If Σ can be stabilized by output injection (i.e., by feeding the output back to the system through an extra input, as in Figure 6.5, then we call Σ detectable (we use duality to avoid a longer treatment analogous to that for state feedback stabilizability):

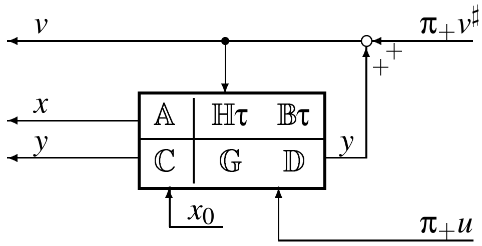


Figure 6.5: Output injection connection

Definition 6.6.21 (Detectability) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & | & \mathbb{B} \\ \hline \mathbb{C} & | & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. *A pair* $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ *is an* admissible output injection pair *for* Σ *if the extended system* $\left[\begin{smallmatrix} \mathbb{A} & | & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & | & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$ *is a* WPLS *and* $I - \mathbb{G} \in \mathcal{G}\mathrm{TIC}_\infty$ *(i.e.,* $L = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ *is admissible for* $\left[\begin{smallmatrix} \mathbb{A} & | & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & | & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$*).*

Such a pair $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ *is* stabilizing*if the resulting closed loop system*

$$\Sigma_\sharp := \left[\begin{array}{c|cc} \mathbb{A} + \mathbb{H}\tau\widetilde{\mathbb{M}}\mathbb{C} & \mathbb{H}\widetilde{\mathbb{M}} & \mathbb{B} + \mathbb{H}\widetilde{\mathbb{M}}\mathbb{D} \\ \hline \widetilde{\mathbb{M}}\mathbb{C} & \widetilde{\mathbb{M}} - I & \widetilde{\mathbb{M}}\mathbb{D} \end{array}\right] \tag{6.168}$$

(here $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$*) is stable, i.e., if* L *is stabilizing; in this case, we call* Σ detectable.

If $\left[\begin{smallmatrix} A & | & \mathtt{H} \\ \hline C & | & 0 \end{smallmatrix}\right]$ *generate a WR WPLS* $\left[\begin{smallmatrix} \mathbb{A} & | & \mathbb{H} \\ \hline \mathbb{C} & | & \mathbb{G} \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ *is an admissible (resp. stabilizing) output injection pair for* Σ*, then* $\mathtt{H}$ *is called a* WR admissible (resp. stabilizing) output injection operator *for* Σ.

Admissible output injection and state feedback pairs $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ *and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *are called* jointly admissible *for* Σ *if they are parts of a single* WPLS

$$\Sigma_{\mathrm{Total}} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \\ \mathbb{K} & \mathbb{E} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}(Y \times U, H, Y \times U) \tag{6.169}$$

(with some interaction operator $\mathbb{E} \in \mathrm{TIC}_\infty(Y,U)$*). If the closed-loop systems of* Σ_{Total} *corresponding to* $L = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ *and* $\widetilde{L} = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *are stable, then the pairs are* jointly stabilizing *and* Σ *is* jointly stabilizable and detectable.

We use prefices and suffices as in Definitions 6.6.10 and 6.6.4 above (e.g., strongly l.c.-detectable *means having a admissible pair* $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ *s.t.* $\Sigma_\sharp$ *is strongly stable and* $\widetilde{\mathbb{M}}$ *and* $\widetilde{\mathbb{M}}\mathbb{D}$ *are l.c.). Prefices preceding the word "jointly" apply to both closed-loop systems (hence to both pairs).*

The two closed-loop systems corresponding to jointly admissible pairs are

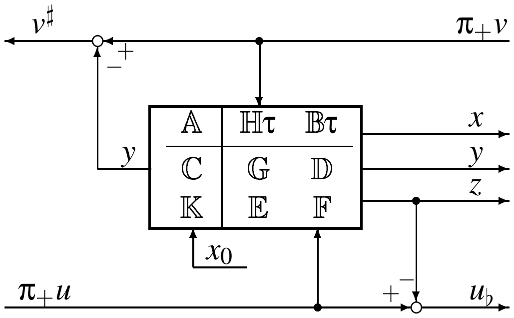


Figure 6.6: The extended system Σ_{Total}

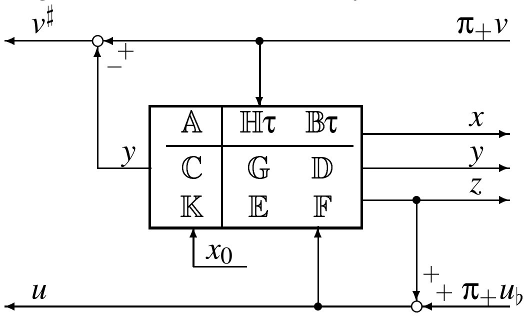


Figure 6.7: The closed-loop system $(\Sigma_{\text{Total}})_L$

easily seen to be given by

$$(\Sigma_{\text{Total}})_L = \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}\tau(I-\mathbb{F})^{-1}\mathbb{K} & \mathbb{H}+\mathbb{B}(I-\mathbb{F})^{-1}\mathbb{E} & \mathbb{B}(I-\mathbb{F})^{-1} \\ \hline \mathbb{C}+\mathbb{D}(I-\mathbb{F})^{-1}\mathbb{K} & \mathbb{G}+\mathbb{D}(I-\mathbb{F})^{-1}\mathbb{E} & \mathbb{D}(I-\mathbb{F})^{-1} \\ (I-\mathbb{F})^{-1}\mathbb{K} & (I-\mathbb{F})^{-1}\mathbb{E} & (I-\mathbb{F})^{-1}-I \end{array}\right], \tag{6.170}$$

$$(\Sigma_{\text{Total}})_{\widetilde{L}} = \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{H}\tau(I-\mathbb{G})^{-1}\mathbb{C} & \mathbb{H}(I-\mathbb{G})^{-1} & \mathbb{B}+\mathbb{H}(I-\mathbb{G})^{-1}\mathbb{D} \\ \hline (I-\mathbb{G})^{-1}\mathbb{C} & (I-\mathbb{G})^{-1}-I & (I-\mathbb{G})^{-1}\mathbb{D} \\ \mathbb{K}+\mathbb{E}(I-\mathbb{G})^{-1}\mathbb{C} & \mathbb{E}(I-\mathbb{G})^{-1} & \mathbb{F}+\mathbb{E}(I-\mathbb{G})^{-1}\mathbb{D} \end{array}\right]. \tag{6.171}$$

Any [strongly/exponentially] stable Σ is [strongly/exponentially] jointly r.c.-stabilizable and l.c.-detectable (take $\mathbb{F} = 0 = \mathbb{G} = \mathbb{K} = \mathbb{H} = \mathbb{E}$). Because of the duality explained in Lemma 6.7.2, we omit most left definitions and left results.

Detectability means roughly that the unstable parts of the system can be detected in the output. Observability means that the whole state space can be observed in the output. The latter condition sounds stronger, and, indeed, an observable system is exponentially detectable, by p. 91 of [LR], if $\dim H < \infty$. In general this is not the case (recall the word "roughly" above): for any $\mathbb{D} \in \mathrm{TIC}(U,Y)$, the (infinite-dimensional) system (6.11) is stable and exactly 0-observable (in infinite time) but not even estimatable (see Definition 6.7.3).

Detectability does not imply observability even for $\dim H < \infty$ (just take any exponentially stable $\mathbb{A}$ with $\mathbb{C} = 0$). By duality, the relations between reachability and stabilizability are analogous.

Next we shall list a few facts on joint stabilizability and detectability. We start with a rather obvious remark:

Lemma 6.6.22 (Jointly stabilizing ⇒ separately) *If* $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ *and* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *are jointly admissible [stabilizing] for some WPLS* Σ*, then* $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ *and* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *are admissible [stabilizing] for* Σ*; all prefices and suffices apply.*

Proof: 1° *Admissibility:* Joint admissibility is obviously equivalent to the invertibility of $I - \mathbb{F}$ and $I - \mathbb{G}$ in TIC_∞, hence the admissibility claim holds.

2° *Stabilization:* Compare (6.170) to (6.133) and (6.171) to (6.168) to obtain the claim in brackets and the claim on prefices and suffices (note that by "r.c.-" we refer to $\mathbb{M} := (I - \mathbb{F})^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ and by "l.c.-" to $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$ and $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$, is in Definitions 6.6.10 and 6.6.21). □

The converse is not true:

Example 6.6.23 (Stabilizing but not jointly stabilizing) If A, B and C are as in Example 6.3.24, then the output injection pair (right column) in $\left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right)$ and the state feedback pair (lower row) in $\left(\begin{array}{c|c} A & 0 \\ \hline C & 0 \end{array}\right)$ are admissible for $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$, by Lemma 6.3.16 and Proposition 6.3.1(c), but they are not jointly admissible, by Example 6.3.24 (there is no suitable interaction operator $\mathbb{E}$).

By replacing A by $A + \omega$ for a suitable $\omega \in \mathbf{R}$ (see Remark 6.1.9), we can make the two pairs stabilizing but yet not jointly stabilizing (not even jointly admissible). ◁

Nevertheless, we do not know whether there is a system that is [exponentially] stabilizable and detectable but not [exponentially] jointly stabilizable and detectable. [Neither do we know whether optimizability and estimatability (see Definition 6.7.3) is strictly weaker than the above two conditions. All these conditions are equivalent if the system is sufficiently regular, by Theorem 7.2.4(c).]

Lemma 6.6.24 *We have "exponentially jointly stabilizable and detectable" = "jointly exponentially stabilizable and detectable" and "strongly jointly stabilizable and detectable" = "jointly strongly stabilizable and detectable".*

Proof: Let $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ be jointly stabilizing, so that $I - (\widetilde{L} - L)\mathbb{D}_{\mathrm{Total}} = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ in terms of (6.172). By Lemma 6.6.3, $((\Sigma_{\mathrm{Total}})_L)_{\widetilde{L}-L} = (\Sigma_{\mathrm{Total}})_{\widetilde{L}}$, hence $(\Sigma_{\mathrm{Total}})_L$ is [exponentially] strongly stable iff $(\Sigma_{\mathrm{Total}})_{\widetilde{L}}$ is [exponentially] strongly stable, by Corollary 6.6.9. □

If C is boundedly left-invertible, then one can detect everything about the state from the output; in particular:

Lemma 6.6.25 *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *be s.t.* $TC = I$ *for some* $T \in \mathcal{B}(Y,H)$*. Then* Σ *is exponentially detectable.*

In standard stabilization problems we have $C = \left[\begin{smallmatrix} I \\ * \end{smallmatrix}\right]$, so that one can choose $T := \begin{bmatrix} I & 0 \end{bmatrix}$.

Proof: Take $\mathtt{H} := -rT \in \mathcal{B}(Y,H)$, $G = 0$, $r > \omega$. Then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} \\ \hline \mathbb{C} & \mathbb{G} \end{smallmatrix}\right]$ is well-posed, by Lemma 6.3.16(b), hence so is $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$, and $\mathbb{G}$ is ULR. Moreover, the generator of $\mathbb{A}_\sharp$ is $A + \mathtt{H}C_s = A - rI$, by Proposition 6.6.18(b3), hence $\mathbb{A}_\sharp$ is exponentially stable, by Lemma A.4.2(g2). □

In the classical case, any admissible state feedback and output-injection pairs are jointly admissible:

Lemma 6.6.26 (K and $\mathtt{H}$ jointly) *Let $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ be admissible (resp. exponentially stabilizing) pairs for Σ.*

If K or $\mathtt{H}$ is bounded, then $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly admissible (resp. jointly exponentially r.c.- and l.c.-stabilizing).

Proof: Let K be bounded (the other case is analogous). Then $\begin{bmatrix} K \mid 0 & F \end{bmatrix}$ extends $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$ to a WPLS, by Lemma 6.3.16(c), hence $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly admissible (resp. and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is exponentially r.c.-stabilizing and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ is exponentially l.c.-stabilizing, by Theorem 6.6.28 (and Remark 6.1.9). □

In the uniformly regular case, jointly stabilizing pairs can be replaced by jointly stabilizing K and $\mathtt{H}$:

Lemma 6.6.27 *If $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly stabilizing for $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ $\mathrm{WPLS}(U,H,Y)$ with interaction operator $\mathbb{E}$, then so are $\begin{bmatrix} R\mathbb{K} \mid I - R(I-\mathbb{F}) \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ with $R\mathbb{E}$, and so are also $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ and $\begin{bmatrix} \mathbb{H}S \mid I - (I-\mathbb{G})S \end{bmatrix}$ with $\mathbb{E}S$, for any $R \in \mathcal{GB}(U)$, $S \in \mathcal{GB}(Y)$. All prefices and suffices apply.*

In particular, if Σ_{Total} is UR, then we can have $F = 0$, $G = 0$ and $E = 0$.

Proof: 1° We observe from equations (6.170)–(6.171) that corresponding closed-loop systems are equal except that $\mathbb{B}_L \mapsto \mathbb{B}_L R^{-1}$, $\mathbb{D}_L \mapsto \mathbb{D}_L R^{-1}$ and $\mathbb{M}_L \mapsto \mathbb{M}_L R^{-1}$ (hence $\mathbb{F}_L \mapsto I - (I - \mathbb{F}_L)R^{-1} = \mathbb{F}_L R^{-1} + I - R^{-1}$), and $(\Sigma_{\text{Total}})_{\widetilde{L}}$ is affected analogously.

2° If Σ_{Total} is UR, then $I - F, I - G \in \mathcal{GB}$, by Proposition 6.3.1(b1), hence we can have $F = 0$ and $G = 0$ (take $R = (I-F)^{-1}$, $S = (I-G)^{-1}$ in 1°). Moreover, if we replace $\mathbb{E}$ by $\mathbb{E} - E$, then we add the elements of the third column of (6.170) plus $\left[\, 0 \mid 0 \mid I \,\right]^{\mathrm{T}}$, times $-E$, to the second column of (6.170), hence the stability of (6.170) is not affected. An analogous claim holds for (6.171). Thus, the new pairs (those with $F = 0$, $G = 0$) with $\mathbb{E} - E$ are stabilizing exactly in the same sense as the old ones. □

We recall the main result of [S98a]:

Theorem 6.6.28 (d.c.f. ⇔ jointly) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then the following are equivalent:*

(i) $\mathbb{D}$ has a d.c.f.;

(ii) $\mathbb{D}$ has a jointly I/O-stabilizable and I/O-detectable realization;

(iii) $\mathbb{D}$ has a strongly jointly r.c.-stabilizable and l.c.-detectable realization.

Moreover, any jointly I/O-stabilizing pairs (if any) for any realization of $\mathbb{D}$ are jointly r.c.- and l.c.-I/O-stabilizing.

Recall from the definition that (iii) means that both (6.170) and (6.171) become strongly stable and that $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is a l.c.f., where $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$, $\widetilde{\mathbb{M}} := (I-\mathbb{G})^{-1}$.

If Σ is the corresponding realization and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ corresponds to a stabilizing SR $K \in \mathcal{B}(H_1, U)$, then $\widehat{\mathbb{M}}(s) = I + K_s(s-A)^{-1}B$ and $\widehat{\mathbb{N}}(s) = D + (C_s + DK_s)(s-A)^{-1}B$ are SR, by Proposition 6.6.18(b3). An analogous claim obviously holds for the whole d.c.f. when also $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ corresponds to a SR operator. (If K and H are merely WR, then the same formulae can be applied but the factorization need not be WR.) See Section 5 of [WC] and Section 3 of [CWW96] for analogous results (in the SR exponentially stabilizing case, which can be obtained by shifting the above result).

Proof: This follows from the proof of Theorem 4.4 of [S98a]; we sketch the proof below.

1° *"(iii)⇒(ii)":* This is trivial.

2° *"(ii)⇒(i)":* Assume (ii) and denote the I/O maps of (6.170) and (6.170) by $\begin{bmatrix} \mathbb{G}_L & \mathbb{D}_L \\ \mathbb{E}_L & \mathbb{F}_L \end{bmatrix}$ and $\begin{bmatrix} \mathbb{G}_{\widetilde{L}} & \mathbb{D}_{\widetilde{L}} \\ \mathbb{E}_{\widetilde{L}} & \mathbb{F}_{\widetilde{L}} \end{bmatrix}$, respectively. Then the $\mathrm{TIC}(U \times Y)$ maps

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = \begin{bmatrix} I+\mathbb{F}_L & -\mathbb{E}_L \\ \mathbb{D}_L & I-\mathbb{G}_L \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I-\mathbb{F}_{\widetilde{L}} & \mathbb{E}_{\widetilde{L}} \\ -\mathbb{D}_{\widetilde{L}} & I+\mathbb{G}_{\widetilde{L}} \end{bmatrix} \tag{6.172}$$

are inverses of each other; here $\Sigma_\sharp$ is the system (6.168) (see [S98a] for details (or use direct computation); note that there $\mathbb{Y} =$"$\mathbb{E}_\flat$"$= -\mathbb{E}_L$ and $\mathbb{Y} =$"$\mathbb{E}_\sharp$"$= -\mathbb{E}_{\widetilde{L}}$ because of different signs in the Bezout equations of [S98a]). Thus, if the pairs are jointly I/O-stabilizing, then (6.172) defines a d.c.f. of $\mathbb{D}$ (and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are jointly r.c.- and l.c.-I/O-stabilizing). Consequently, we have (i) holds.

2° *"(i)⇒(iii)":* We work as in Lemma 6.6.29: Let (6.109) be a d.c.f. of $\mathbb{D}$. Start with, e.g., a strongly stable realization (say, Σ_L) of

$$\begin{bmatrix} \mathbb{G}_L & \mathbb{D}_L \\ \mathbb{E}_L & \mathbb{F}_L \end{bmatrix} := \begin{bmatrix} I-\mathbb{X} & \mathbb{N} \\ -\mathbb{Y} & \mathbb{M}-I \end{bmatrix} \tag{6.173}$$

(cf. Definition 6.1.6), where Close it with the output feedback operator $L' := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$ (which is admissible since $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$) to obtain a WPLS $\Sigma_{\mathrm{Total}} := (\Sigma_L)_{L'}$; drop the middle column and bottom line of Σ_{Total} to obtain $\Sigma \in \mathrm{WPLS}(U, H, Y)$.

The additional operators in Σ_{Total} constitute strongly jointly stabilizing and detecting pairs and an interaction operator as in Definition 6.6.21: indeed, $(\Sigma_{\mathrm{Total}})_{-L'} = \Sigma_L$, and, by Lemma 6.6.3, (6.171) is strongly stable iff

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I-\mathbb{X} & \mathbb{N} \\ -\mathbb{Y} & \mathbb{M}-I \end{bmatrix} = \begin{bmatrix} \mathbb{X} & -\mathbb{N} \\ -\mathbb{Y} & \mathbb{M} \end{bmatrix} \tag{6.174}$$

is in $\mathcal{G}$TIC, by Corollary 6.6.9, and this is the case, by assumption (see (6.109)). □

By same methods, we obtain the following:

Lemma 6.6.29 *If(f) $\mathbb{D} \in \mathrm{TIC}_\infty$ has a right factorization [[q.]r.c.f.] $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, then $\mathbb{D}$ has a strongly [[q.]r.c.-]stabilizable realization.*

The converse is also true, by Definition 6.6.10. If $\mathbb{M} \in \mathrm{UR}$ or $\mathbb{M}, \mathbb{M}^{-1} \in \mathrm{SR}$, we can take $M = I$ above, by Proposition 6.3.1(a3)&(b1) and Lemma 6.4.5, i.e., then the feedback pair $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ in the proof below has no feedthrough: $\left[\begin{array}{c|c} K & I - M^{-1} \end{array}\right] = \left[\begin{array}{c|c} K & 0 \end{array}\right]$.

Proof: Take a strongly stable realization $\Sigma_\flat$ of $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M}-I \end{smallmatrix}\right]$ (e.g. $\Sigma_\flat := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]\pi_- \\ \hline I & \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M}-I \end{smallmatrix}\right] \end{array}\right] \in \mathrm{WPLS}_0(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y)$), and close it with $L := \begin{bmatrix} 0 & -I \end{bmatrix}$ to get a realization

$$\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right] := (\Sigma_\flat)_L \in \mathrm{WPLS}. \tag{6.175}$$

By Lemma 6.6.3, $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ is strongly [[q.]r.c.-]stabilizing for $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$. □

Notes

All definitions are basically from [S98a]; the suffices, some prefices (such as r.c.-stabilization) and the concept "compatible state feedback (or output injection) operator" are new. The definition of stabilizability in [WC] and [CWW96] is equivalent to the existence of a SR exponentially stabilizing state feedback operator; a dual remark applies to detectability.

Proposition 6.6.2 is [S98a, Proposition 3.2], which is a variant of [W94b, Proposition 3.6] (the semigroup part of this was contained already in [Sal87]).

Lemma 6.6.3 is [W94b, Remark 6.5]. Part of Lemmas 6.6.7 and 6.6.8 and Corollary 6.6.9 are based on Lemma 21 of [S97b] and on its proof.

Most of parts (a1)–(a3), (b3), (c1)–(c5) and (d4) of Proposition 6.6.18 are based on [W94b] or on the methods used in its proofs. Most of (b1) and (d1) are from [Mik97a]; the claims on B_L (including V and the corresponding identification) were added in [Sbook].

Example 6.6.19 is based on [SW01b]. Theorem 6.6.28 is essentially Theorem 4.4 of [S98a] (Theorems 3.2 and 3.4 of [CWW96] present an independent variant for regular WPLSs); the proof of Lemma 6.6.29 is analogous.

A classical WPLS reference for output feedback is [W94b], which contains the rudiments of static feedback and state feedback; the most complete reference at present is [Sbook, Chapters 7&8], whose results are partially contained in [S98a]. Most existing literature treats exponential stability rather than stability; however, the results on the latter always imply analogous results on the former, but the converse does not hold.

6.7 Further feedback results

Stability itself is nothing else than a more sluggish motion.

In this section, we present further results on feedback; especially on stabilizability. We also define and study optimizability and estimatability, which are weak forms of exponential stabilizability and exponential detectability, respectively.

For the reader to distinguish between the several stability concepts introduced above and in the next chapter, we give here a summary of all such concepts:

Summary 6.7.1 (Stabizability concepts) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\omega \in \mathbf{R}$.

(a) *[6.6.4] An operator* $L \in \mathcal{B}(Y,U)$ *is called a* stabilizing (static) output feedback operator *for* Σ *if* L *is admissible* ($I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$) *and the resulting closed-loop system* $\Sigma_L = \left[\begin{smallmatrix} \mathbb{A}_L & \mathbb{B}_L \\ \mathbb{C}_L & \mathbb{D}_L \end{smallmatrix}\right]$ *is stable.*

If such an L *exists, we call* Σ stabilizable by static output feedback.

Whenever L *is s.t.* $\mathbb{B}_L$ *is stable,* L *is called* $\mathbb{B}$-stabilizing *for* Σ*; the applies also to the other components of* Σ.

Even without reference to Σ, *we say that* L *is* stabilizing *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *when* L *is admissible and* $\mathbb{D} := \mathbb{D}(I - L\mathbb{D})^{-1}$ *is stable.*

(b) *[6.6.10] A pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is called a* stabilizing state feedback pair *for* Σ *if the extended system* Σ_{ext} *is a* WPLS *and* $L := \begin{bmatrix} 0 & I \end{bmatrix}$ *stabilizes* Σ_{ext}.

If such a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *exists, we call* Σ stabilizable.

When we add the prefix "[q.]r.c.-" *(resp. suffix* "in $\mathcal{A}$"*) (e.g.,* "$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-stabilizing"), we mean that* $\mathbb{N}$ *and* $\mathbb{M}$ *are [q.]r.c. (resp.* $\mathbb{N}, \mathbb{M} \in \mathcal{A}$*), where* $\mathbb{M} := (I - \mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$.

The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is* exponentially [q.]r.c.-stabilizing *for* Σ *if* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is exponentially stabilizing and* $\mathbb{N}$ *and* $\mathbb{M}$ *are exponentially [q.]r.c.; equivalently, if* $\mathcal{T}_\omega \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-stabilizing for* $\mathcal{T}_\omega \Sigma$ *for some* $\omega > 0$ *(see Remark 6.1.9).*

(c) *[6.6.21] A pair* $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *is an* stabilizing output injection pair *for* Σ *if* $\left[\begin{array}{c|c} \mathbb{H}^{\mathrm{d}} & \mathbb{G}^{\mathrm{d}} \end{array}\right]$ *is a stabilizing state feedback pair for* Σ.

If such a pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *exists, we call* Σ detectable. *We use prefices (e.g., "[q.]l.c.") as in (b).*

(d) *[6.6.21] The output injection and state feedback pairs* $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *are called* jointly stabilizing *for* Σ *if they are part of a single* WPLS *(6.169) and both* $L = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ *and* $L = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *stabilize this* WPLS.

If such pairs exist, we call Σ jointly stabilizable and detectable.

By dynamic feedback (DF) we mean output feedback similar to that defined in (a) but with a dynamic controller L, *i.e.,* $L \in \mathrm{TIC}_\infty(Y,U)$ *need not be static. In dynamic partial feedback (DPF, aka. measurement feedback) we mean the* *situation where the input and output of the controller are connected only to a part of the output and input of the system to be controlled. Maps with internal loop are a generalized concept of* TIC_∞ *maps. This concept makes the algebraic stabilization theory more complete, and it has also a reasonable physical interpretation.*

These concepts are studied in Chapter 7, but we include them in this summary for easy comparison; here also Ξ *is an arbitrary Hilbert space:*

(e) **(DF)** *[7.1.1] A map* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ *is a* stabilizing (DF-)controller *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *if* $L = I$ *is stabilizing for* $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$.

If such a $\mathbb{Q}$ *exists, we say that* $\mathbb{D}$ *is* DF-stabilizable *and that* $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$.

(e') **(DF-IL)** *[7.2.1] A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is a* stabilizing (DF-)controller for with internal loop $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *if* $L = I$ *is stabilizing for (7.20).*

If such an $\mathbb{O}$ *exists, we say that* $\mathbb{D}$ *is* DF-stabilizable with internal loop *and that* $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ with internal loop.

(f) **(DPF)** *[7.3.1] A map* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ *is a* stabilizing DPF-controller *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *if* $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right]$ *is a stabilizing DF-controller for* $\mathbb{D}$.

(f') **(DPF-IL)** *[7.3.1] A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is a* stabilizing (DPF-)controller with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *if (7.58) is an admissible [stabilizing] (DF-)controller for* $\mathbb{D}$ *with internal loop.*

If such an $\mathbb{O}$ *exists, we say that* $\mathbb{D}$ *is* DPF-stabilizable with internal loop *and that* $\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ with internal loop.

In (e)–(f'), one can make analogous definitions with $\mathbb{D}$ *replaced by its realization [and* $\mathbb{Q}$ *or* $\mathbb{O}$ *replaced by its realization], see Definitions 7.1.1, 7.2.1 and 7.3.1.*

When we say that L is strongly internally ω-stabilizing *for* Σ*, we mean that we do not require the corresponding closed loop system* Σ_L*, to be stable but strongly internally* ω*-stable; the same applies to all other stability concepts (prefices) of Definition 6.1.3.*

Same prefices are used for stabilizability by static output feedback (e.g., "Σ *is strongly internally* ω*-stabilizable by static output feedback"), and these are inherited by definitions (b)–(f'), where also further prefices and suffices can be used (see 2. and 4. of Definition 6.6.10). In (d), prefices preceding the word "jointly" apply to both pairs.*

In (a)–(f'), we use the word admissible *instead of stabilizing if L is (merely) admissible. The word* stabilizes *means "is stabilizing for". See Remark 6.7.19 for* ω*-stabilization (*$\omega \in \mathbf{R}$*).*

See Definition 7.2.11 for maps with a *coprime* (that is, a d.c., r.c., or l.c.) internal loop. Remark 6.7.19 explains ω-stabilization further. We usually say "stabilizing" instead of "admissible stabilizing".

The concepts of Summary 6.7.1 are invariant under duality:

Lemma 6.7.2 (Duality) *Make the assumptions of Summary 6.7.1.*

Then the properties (a)–(f') of the summary are invariant to the following extent:

(a) L is admissible for Σ iff L^{d} is admissible for Σ^{d}.

If L is admissible, then $(\Sigma^{\mathrm{d}})_{L^{\mathrm{d}}} = (\Sigma_L)^{\mathrm{d}}$. Thus, e.g., L is [exponentially] stabilizing for Σ iff L is [exponentially] stabilizing for Σ^{d}, etc.

(b)&(c) A pair $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ is an admissible output injection pair for Σ iff $\left[\begin{array}{c|c} \mathbb{H}^{\mathrm{d}} & \mathbb{G}^{\mathrm{d}} \end{array}\right]$ is an admissible state feedback pair for Σ^{d}.

Moreover, the corresponding closed-loop systems (see (6.168) and (6.133)) are duals of each other, hence $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ is [exponentially / exponentially [q.]l.c.- / [q.]l.c.-]stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{H}^{\mathrm{d}} & \mathbb{G}^{\mathrm{d}} \end{array}\right]$ is [exponentially / exponentially [q.]r.c.- / [q.]r.c.-]stabilizing for Σ^{d}.

Thus, Σ is stabilizable iff Σ^{d} is detectable, etc.

(d) The pairs $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are jointly admissible (resp. [exponentially] jointly stabilizing) for Σ iff $\left[\begin{array}{c|c} \mathbb{H}^{\mathrm{d}} & \mathbb{G}^{\mathrm{d}} \end{array}\right]$ and $\left[\frac{\mathbb{K}^{\mathrm{d}}}{\mathbb{F}^{\mathrm{d}}}\right]$ are jointly admissible (resp. [exponentially] jointly stabilizing) for Σ^{d}. The corresponding closed-loop systems are duals of each other modulo permutations of the two I/O rows and columns.

Thus, Σ is jointly stabilizable and detectable iff Σ^{d} is jointly stabilizable and detectable, etc.

(e)–(f') Analogously, $\mathbb{Q}$ is admissible for $\mathbb{D}$ in the sense of (e) (resp. (e'), (f), (f')) iff $\mathbb{Q}^{\mathrm{d}}$ is admissible for $\mathbb{D}_{\mathrm{d}}$ in the sense of (e) (resp. (e'), (f), (f')).

Moreover, the corresponding closed-loop systems are duals of each other modulo permutations of I/O rows and columns. Thus, $\mathbb{Q}$ is [exponentially] stabilizing for $\mathbb{D}$ in the sense of (e) (resp. (e'), (f), (f')) iff $\mathbb{Q}^{\mathrm{d}}$ is [exponentially] stabilizing for $\mathbb{D}_{\mathrm{d}}$ in the sense of (e) (resp. (e'), (f), (f')).

Analogous remarks apply to realizations of $\mathbb{D}$ (and $\mathbb{Q}$).

By $\mathbb{D}_{\mathrm{d}}$ we mean $\mathbb{D}^{\mathrm{d}}$ in case of (e) or (e') and $\begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$ in case of (f) or (f') (note that these correspond to $\mathbb{O}^{\mathrm{d}}$, not to $\mathbb{O}_{\mathrm{d}}$, in Definitions 7.2.1 and 7.3.1). Cf. also Proposition 7.2.5(d), Lemma 7.2.6 and Proposition 7.3.4(d).

Because of the duality between the closed-loop systems, also most prefices and suffices (e.g., "internally", "I/O-", "weakly", "ω-", and "in $\mathcal{A}$" if $\mathcal{A} = \mathcal{A}^{\mathrm{d}}$) are preserved (although "output" becomes "input", "[q.]r.c." becomes "[q.]l.c." etc.). However, the "strong" properties (e.g., "strongly detectable") are exceptions to this duality, because the adjoint of a strongly stable system need not be strongly stable. In the following we will apply these facts without further mention.

Proof: (a) $(I - \mathbb{D}L)^{\mathrm{d}} = I - L^{\mathrm{d}}\mathbb{D}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $I - \mathbb{D}L \in \mathcal{G}\mathrm{TIC}_\infty$ (because $(\cdot)^{\mathrm{d}} \in \mathcal{GB}(\mathrm{TIC}_\omega)$ for any $\omega \in \mathbf{R}$) but also iff $I - \mathbb{D}^{\mathrm{d}}L^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, L is admissible iff L^{d} is. Obviously, $(\Sigma^{\mathrm{d}})_{L^{\mathrm{d}}} = (\Sigma_L)^{\mathrm{d}}$. The remaining claims follow from this.

(b)–(d) These follow from (a), because the extended systems are duals of each other modulo the permutation of I/O rows and/or columns (rows for (b), columns for (c) and both for (d)).

(e)–(f') (Here $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ or $\mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$.)

These follow from (a), because the dual of (7.21) is equal to its counterpart for Σ^{d} and $\widetilde{\Sigma}^{\mathrm{d}}$ modulo the permutation of its two first I/O rows and columns. The well-posed case (i.e., the case without internal loop) is a special case of this (alternatively, use the same proof with (7.4 in place of (7.21)).

(This applies to "$\mathbb{Q}$ stabilizes $\mathbb{D}$", "$\mathbb{Q}$ stabilizes Σ" and "$\widetilde{\Sigma}$ stabilizes Σ", in the sense of any of Definitions 7.1.1, 7.2.1 and 7.3.1, with or without internal loop.)

(See Proposition 7.2.5(d) and Proposition 7.3.4(d) for alternative partial proofs.) □

Optimizability is the weakest reasonable extension of the finite-dimensional concept exponential stabilizability. In the infinite-dimensional theory, the former concept often takes the place of exponential stabilizability (as formulated in Definition 6.6.10), hence we shall study this concept briefly:

Definition 6.7.3 (Optimizability and estimatability) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

If for each $x_0 \in H$ *there is* $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ *s.t.* $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ *is in* L^2*, then we call* Σ optimizable. *We call* Σ estimatable *if* Σ^{d} *is optimizable.*

Obviously, Σ is optimizable iff $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable (one often says that (A,B) is optimizable).

It follows from the definition that Σ is optimizable iff the cost $\|x\|_2^2 + \|u\|_2^2$ is finite for each $x_0 \in H$ there is $u \in \mathrm{L}^2(\mathbf{R}_+;U)$. Therefore, the assumption that a system is optimizable is often called the finite cost condition. The concept "$\mathcal{U}_{\mathrm{exp}}^{\Sigma}(x_0) \neq \emptyset$ for all $x_0 \in H$" of Section 8.3 is also equivalent to optimizability.

Exponential stabilizability (by state feedback) implies optimizability. In fact, even exponential stabilizability by (static or dynamic, even partial and/or with internal loop) output feedback implies optimizability, by Lemma 6.7.6, Theorem 7.2.3(c1) and Theorem 7.3.11(c1). Therefore, optimizability is a necessary condition for the solvability of any standard control problem over exponentially stabilizing state or output feedback controllers.

If, e.g., B is bounded (see Theorem 9.2.12 for weaker sufficient conditions), then optimizability is equivalent to exponential stabilizability (this is also the case for any discrete-time systems, by Proposition 13.3.14). However, it is not known whether this equivalence holds for general WPLSs (or equivalently, by Remark 6.9.5, for general Pritchard–Salamon systems). The situation is analogous with the dual properties, estimatability and exponential detectability.

The following is obvious:

Lemma 6.7.4 *If* $\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \end{array}\right] \in \mathrm{WPLS}$ *is s.t.* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ *is optimizable, then so is* $\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \end{array}\right]$. □

By duality, if $\begin{bmatrix}\mathbb{A}\\\hline\mathbb{C}_1\\\mathbb{C}_2\end{bmatrix} \in$ WPLS is s.t. $\begin{bmatrix}\mathbb{A}\\\hline\mathbb{C}_1\end{bmatrix}$ is estimatable, then $\begin{bmatrix}\mathbb{A}\\\hline\mathbb{C}_1\\\mathbb{C}_2\end{bmatrix}$ estimatable. Next we prove the two lemmas mentioned above.

Lemma 6.7.5 *Any exponentially stabilizable system is optimizable.*

Hence any exponentially detectable system is estimatable, by duality.

Proof: Let $\begin{bmatrix}\mathbb{K} & | & \mathbb{F}\end{bmatrix}$ be exponentially stabilizing for Σ. By Lemma 6.1.10(c1), $\mathrm{L}^2 \ni \mathbb{A}_\sharp x_0 = \mathbb{A}x_0 + \mathbb{B}\tau\mathbb{K}_\flat x_0$ and $\mathbb{K}_\flat x_0 \in L^2$ for all $x_0 \in H$ (see (6.133)), hence Σ is optimizable. (A second proof: apply Lemma 6.7.6 to Σ_{ext}.) $\square$

Lemma 6.7.6 (Σ_L exp. stable $\Longrightarrow$ Σ is opt. & est.) *If there is an exponentially stabilizing output feedback operator for $\Sigma \in$ WPLS, then Σ is optimizable and estimatable.*

We extend this (and partially the converse) for dynamic output feedback in Theorem 7.2.3(c1) and Theorem 7.3.12.

Proof: Now (6.125) is exponentially stable for some L, hence, for any $x_0 \in H$, the function $u := L\mathbb{C}_L x_0 \in \mathrm{L}^2$ satisfies $\mathbb{A}x_0 + \mathbb{B}\tau u = \mathbb{A}_L \in \mathrm{L}^2$. Thus, Σ is optimizable. By duality (see Lemma 6.7.2(a)), Σ is also estimatable. $\square$

We have "$u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$" for estimatable systems:

Theorem 6.7.7 ($u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$) *Let $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ be estimatable. Then there is $M < \infty$ s.t. if $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ and $x_0 \in H$ are s.t. $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, then $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2 \cap \mathcal{C}_0$ and $\|x\|_2 \le M(\|x_0\|_H + \|u\|_2 + \|y\|_2)$.*

Proof: The proof will be given in Section 8.3, see Lemma 8.3.20. (Except for the $\mathcal{C}_0$ property, this would follow from Theorem 13.3.15 and Theorem 13.4.4(a3)&(e3)). $\square$

We need two more implications between the signals:

Lemma 6.7.8 ($u, x \in \mathrm{L}^2 \Rightarrow y \in \mathrm{L}^2$) *Let $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. If $u, x \in \pi_+\mathrm{L}^2$, then $y \in \pi_+\mathrm{L}^2$, where $x_0 \in H$ is arbitrary and $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\tau\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]\left[\begin{smallmatrix}x_0\\ u\end{smallmatrix}\right]$. In fact, there is $M = M_\Sigma < \infty$ s.t. $\|y\|_2 \le M(\|u\|_2 + \|x\|_2 + \|x_0\|_H)$.*

Conversely, if $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, $\omega \in \mathbf{R}$, $x, y \in \mathrm{L}^2$ and $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, then $u \in \mathrm{L}^2$.

By combining the above lemma with Theorem 6.7.7, we see that "$u, x \in \mathrm{L}^2 \Rightarrow y \in \mathrm{L}^2$" holds for arbitrary WPLSs, "$x, y \in \mathrm{L}^2 \Rightarrow u \in \mathrm{L}^2$" for WPLSs with $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, and "$u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$" for estimatable WPLSs.

Proof: By the second inequality of Theorem 13.4.4(a3), we have

$$\|x\|_{\ell^2_{r'}(\mathbf{N};H)} + \|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(\mathbf{N};U)} \le M'(\|x_0\|_H + \|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)}) \tag{6.176}$$

for some $M' = M'_\Sigma < \infty$. Combine this with Lemma 13.3.18 to obtain the first claim (thus, our M depends on Σ only).

The converse for $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$ follows from the fact that x and u are the state and output of the closed-loop system $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}-I \end{array}\right]_{-I}$ (the "flow-inverted system") with input y and initial state x_0 (it is straightforward to verify this, see Example 6.2.4 of [Sbook]). $\square$

For an exponentially detectable system, output-stabilization is equivalent to exponential stabilization:

Lemma 6.7.9 *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be estimatable.*

If a state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *or an output injection pair* $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *or a static output feedback operator L output-stabilizes Σ, then it stabilizes Σ exponentially.*

Proof: 1° *Output feedback:* Let L output-stabilize Σ. Let $x_0 \in H$. Set $u := L\mathbb{C}_L x_0 \in \mathrm{L}^2$, $x := \mathbb{A}_L x_0$, $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] := y := \mathbb{C}_L x_0 \in \mathrm{L}^2$, so that

$$x = \mathbb{A}x_0 + \mathbb{B}\tau u, \quad y = \mathbb{C}x_0 + \mathbb{D}u, \quad y_1 = \mathbb{C}_1 x_0 + \mathbb{D}_1 u, \tag{6.177}$$

by (6.126). Consequently, $x \in \mathrm{L}^2$, by Theorem 6.7.7. Because x_0 was arbitrary, $\mathbb{A}_L$ is exponentially stable, by Lemma A.4.5.

2° *State feedback or output injection:* Apply 1° with the extended system (which is also estimatable, by Lemma 6.7.4) in place of Σ. $\square$

Next we explore the connection between stability and stabilizability (see Definitions 6.1.3, 6.6.10 and 6.6.21):

Theorem 6.7.10 *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$.

(a) **(Stability)** *The following are equivalent:*

(i) Σ is stable (i.e., $\Sigma \in \mathrm{WPLS}_0$);
(ii) $\mathbb{D}$ is stable and Σ is detectable and output-stabilizable;
(iii) $\mathbb{D}$ is stable and Σ is q.r.c.-stabilizable;
(iv) $\mathbb{D}$ is stable and Σ is q.l.c.-detectable.

(b) **(SOS-Stability)** *The following are equivalent:*

(i) Σ is SOS-stable (i.e., $\Sigma \in \mathrm{SOS}$);
(ii) $\mathbb{D}$ is stable and Σ is output-stabilizable;
(iii) $\mathbb{D}$ is stable and Σ is q.l.c.-output-detectable.

(c) **(Strong stability)** *The following are equivalent:*

(i) Σ is strongly stable;
(ii) Σ is stable and strongly detectable;
(iii) $\mathbb{D}$ is stable, Σ is output-stabilizable, and strongly detectable;
(iv) $\mathbb{D}$ is stable and Σ is strongly q.r.c.-stabilizable;
(v) $\mathbb{D}$ is stable and Σ is strongly q.l.c.-detectable.

(d) **(Exponential stability)** *The following are equivalent:*

(i) Σ is exponentially stable (i.e., $\Sigma \in \mathrm{WPLS}_\omega$ for some $\omega < 0$);
(ii) $\mathbb{B}$ is stable (or $(s-A)^{-1}B \in \mathrm{H}^\infty$) and Σ is optimizable;
(iii) $\mathbb{C}$ is stable (or $C(s-A)^{-1} \in \mathrm{H}^\infty$) and Σ is estimatable;
(iv) $\mathbb{D}$ is stable, and Σ is optimizable and input-detectable;
(v) $\mathbb{D}$ is stable and Σ is output-stabilizable and estimatable;
(vi) $\mathbb{D}$ is stable and Σ is optimizable and q.r.c.-stabilizable;
(vii) $\mathbb{D}$ is stable and Σ is estimatable and q.l.c.-detectable;
(viii) $\mathbb{D}$ is stable and Σ is optimizable and estimatable.

Proof: In (a)–(d), obviously (i) always implies all the other conditions (use, e.g., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$), hence we only prove the converse claims.

We give more detailed proofs for part (b); the other proofs are given briefly (and they are more or less analogous to those in (b)).

(b) "(ii)⇒(i)": Assume (ii), i.e., that $\mathbb{D}$, $\mathbb{C}_\flat$ and $\mathbb{K}_\flat$ are stable in (6.133) (for some $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$). Then also $\mathbb{C} = \mathbb{C}_\flat - \mathbb{D}\mathbb{K}_\flat$ is stable, hence (i) holds.

"(iii)⇒(i)": Assume (iii), i.e., that $\mathbb{D}$ is stable and there is an admissible output injection pair $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ for Σ s.t. $\widetilde{\mathbb{M}}\mathbb{C}$ is stable, where $\widetilde{\mathbb{M}} := (I-\mathbb{G})^{-1}$, and $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$ and $\widetilde{\mathbb{M}}$ are q.l.c. Then also $\mathbb{D} = I^{-1}(\mathbb{D})$ is a q.l.c.f. of $\mathbb{D}$, hence $\widetilde{\mathbb{M}} = \widetilde{\mathbb{M}}I \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(d). Consequently, $\mathbb{C} = \widetilde{\mathbb{M}}^{-1}(\widetilde{\mathbb{M}}\mathbb{C})$ is stable, hence (i) holds.

(a) "(ii)$\Longrightarrow$(i)": Assume (ii). Then $\mathbb{C}$ is stable, by (b). From (6.168) we see that also $\mathbb{B} = \mathbb{B}_\sharp - \mathbb{H}_\sharp\mathbb{D}$ and $\mathbb{A} = \mathbb{A}_\sharp - \mathbb{H}_\sharp\tau\mathbb{C}$ are stable.

"(iii)⇒(i)": This follows from Lemma 6.6.17(a)&(c). "(iv)⇒(i)" is the dual result of this.

(c) "(ii)⇒(i)": This follows from Lemma 6.6.8 (where "Σ":=(6.168) and $L := \begin{bmatrix}-I & 0\end{bmatrix}$). (Note that an analogous stabilizability result would require $\mathbb{K}$ to be stable.) "(iii)⇒(ii)" & "(v)⇒(ii)": These hold by (a). "(iv)⇒(i)": This follows from Lemma 6.6.17(a)&(c).

(d) "(iii)⇒(i)": If Σ is estimatable and $C(s-A)^{-1} \in \mathrm{H}^\infty$, then Σ is exponentially stable, by Proposition 6.2 of [WR00] (use discretization and Theorem 13.3.13 for an alternative proof; this also applies to "(viii)⇔(i)").

Assume then that $\mathbb{C}$ is stable and Σ is estimatable. By Theorem 6.7.7, $\mathbb{A}x_0 \in \mathrm{L}^2$ for all $x_0 \in H$ (take $u = 0$ and note that $y := \mathbb{C}x_0 \in \mathrm{L}^2$), hence $\mathbb{A}$ is exponentially stable, by Lemma A.4.5. An alternative proof (for the dual claim (ii)⇒(i)) is obtained by slightly modifying the proof of Proposition 6.1 of [WR00] (combined to the dual of Theorem 6.2.11(c2)).

"(v)⇒(iii)": Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be output-stabilizing for Σ. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing, by Lemma 6.7.9, hence Σ is stable, by (a)(ii).

"(ii)⇒(i)"&"(iv)⇒(ii)": These are the duals of "(iii)⇒(i)" and "(v)⇒(iii)" (note also that $(s-A)^{-1}B \in \mathrm{H}^\infty \Leftrightarrow \mathbb{B}\tau \in \mathrm{TIC}$).

"(vi)⇒(ii)": ["(vii)⇒(iii)"]: This follows from implication (iii)[(iv)]⇒(i) of (a).

"(viii)⇔(i)": This is Theorem 6.3 of [WR00]. □

Any admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for a system Σ preserves stabilizability in the following way: if $\Sigma_\flat$ is the corresponding closed-loop system and $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ is stabilizable in some sense, then the components $\left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$ of $\Sigma_\flat$ can be stabilized exactly in the same sense (see (a) below). This only applies two the top two rows of the closed-loop systems; we need coprimeness or exponential stabilization in order to guarantee that also the third row of the closed-loop system The situation with output injection is analogous (see (b)), whereas static output feedback (see (c)) preserves anything:

Lemma 6.7.11 (Stabilizability preserved) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

(a) ($\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$) *Optimizability, exponential stabilizability, and all* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-stabilizability properties (see (a')) are invariant under admissible state feedback. Moreover, if* Σ *is estimatable, then so is* $\Sigma_\flat$ *(of (6.178)).*

(a') The stabilizability of the "$\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *part" of a system is invariant under admissible state feedback:*

Let $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *and* $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *be admissible state feedback pairs for* Σ*, and let*

$$\Sigma_\flat := \left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\\ \mathbb{K}_\flat & \mathbb{F}_\flat\end{array}\right] \quad \text{and} \quad \Sigma_{\flat'} := \left[\begin{array}{c|c}\mathbb{A}_{\flat'} & \mathbb{B}_{\flat'}\\ \hline \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'}\\ \mathbb{K}^2_{\flat'} & \mathbb{F}^2_{\flat'}\end{array}\right] \tag{6.178}$$

be the corresponding closed-loop systems, respectively.

Then the state feedback pair $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *defined by (6.180) is admissible for* $\Sigma_\flat$*, and the two top rows of the corresponding closed-loop system are the two top rows* $\left[\begin{smallmatrix}\mathbb{A}_{\flat'} & \mathbb{B}_{\flat'}\\ \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'}\end{smallmatrix}\right]$ *of* $\Sigma_{\flat'}$.

Moreover, with additional assumptions, this pair $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *stabilizes even more (here* $\Sigma^1_\flat := \left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$*):*

(a1) Let $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *be q.r.c.-I/O-stabilizing for* Σ*. Then* $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *is [q.r.c.-]I/O-stabilizing for* Σ *iff* $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *is [[q]r.c.-]I/O-stabilizing for* $\Sigma^1_\flat$ *[(iff* $I - \mathbb{F}_\natural \in \mathcal{G}\mathrm{TIC}$*)].*

(a2) Let $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *be q.r.c.-SOS-stabilizing for* Σ*. Then* $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *is [q.r.c.-]SOS-stabilizing for* Σ *iff* $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *is [stable and r.c.-]SOS-stabilizing for* $\Sigma^1_\flat$.

Thus, $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *[q.r.c.-]stabilizes* Σ *(resp. weakly, strongly) iff* $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *[[q.]r.c.-]stabilizes* $\Sigma^1_\flat$ *(resp. weakly, strongly).*

(a3) The pair $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *is exponentially stabilizing for* Σ *iff* $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *is exponentially stabilizing for* $\Sigma^1_\flat$.

(a4) Let $\left[\ \mathbb{K}^2 \mid \mathbb{F}^2\ \right]$ *be exponentially [q.]r.c.-stabilizing for* Σ*, and let* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *be I/O-stabilizing for* Σ.

Then $\left[\ \mathbb{K}_\natural \mid \mathbb{F}_\natural\ \right]$ *is exponentially r.c.-stabilizing for* $\Sigma^1_\flat$*, and* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *is exponentially [q.]r.c.-stabilizing for* Σ.

$$
\begin{tikzcd}
 & \Sigma \arrow[dl, "{\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]}"'] \arrow[dr, "{\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]}"] & \\
\Sigma_\flat \arrow[rr, "{\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]}"'] & & \Sigma_{\flat'}, \Sigma_{\circlearrowleft}
\end{tikzcd}
$$

Figure 6.8: The setting of Lemma 6.7.11(a')

(a4') Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be exponentially [[q.]r.c.-]stabilizing for* Σ. *Then* $\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]$ *is exponentially [[q.]r.c.-]stabilizing for* Σ *iff* $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *is I/O-stabilizing (or input-stabilizing) for* $\Sigma_\flat^1$ *(in which case* $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *is exponentially stable and exponentially r.c.-stabilizing for* $\Sigma_\flat^1$*).*

(a5) Let $\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]$ *be exponentially stabilizing and [q.]r.c.-stabilizing for* Σ*, and let* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be I/O-stabilizing for* Σ.

Then $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *is exponentially r.c.-stabilizing for* $\Sigma_\flat^1$*, and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is exponentially stabilizing and [q.]r.c.-stabilizing for* Σ.

(a6) Claims (a1)–(a5) also hold with $\Sigma_\flat$ *in place of* $\Sigma_\flat^1$.

(b) $\left(\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]\right)$ *Estimatability, exponential detectability, and all* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$*-detectability properties (cf. (a)) are invariant under admissible output injection. Moreover, if* Σ *is optimizable, then so is* $\Sigma_\sharp$ *(of (6.168).*

(c) $(\boldsymbol{\Sigma_L})$ *All different versions of stabilizability and detectability listed in Summary 6.7.1(a)–(d) (including prefices and suffices except those corresponding to 4. of Definition 6.6.10), as well as optimizability and estimatability are preserved under admissible static output feedback, i.e., the systems* Σ *and* Σ_L *of Proposition 6.6.2 are output feedback stabilizable, stabilizable, detectable or jointly stabilizable and detectable exactly in the same sense.*

Proof: (a) All $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$-stabilizability properties of Σ and $\Sigma_\flat$ are equal, by the first claim in (a'). By (a3), Σ is (equivalently, $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is) exponentially stabilizable iff $\Sigma_\flat$ is (interchange their roles for the converse).

In discrete time, optimizability is equivalent to exponential stabilizability, by Proposition 13.3.14, hence invariant under state feedback. By Theorem 13.4.4(e3)&(e1), optimizability and state feedback are invariant under discretization, hence optimizability is invariant under state feedback in continuous time too.

Finally, for estimatability (of $\Sigma_\flat$, not $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$), we can use discretization as above (since exponential detectability is preserved in discrete time: $A + \mathtt{H}C = A + BMK + \begin{bmatrix} \mathtt{H} & \widetilde{H} \end{bmatrix} \begin{bmatrix} C+DMK \\ MK \end{bmatrix}$, where $\widetilde{H} := -\mathtt{H}D - B$). Alternatively, we can establish the dual claim by noting that if $u \in \mathrm{L}^2$ is s.t. $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2$, then $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, by Lemma 6.7.8. But $\mathbb{A}_\sharp x_0 + \mathbb{B}_\sharp \tau u + \mathbb{H}_\sharp \tau(-y) = x \in \mathrm{L}^2$. Consequently, if Σ is estimatable, then so is $\Sigma_\sharp$, for any output injection pair $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$.

(a') Extend Σ to Σ_{Ext2} with these two state feedback pairs, and let $\Sigma_{\flat 2}$ be the closed-loop system of Σ_{Ext2} corresponding to $L := \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, i.e.,

$$\Sigma_{\mathrm{Ext2}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \\ \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right], \quad \Sigma_{\flat 2} := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \\ \mathbb{K}^2_\flat & \mathbb{F}^2_\flat \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}+\mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C}+\mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M}-I \\ \mathbb{K}^2+\mathbb{F}^2\mathbb{M}\mathbb{K} & \mathbb{F}^2\mathbb{M} \end{array}\right], \tag{6.179}$$

where $\mathbb{M} := \mathbb{X}^{-1} := (I-\mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$. Because $L' := \begin{bmatrix} 0 & 0 & I \end{bmatrix}$ makes the first, second and fourth rows of Σ_{Ext2} equal to $\Sigma_{\flat'}$, it follows from Proposition 6.6.3 that $L'-L$ does the same for $\Sigma_{\flat 2}$. Therefore, the state feedback pair (here $\mathbb{M}^2 := (I-\mathbb{F}^2)^{-1}$)

$$\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] := \left[\begin{array}{c|c} \mathbb{K}^2_\flat - \mathbb{K}_\flat & \mathbb{F}^2_\flat - \mathbb{F}_\flat \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}^2 - \mathbb{X}^2\mathbb{M}\mathbb{K} & I - \mathbb{X}^2\mathbb{M} \end{array}\right] \tag{6.180}$$

also does the same for $\Sigma_{\flat 2}$, in particular, it is admissible for $\Sigma_\flat$, and the corresponding closed-loop system

$$\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} \mathbb{A}_{\flat'} & \mathbb{B}_{\flat'} \\ \hline \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'} \\ \mathbb{K}_{\flat\circlearrowleft} & \mathbb{F}_{\flat\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}_\flat + \mathbb{B}_\flat\tau\mathbb{K}_{\circlearrowleft} & \mathbb{B}_\flat\mathbb{M}_\natural \\ \hline \mathbb{C}_\flat + \mathbb{D}_\flat\mathbb{K}_{\circlearrowleft} & \mathbb{D}_\flat\mathbb{M}_\natural \\ \mathbb{K}_\flat + \mathbb{F}_\flat\mathbb{K}_{\circlearrowleft} & \mathbb{F}_\flat\mathbb{M}_\natural \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural - I \end{array}\right], \tag{6.181}$$

where $\mathbb{M}_\natural := (I-\mathbb{F}_\natural)^{-1}$. In particular, the two top rows of $\Sigma_{\circlearrowleft}$ are equal to the two top rows of $(\Sigma_{\flat 2})_{(L'-L)}$, i.e., to those of $\Sigma_{\flat'}$.

Indeed, the $\begin{bmatrix} x_0 \\ u_L \end{bmatrix} \mapsto \begin{bmatrix} x_0 \\ u \end{bmatrix}$ map "$\begin{bmatrix} I & 0 \\ -L\mathbb{C} & I-L\mathbb{D} \end{bmatrix}^{-1}$" (cf. (6.127)) of this state feedback connection is equal to that corresponding to $L'-L$ with $\Sigma_\flat$, i.e., to $\begin{bmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbb{K}_{\circlearrowleft} & \mathbb{M}_\natural \end{bmatrix}$ see also formula (6.134).

The fourth row of $\Sigma_{\circlearrowleft}$ is given by

$$\left[\begin{array}{c|c} \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] \begin{bmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix} = \left[\begin{array}{c|c} \mathbb{M}_\natural\mathbb{K}^2 - \mathbb{K} & \mathbb{M}_\natural - I \end{array}\right] \tag{6.182}$$

$$= \left[\begin{array}{c|c} \mathbb{X}\mathbb{M}^2\mathbb{K}^2 - \mathbb{K} & \mathbb{X}\mathbb{M}^2 \end{array}\right], \tag{6.183}$$

since $\mathbb{M}_\natural = (I-\mathbb{F}_\natural)^{-1} = (\mathbb{X}^2\mathbb{M})^{-1} = \mathbb{X}\mathbb{M}^2$.

(a1) Now $\mathbb{D} = \mathbb{D}_\flat\mathbb{M}^{-1}$ is a q.r.c.f. The maps $\mathbb{D}_{\flat'}$ and $\mathbb{M}^2$ are [q.r.c. and] stable, iff $\mathbb{M}_\natural = \mathbb{M}^{-1}\mathbb{M}^2 \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ $[\cap\mathcal{G}\mathrm{TIC}(U)]$, by Lemma 6.4.5(b)[(c)]. But $\mathbb{F}_{\circlearrowleft} + I = \mathbb{M}_\natural \in \mathrm{TIC}(U)[\cap\mathcal{G}\mathrm{TIC}(U)]$ iff $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is [r.c.]-I/O-stabilizing [(equivalently, [q.r.c.]-I/O-stabilizing), by Lemma 6.6.17(a)].

(Actually $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ will then [r.c.-]I/O-stabilize the whole $\Sigma_{\flat 2}$, because also the additional row $\left[\begin{array}{c|c} \mathbb{K}^2_\flat & \mathbb{F}^2_\flat \end{array}\right] \begin{bmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix}$ is equal to $\left[\begin{array}{c|c} \mathbb{K}^2_{\flat'} & \mathbb{F}^2_{\flat'} \end{array}\right]$, hence I/O-stable, by assumption. A similar comment applies to (a2).)

(a2) Because the second row of $\Sigma_{\circlearrowleft}$ (i.e., of $\Sigma_{\flat'}$) is now stable, and $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is [r.c.-]I/O-stabilizing for $\Sigma_\flat$ under either assumption, by (a1), we only have to show that $\mathbb{K}_{\circlearrowleft}$ is stable iff $\mathbb{K}_{\flat'}$ is stable (since $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is q.r.c.-SOS-stabilizing iff it is stable and r.c.-SOS-stabilizing, by Lemma 6.6.17(b)).

We have

$$\mathbb{N}\mathbb{K}_{\circlearrowleft} = (\mathbb{D}\mathbb{M})(\mathbb{M}_{\natural}\mathbb{K}^2 - \mathbb{K}) = \mathbb{D}\mathbb{M}^2\mathbb{K}^2 - \mathbb{D}\mathbb{M}\mathbb{K} = \mathbb{C}_{\flat'} - \mathbb{C}_{\flat}, \quad \text{and} \tag{6.184}$$

$$\mathbb{M}\mathbb{K}_{\circlearrowleft} = \mathbb{M}\mathbb{M}_{\natural}\mathbb{K}_{\natural} = \mathbb{M}\mathbb{M}_{\natural}(\mathbb{K}^2 - (\mathbb{M}^2)^{-1}\mathbb{M}\mathbb{K}) = \mathbb{M}^2\mathbb{K}^2 - \mathbb{M}\mathbb{K} = \mathbb{K}^2_{\flat'} - \mathbb{K}_{\flat}. \tag{6.185}$$

Since $\mathbb{C}_{\flat'}$, $\mathbb{C}_{\flat}$ and $\mathbb{K}_{\flat}$ are stable under either assumption, it follows that $\mathbb{M}\mathbb{K}_{\circlearrowleft}$ and $\mathbb{N}\mathbb{K}_{\circlearrowleft}$ are stable iff $\mathbb{K}^2_{\flat'}$ is stable. But this holds iff $\mathbb{K}_{\circlearrowleft}$ is stable, by Lemma 6.5.2(c2), as required.

(The "Thus" comment follows from the fact that the first row $\left[\begin{array}{c|c} \mathbb{A}_{\flat'} & \mathbb{B}_{\flat'} \end{array}\right]$ is the same for $\Sigma_{\circlearrowleft}$ and $\Sigma_{\flat'}$.)

(a3) Now $\Sigma_{\flat'}$ is exponentially stable, hence so are $\mathbb{A}_{\flat'} = \mathbb{A}_{\circlearrowleft}$ and $\Sigma_{\circlearrowleft}$, by Lemma 6.1.10.

(a4) 1° *"Only if":* By (a3), $\mathbb{M}_{\natural} = \mathbb{F}_{\circlearrowleft} + I$, $\Sigma_{\flat'}$ and $\Sigma_{\circlearrowleft}$ are exponentially stable. But $\mathbb{M}_{\natural}^{-1} = (\mathbb{M}^2)^{-1}\mathbb{M}$ is stable, by Lemma 6.4.5(b) (with $\mathbb{U} := \mathbb{M}_{\natural}^{-1}$). Therefore, by Lemma 2.2.7, there is $\varepsilon > 0$ s.t. $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U)$ and $\Sigma_{\circlearrowleft} \in \mathrm{WPLS}_{-\varepsilon}$.

It follows that $\mathbb{M} = \mathbb{M}^2\mathbb{M}_{\natural}^{-1}$ and $\mathbb{D}_{\flat} = \mathbb{D}_{\flat'}\mathbb{M}_{\natural}^{-1}$ are exponentially [q.]r.c., by Lemma 6.4.5(c) (shifted by $-\varepsilon$; cf. Remark 2.1.6).

Moreover, $\mathbb{M}_{\natural}$ is exponentially r.c. with any $\mathrm{TIC}_{-\varepsilon}(U, *)$ operator (in particular, with the other I/O components of $\Sigma_{\circlearrowleft}$), hence $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Sigma_{\flat}$.

By Theorem 6.7.10(d)(vi), $\Sigma_{\flat}$ is exponentially stable, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially [q.]r.c.-stabilizing for Σ.

2° *"If":* Since $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially stable (since $\mathbb{A}_{\flat}$ is), it follows that $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}$ (even exponentially, as in 1°). The rest follows as in 1°.

(a4') Since $\mathbb{A}_{\flat'} = \mathbb{A}_{\circlearrowleft}$ is common for $\Sigma_{\flat'}$ and $\Sigma_{\circlearrowleft}$, one of them is exponentially stabilizing iff the other is. By Theorem 6.7.15(b1), $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially r.c.-stabilizing iff it is I/O-stabilizing (or input-stabilizing) (since $\Sigma_{\flat}$ is exponentially stable).

Assume that this is the case. Then $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U)$ for some $\varepsilon > 0$, so that $\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]$ is exponentially [q.]r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is, by Lemma 6.4.5(c).

(a5) The proof of (a4) applies, except that now we have to apply the original (unshifted) version of Lemma 6.4.5(c).

(a6) Equation (6.181) shows that also $\left[\begin{array}{c|c} \mathbb{K}_{\flat\circlearrowleft} & \mathbb{F}_{\flat\circlearrowleft} \end{array}\right]$ of (6.181) is stable (resp. I/O-stable) whenever $\left[\begin{array}{c|c} \mathbb{K}_{\flat} & \mathbb{F}_{\flat} \end{array}\right]$ and $\left[\begin{array}{c|c} \mathbb{K}_{\circlearrowleft} & \mathbb{M}_{\natural} \end{array}\right]$ are stable (resp. I/O-stable).

Therefore, (a3)–(a5) and the "only if" parts of (a1)–(a2) hold for $\Sigma_{\flat}$ in place of $\Sigma^1_{\flat}$ (for the [q.]r.c. claims this is trivial since in those cases we always have (exponentially in (a4)) $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}$, as shown in the proof of (a1), so that in those cases $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is r.c.-I/O-stabilizing whenever it is I/O-stabilizing (exponentially r.c.-stabilizing in (a4))). For the "if" parts of (a1)–(a2), we note that

$$\begin{bmatrix} \mathbb{D}_{\flat'} \\ \mathbb{F}_{\flat'\circlearrowleft} \\ \mathbb{M}_{\natural} \end{bmatrix} = \begin{bmatrix} \mathbb{D}_{\flat'} \\ (\mathbb{M}-I)\mathbb{M}_{\natural} \\ \mathbb{M}_{\natural} \end{bmatrix} \tag{6.186}$$

is [quasi-]left-invertible over TIC iff $\begin{bmatrix} \mathbb{D}_{\flat'} \\ \mathbb{M}_{\natural} \end{bmatrix}$ is [quasi-]left-invertible over TIC.

(b) This follows from (a) by duality, i.e., by taking causal adjoints (note that "strongly stable" maps to "strongly* stable").

(c) The preservation of optimizability and estimatability is shown in Theorem 7.3 of [WR00]; for the rest we deduce as follows:

Let L be an admissible static output feedback operator for Σ.

1° For (a) of Summary 6.7.1, the claim follows from Lemma 6.6.3 (what K is for Σ, that $K-L$ is for Σ_L).

2° Part (b) of Summary 6.7.1: Use the notation of Proposition 6.6.2 and Definition 6.6.10 and L and let $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ be admissible for Σ. Set

$$\left[\ \mathbb{K}_L \ \middle|\ \mathbb{F}_L\ \right] := \left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right] \left[\begin{smallmatrix} I & 0 \\ L\mathbb{C}_L & (I-L\mathbb{D})^{-1} \end{smallmatrix}\right] \tag{6.187}$$

(cf. (6.126); thus this is the bottom line of $(\Sigma_{\text{ext}})_{[L\ 0]}$). Then $\left[\ \mathbb{K}_L \ \middle|\ \mathbb{F}_L\ \right]$ is admissible for Σ_L. Now (we leave the details to the reader)

$$\left[\ \mathbb{K}_{\natural} \ \middle|\ \mathbb{F}_{\natural}\ \right] := \left[\ \mathbb{K}_L - L\mathbb{C}_L \ \middle|\ \mathbb{F}_L - L\mathbb{D}_L\ \right] \tag{6.188}$$

is admissible for Σ_L, because

$$(I-\mathbb{F}_{\natural})^{-1} = (\mathbb{X}(I+L\mathbb{D}_L))^{-1} = (I-L\mathbb{D})\mathbb{M} \in \mathcal{G}\mathrm{TIC}_{\infty}(U). \tag{6.189}$$

Therefore, $\left[\begin{smallmatrix} I & 0 \\ -\mathbb{K}_{\natural} & I-\mathbb{F}_{\natural} \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & 0 \\ (I-L\mathbb{D})\mathbb{M}\mathbb{K} & (I-L\mathbb{D})\mathbb{M} \end{smallmatrix}\right]$. Consequently, the combined $\left[\begin{smallmatrix} x_0 \\ u_\flat \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x_0 \\ u_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$ map is given by

$$\begin{bmatrix} I & 0 \\ L\mathbb{C}_L & (I-L\mathbb{D})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ (I-L\mathbb{D})\mathbb{M}\mathbb{K} & (I-L\mathbb{D})\mathbb{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbb{M}\mathbb{K} & \mathbb{K} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\mathbb{K} & I-\mathbb{F} \end{bmatrix}. \tag{6.190}$$

Consequently, the closed-loop system of Σ_L with the state feedback pair $\left[\ \mathbb{K}_{\natural} \ \middle|\ \mathbb{F}_{\natural}\ \right]$ is given by

$$\left[\begin{array}{c|c} \mathbb{A}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \\ \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]_{[L\ 0]} = \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat - L\mathbb{C}_\flat & \mathbb{F}_\flat - L\mathbb{D}_\flat \end{array}\right]. \tag{6.191}$$

Thus, if $\Sigma_\flat$ is stable in some sense, then so is (6.191). Moreover, $\mathbb{D}_\flat$ and $(\mathbb{F}_\flat+I)-L\mathbb{D}_\flat$ are [q.]r.c. iff $\mathbb{N} := \mathbb{D}_\flat$ and $\mathbb{M} := (\mathbb{F}_\flat+I)$ are [q.]r.c., by Lemma 6.5.1(d). Obviously, $\mathbb{D}_\flat, \mathbb{F}_\flat+I \in \widetilde{\mathcal{A}}$ iff $\mathbb{D}_\flat, (\mathbb{F}_\flat+I)-L\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$. Thus, all prefices and suffices are preserved except those of 4. of Definition 6.6.10.

(Remark: the prefices in 4. of Definition 6.6.10 need not preserve, i.e., $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ might be, e.g., SR or exponentially stable even if $\left[\ \mathbb{K}_{\natural} \ \middle|\ \mathbb{F}_{\natural}\ \right]$ were not. Our claim concerns only the stabilizability of Σ and Σ_L, including prefices and suffices others than those concerning the direct properties of $\mathbb{K}$ and $\mathbb{F}$.)

3° Part (c) of Summary 6.7.1: this is analogous to 2° (use $\left[\frac{\mathbb{H}_L-\mathbb{B}_L L}{\mathbb{G}_L-\mathbb{D}_L L}\right]$).

4° Part (d) of Summary 6.7.1: Let Σ and (6.169) be as in Definition 6.6.21, let $L\in\mathcal{B}(Y,U)$ be admissible for Σ and let $\left[\begin{array}{c|cc}\mathbb{A}_L & \mathbb{H}_L & \mathbb{B}_L\\ \mathbb{C}_L & \mathbb{G}_L & \mathbb{D}_L\\ \mathbb{K}_L & \mathbb{E}_L & \mathbb{F}_L\end{array}\right]$ be the closed-loop system of (6.169) induced by the output feedback operator $\left[\begin{smallmatrix}L&0\\0&0\end{smallmatrix}\right]$. Then one can easily verify that $\left[\begin{array}{c|c}\mathbb{K}_L-L\mathbb{C}_L & \mathbb{F}_L-L\mathbb{D}_L\end{array}\right]$ and $\left[\frac{\mathbb{H}_L-\mathbb{B}_L L}{\mathbb{G}_L-\mathbb{D}_L L}\right]$ are jointly admissible (with the interaction operator $\mathbb{E}_L' := \mathbb{E}_L - L\mathbb{G}_L + \mathbb{F}_L L - L\mathbb{D}_L L$) for $\Sigma_L := \left[\begin{array}{c|c}\mathbb{A}_L & \mathbb{B}_L\\ \hline \mathbb{C}_L & \mathbb{D}_L\end{array}\right]$ (see the proof of [Sbook, Lemma 8.2.7] for a heuristic derivation for the formulae).

Straightforward computations show that the closed-loop system corresponding to output feedback operator $\left[\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right]$ is

$$\left[\begin{array}{c|cc}\mathbb{A}_L & \mathbb{H}_L-\mathbb{B}_L L & \mathbb{B}_L\\ \hline \mathbb{C}_L & \mathbb{G}_L-\mathbb{D}_L L & \mathbb{D}_L\\ \mathbb{K}_L-L\mathbb{C}_L & \mathbb{E}_L-L\mathbb{F}_L+\mathbb{G}_L L & \mathbb{F}_L\end{array}\right]_{\left[\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right]} = \left[\begin{array}{c|cc}\mathbb{A}_\flat & \mathbb{H}_\flat-\mathbb{B}_\flat L & \mathbb{B}_\flat\\ \hline \mathbb{C}_\flat & \mathbb{G}_\flat-\mathbb{D}_\flat L & \mathbb{D}_\flat\\ \mathbb{K}_\flat-L\mathbb{C}_\flat & \mathbb{E}_\flat-L\mathbb{F}_\flat-\mathbb{G}_\flat L & \mathbb{F}_\flat\end{array}\right]. \tag{6.192}$$

Thus, if $\left[\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right]$ is stabilizing for (6.169), then it is stabilizing for Σ_{ext}^L; the same applies all prefices and suffices (except for the ones concerning the stability and regularity of $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$; cf. 2° above). A similar computation shows that the same applies the output feedback operator $\left[\begin{smallmatrix}I&0\\0&0\end{smallmatrix}\right]$.

Therefore, the system Σ_L inherits the joint stabilizability properties of Σ. □

We often need to apply part (a) of the above lemma with Σ and $\Sigma_\flat$ interchanged. Let us make this explicit:

Lemma 6.7.12 *Let* $\left[\begin{array}{c|c}\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}\end{array}\right]$ *be an admissible state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$*, and let* $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *be an admissible state feedback pair for* $\Sigma_\flat$ *with closed-loop system* $\Sigma_\circlearrowleft$*. Then*

$$\left[\begin{array}{c|c}\mathbb{K}' & \mathbb{F}'\end{array}\right] = \left[\begin{array}{c|c}\mathbb{K}_\natural+\mathbb{X}_\natural\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}+\mathbb{F}_\natural-\mathbb{F}_\natural\widetilde{\mathbb{F}}\end{array}\right] = \left[\begin{array}{c|c}\mathbb{X}'\mathbb{K}_\flat+\mathbb{K}_\natural & \mathbb{F}'\end{array}\right] \tag{6.193}$$

is an admissible state feedback pair for Σ *with closed-loop system*

$$\Sigma_\circlearrowleft' := \left[\begin{array}{c|c}\mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft\\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft\\ \mathbb{K}_\circlearrowleft' & \mathbb{F}_\circlearrowleft'\end{array}\right] = \left[\begin{array}{c|c}\mathbb{A}_\flat+\mathbb{B}\mathbb{M}'\tau\mathbb{K}_\natural & \mathbb{B}\mathbb{M}'\\ \hline \mathbb{C}_\flat+\mathbb{N}'\mathbb{K}_\natural & \mathbb{N}'\\ \mathbb{K}_\flat+\mathbb{M}'\mathbb{K}_\natural & \mathbb{M}'-I\end{array}\right], \tag{6.194}$$

where $\mathbb{X}' := I-\mathbb{F}'$*,* $\mathbb{M}' := (\mathbb{X}')^{-1}$*,* $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat\mathbb{X}_\natural^{-1}$*,* $\mathbb{X}_\natural := I-\mathbb{F}_\natural$*,* $\widetilde{\mathbb{X}} := I-\widetilde{\mathbb{F}}$*. Moreover,* $\mathbb{X}' = \mathbb{X}_\natural\widetilde{\mathbb{X}}$*,* $\mathbb{K}_\circlearrowleft' = \mathbb{K}_\flat+\widetilde{\mathbb{M}}\mathbb{K}_\circlearrowleft$ *and* $\mathbb{M}' = \widetilde{\mathbb{M}}\mathbb{M}_\natural$*.*

In particular, $\Sigma_\circlearrowleft'$ is equal to $\Sigma_\circlearrowleft$ except for $\mathbb{K}_\circlearrowleft'$ and $\mathbb{F}_\circlearrowleft'$. Also Lemma 9.12.3(a)–(c) apply (with same proofs); see also Proposition 6.6.18(f).

Proof: Apply Lemma 6.7.11(a) with substitutions $\Sigma\mapsto\Sigma_\flat$, $\Sigma_\flat\mapsto\Sigma$, $\Sigma_{\flat'}\mapsto\Sigma_\circlearrowleft$, so that "$\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$"$=\left[\begin{array}{c|c}-\mathbb{K}_\flat & -\mathbb{F}_\flat\end{array}\right]$, by Lemma 6.6.14,

$$
\begin{tikzcd}
 & \Sigma \arrow[dl, "\left[\begin{array}{c|c}\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}\end{array}\right]"'] \arrow[dr, "\left[\begin{array}{c|c}\mathbb{K}' & \mathbb{F}'\end{array}\right]"] & \\
\Sigma_\flat \arrow[rr, "\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]"'] & & \Sigma_{\circlearrowleft}, \Sigma'_{\circlearrowleft}
\end{tikzcd}
$$

Figure 6.9: The setting of Lemma 6.7.12

"$\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$"$= \left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ and "$\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$"$= \left[\begin{array}{c|c}\mathbb{K}' & \mathbb{F}'\end{array}\right]$. $\square$

In optimization problems, one often first stabilizes a system exponentially and then optimizes the exponentially stable closed-loop system with the aid of a spectral factorization.

Since optimization should be independent on preliminary stabilization, one would expect the same for the existence of a spectral factorization; this is indeed the case:

Lemma 6.7.13 (Exp. stabilized SpF) *Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ and $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ be exponentially stabilizing state feedback pairs for $\Sigma \in \mathrm{WPLS}(U,H,Y)$ with closed-loop systems $\Sigma_\flat$ and $\Sigma_{\circlearrowleft}$, respectively.*

If $\mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft} = \widetilde{\mathbb{X}}^* S \widetilde{\mathbb{X}}$ for some $J = J^* \in \mathcal{B}(Y)$, $S \in \mathcal{GB}(U)$ and $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U)$, then $\mathbb{D}_\flat^* J \mathbb{D}_\flat = (\mathbb{X}')^* J \mathbb{X}'$, where $\mathbb{X}' := \widetilde{\mathbb{X}}(I - \mathbb{F}^2)(I - \mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U)$.

Proof: By Lemma 6.7.11(a'), $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$:=(6.180) is exponentially stable and exponentially stabilizing for $\Sigma_\flat$ (since $\mathbb{A}_\flat$ and $\mathbb{A}_{\circlearrowleft}$ are exponentially stable; see Lemma 6.1.10), and it leads to closed-loop system with same top two rows. In particular, $\mathbb{X}_\natural := I - \mathbb{F}_\natural = (I - \mathbb{F}^2)(I - \mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U)$, and $\mathbb{D}_{\circlearrowleft} = \mathbb{D}_\flat \mathbb{X}_\natural^{-1}$.

Consequently, $\mathbb{D}_\flat^* J \mathbb{D}_\flat = \mathbb{X}_\natural^* \mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft} \mathbb{X}_\natural = (\mathbb{X}')^* J \mathbb{X}'$, where $\mathbb{X}' := \widetilde{\mathbb{X}} \mathbb{X}_\natural$. By Lemma 6.4.7(c), $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{X}' \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$. $\square$

By combining Lemma 6.7.11 and Theorem 6.7.10 we obtain the following result:

Proposition 6.7.14 (An I/O-stabilizing L is stabilizing) *Let $\Sigma \in \mathrm{WPLS}$.*

(a) **(SOS)** *If Σ is SOS-stabilizable, then any I/O-stabilizing static output feedback operator L for Σ is SOS-stabilizing.*

(b) **([Strong] stability)** *Suppose that any of the following conditions holds:*

(1.) Σ is [[exponentially] strongly] q.r.c.-stabilizable.
(2.) Σ is [[exponentially] strongly] q.l.c.-detectable.
(3.) Σ is SOS-stabilizable and [[exponentially] strongly] detectable.
(4.) Σ is detectable and [exponentially] stabilizable.

Then any I/O-stabilizing static output feedback operator L for Σ is [[exponentially] strongly] stabilizing.

(c) **(Exponential stability)** *Suppose that any of the following conditions holds:*

(1.) Σ is optimizable and estimatable;
(2.) Σ is optimizable and input-detectable;
(3.) Σ is estimatable and output-stabilizable;
(4.) Σ is optimizable and q.r.c.-stabilizable;
(5.) Σ is estimatable and q.l.c.-detectable.

Then any I/O-stabilizing static output feedback operator L for Σ is exponentially stabilizing. Conversely, if some I/O-stabilizing output feedback operator for Σ is exponentially stabilizing, then (1.) holds.

Note the "missing strong stabilizability result" in (4.) corresponding to the analogous "results" missing in Theorems 6.7.10(c), 7.2.3 and 7.3.11 (we do not even know whether such "results" are true).

Of course, one can analogously deduce from Theorem 6.7.10, that, e.g., if Σ is estimatable, then an output-stabilizing L is exponentially stabilizing, but the above results will be applied to the I/O-stabilization theory of Chapter 7, hence we are only interested of consequences of I/O-stabilization only.

Proof: (a)&(b)(1.) By Lemma 6.7.11(c), the resulting closed-loop system Σ_L is also [SOS-/strongly/exponentially] q.r.c.-stabilizable. Because it is I/O-stable, by the assumption, it is [SOS-/strongly/exponentially] stable, by Theorem 6.7.10 (note that "q.r.c." is not needed in the SOS case).

The proofs of (b) assuming (2.), (3.) or (4.) are analogous. (We do not know whether the (mere) "strong" version of (4.) holds; cf. Theorem 6.7.10(c)&(d)(iv).)

(c) The proof of (c) is analogous to that of (b), except for the necessity of (1.): If $L \in \mathcal{B}(Y,U)$ is exponentially stabilizing for Σ, then $u := L(I-\mathbb{D}L)^{-1}\mathbb{C}x_0$ is in L^2 and satisfies $\mathbb{A}x_0 + \mathbb{B}u \in \mathrm{L}^2$, by (6.125). By duality (see Lemma 6.7.2(a)), also Σ^{d} is optimizable, i.e., Σ is estimatable. $\square$

We now list similar (often weaker) results on state feedback stabilization:

Theorem 6.7.15 (An I/O-stabilizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is stabilizing) *Let* $\Sigma \in$ WPLS*, let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ.*

(a1) Let Σ be [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [q.]r.c.-stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing for Σ.

(a2) Let Σ be strongly [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is strongly [q.]r.c.-stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing for Σ.

(b1) Let Σ be exponentially [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially [q.]r.c.-stabilizing for Σ iff it is I/O-stabilizing or input-stabilizing.

(b2) Let Σ have a exponentially stabilizing and [q.]r.c.-stabilizing state feedback pair. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing and [q.]r.c.-stabilizing for Σ iff it is I/O-stabilizing or input-stabilizing.

(b3) Let Σ be optimizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is input-stabilizing for Σ.

(c1) Let Σ be output stabilizable (or optimizable) and estimatable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing for Σ iff it is I/O-stabilizing, output-stabilizing or input-stabilizing.

(c2) Let Σ be estimatable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-I/O-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing.

Moreover, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing.

(c3) Let Σ be input-detectable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing.

(d) Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ be [strongly] stable. If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ [strongly] stabilizes $\mathbb{A}$.

(e) If Σ is exponentially stable, then the following are equivalent:

(i) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing
(ii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is input-stabilizing
(iii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing
(iv) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing;
(v) $(I-\mathbb{F})^{-1} \in \mathrm{TIC}$.

We leave the dual results (concerning output injection) to the reader (see [Sbook, Theorem 8.4.11]). Note that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing iff $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}$, where $\mathbb{M} = (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ (hence $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$).

The conclusion of part (b1) is quite strong: any I/O-stabilizing feedback pair for Σ is exponentially q.r.c.-stabilizing (note that the q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ mentioned above is exponentially q.r.c.); this conclusion is based on Lemma 6.1.10(a1).

Recall from Lemma 6.6.13 that an exponentially stable system is obviously exponentially r.c.-stabilizable and estimatable, and a strongly stable system is strongly r.c.-detectable.

The results in (a) are not as strong as their counterparts for static output feedback. The reason is that we had to assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing, because an I/O-stabilizing pair would only guarantee that some pair would stabilize $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by Lemma 6.7.11(a), and that is not enough for Theorem 6.7.10.

Proof of Theorem 6.7.15: The "only if" parts are trivial, so we only prove the "if" parts, with the notation of Definition 6.6.10.

(a1)[&(a2)] By Lemma 6.7.11(a2), some stable pair $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ r.c.-stabilizes $\Sigma_\flat$. By Lemma 6.7.10(a)(iii)[(c)(iv)], $\Sigma_\flat$ is [strongly] stable.

If Σ has a r.c.f., then the q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., by Lemma 6.4.5(c).

(b1)&(b2) These follow from Lemma 6.7.11(a4)&(a5).

(b3) If Σ is exponentially stabilizable, then, by Lemma 6.7.11(a3), $\Sigma_\flat$ is exponentially stabilizable, hence it is exponentially stable, by Theorem 6.7.10(d)(ii).

Assume then that Σ is optimizable. Then $\Delta^{\mathrm{S}}\Sigma_\flat$ is exponentially stable, by the discrete-time version of this claim, hence $\Sigma_\flat$ is exponentially stable.

(c1) By (c2), Σ is exponentially q.r.c.-stabilizable. Therefore, the claims on input- and I/O-stabilization follow from (b1); the claim on output-stabilization follows from (c2).

(If Σ is optimizable and estimatable, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing by discretization of Σ, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and this result (see Proposition 13.3.14).)

(c2) 1° Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be output-stabilizing. By Lemma 13.3.17(b), $\Delta^{\mathrm{S}}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Delta^{\mathrm{S}}\Sigma$, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing for Σ, by Theorem 13.4.4(e1).

2° Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be I/O-stabilizing. By Lemma 13.3.17(c), $\Delta^{\mathrm{S}}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Delta^{\mathrm{S}}\Sigma$, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-I/O-stabilizing for Σ, by Theorem 13.4.4(e1).

(c3) By the dual of (c1), any input-stabilizable $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ is exponentially q.l.c.-stabilizing, hence Σ is exponentially detectable. Therefore, (c3) follows from (c1).

(d) This follows from Lemma 6.6.8. (Note that if $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is SOS-stabilizing, then it is [strongly] stabilizing.)

(e) This follows from (c1) and the fact that $\mathbb{D}$ and $\mathbb{F}$ are necessarily (exponentially) stable (by Lemma 6.1.10(a1)). □

Since SOS-q.r.c.-stabilizability is rather important for optimization theory (unless we require the closed-loop system to be exponentially stable), we summarize below three cases in which this property can be deduced from other stabilizability and detectability properties:

Corollary 6.7.16 (q.r.c.-stabilizable) *If Σ is [strongly/SOS-]stable, or Σ is jointly [strongly/SOS-]stabilizable and I/O-detectable, or Σ is output-stabilizable and estimatable, then Σ is [strongly/SOS-]q.r.c.-stabilizable.* □

(This follows from Lemma 6.6.13 (r.c.), Theorem 6.6.28 (r.c.) and Theorem 6.7.15(c1) (exponentially q.r.c.).)

We note that certain kind of similarity transforms do not affect the properties of a system:

Lemma 6.7.17 (Permutations) *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. Let also U', H' and Y' be Hilbert spaces, $F \in \mathcal{GB}(U',U)$, $E \in \mathcal{GB}(H',H)$, and $G \in \mathcal{GB}(Y',Y)$. Then the following systems are also WPLSs and have the exactly the same stability, stabilizability and detectability properties (those listed in Summary 6.7.1, including prefices and suffices and optimizability and estimatability) as Σ has:*

$$\left[\begin{array}{c|c} E^{-1}\mathbb{A}E & E^{-1}\mathbb{B} \\ \hline \mathbb{C}E & \mathbb{D} \end{array}\right], \quad \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}F \\ \hline \mathbb{C} & \mathbb{D}F \end{array}\right], \quad \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline G\mathbb{C} & G\mathbb{D} \end{array}\right]. \tag{6.195}$$

If, instead, we do not require F and G to be invertible, then the above systems are still WPLSs and have same stability properties and Σ (but the stabilizability and detectability properties may be weaker or stronger in general).

The same holds for

$$\left[\begin{array}{c|c} L\mathbb{A}R & L\mathbb{B} \\ \hline \mathbb{C}R & \mathbb{D} \end{array}\right] \tag{6.196}$$

if $R \in \mathcal{B}(H', H)$ *and* $L \in \mathcal{B}(H, H')$ *are s.t.* $LR = I_{H'}$ *and* $\mathbb{C}P\mathbb{A}P = \mathbb{C}\mathbb{A}P$, $P\mathbb{A}P\mathbb{B} = P\mathbb{A}\mathbb{B}$ *and* $P\mathbb{A}P\mathbb{A}P = P\mathbb{A}P$, *where* $P := RL \in \mathcal{B}(H)$.

Obviously, invertible E, F and G do not essentially affect the other properties of a system either. In particular, permutations (of the above form) of rows and columns do not affect the properties of a system (matrix).

Proof: (We will use this lemma only for properties (a)–(d) of Summary 6.7.1, so the reader need not worry about that (e)–(f') have not been studied this far.)

We only prove the "if" claims; the converses will follow from this by applying E^{-1} (resp. F^{-1}, G^{-1}) in place of E (resp. F, G).

It is obvious that the three systems of (6.195) are in WPLS_ω if Σ is (for any $\omega \in \mathbf{R}$).

For E, the operators stabilizing/detecting Σ will stabilize/detect the first system in (6.195) in any case.

For F, we may use $F^{-1}L$ in (a) to obtain $\left[\begin{smallmatrix} \mathbb{A}_L & \mathbb{B}_L F \\ \mathbb{C}_L & \mathbb{D}_L F \end{smallmatrix}\right]$ as the closed-loop system; similar remarks prove the other cases too (with the notations of the corresponding definitions, use $\left[\begin{array}{c|c} F^{-1}\mathbb{K} & F^{-1}\mathbb{F}F \end{array}\right]$ in (b), $F^{-1}\mathbb{Q}$ in (c), multiply (7.20) by $H := \left[\begin{smallmatrix} F & 0 \\ 0 & I_{Y\times\Xi} \end{smallmatrix}\right]$ to the right and by its inverse to the left (so that $L = I = H^{-1}IH$ is stabilizing for the resulting system, by (a)) to prove (e'), and so on.

One easily verifies the claim on (6.196) by using Lemma A.4.2(h2). □

In the dynamic output feedback theory of Chapter 7, we will often combine a plant $\Sigma_1 \in \mathrm{WPLS}(U, H, Y)$ and its controller $\Sigma_2 \in \mathrm{WPLS}(Y, H_2, U)$ to form a larger system in the following way (with the third and fourth rows interchanged for convenience):

Lemma 6.7.18 *Let* $\Sigma_i = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U_i, H_i, Y_i)$ *for* $i = 1, 2$. *If* Σ_1 *and* Σ_2 *have some property (with some allowable prefices and suffices) listed in (a)–(d) of Summary 6.7.1, then so does their* parallel connection

$$\Sigma := \left[\begin{array}{cc|cc} \mathbb{A}_1 & 0 & \mathbb{B}_1 & 0 \\ 0 & \mathbb{A}_2 & 0 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & 0 & \mathbb{D}_1 & 0 \\ 0 & \mathbb{C}_2 & 0 & \mathbb{D}_2 \end{array}\right]. \tag{6.197}$$

Moreover, Σ_1 *and* Σ_2 *are optimizable (resp. estimatable) iff* Σ *is optimizable (resp. estimatable).*

Proof: This is obvious, because the required stabilizing/detecting operators for Σ can be combined from those for Σ_i ($i = 1, 2$); e.g., we use $L := \left[\begin{smallmatrix} L_1 & 0 \\ 0 & L_2 \end{smallmatrix}\right]$

and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{cc|cc} \mathbb{K}_1 & 0 & \mathbb{F}_1 & 0 \\ 0 & \mathbb{K}_2 & 0 & \mathbb{F}_2 \end{array}\right]$ (the symbols correspond to those in Summary 6.7.1). □

The concept "ω-stabilization" is sometimes used in the literature. All our stabilization results can be shifted to ω-stabilization results:

Remark 6.7.19 (ω-stabilization) *It follows from Remark 6.1.9 that any feedback or injection X (see Summary 6.7.1) is ω-stabilizing for $\Sigma \in$ WPLS iff $\mathcal{T}_{-\omega}X$ is stabilizing for $\mathcal{T}_{-\omega}\Sigma$ (all prefices and suffices apply).*

In particular, one gets directly a corollary about ω-stabilization of any of the results in this section (or in those to follow).

For example, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. over TIC_ω, then L ω-stabilizes $\mathbb{D}$ iff $(\mathbb{M} - L\mathbb{N})^{-1} \in \mathrm{TIC}_\omega$, by Lemma 6.6.6. These corollaries can be used to deduce results on exponential stabilization. □

Recall that "exponentially" means "for some $\omega < 0$" and that hence "exponentially strongly" is equivalent to "exponentially".

Notes

Most of Lemma 6.7.2 is well known. Optimizability has become popular since (or before) [FLT], estimatability is from [WR00]. Lemmas 6.7.5 and 6.7.6 are from [WR00] (in fact, from earlier conference versions of [WR00]). While writing these notes, we found an independent copy of Lemma 6.7.8 in [WR00]. The implication $u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$ of Theorem 6.7.7 was given in Lemma 14.1 of [ZDG] for finite-dimensional exponentially detectable systems.

In the spring of 1999, the manuscript of this monograph contained (a) and the q.r.c. and q.l.c. parts of Proposition 6.7.14; parts of Theorem 6.7.10 and Lemma 6.7.11(c) were implicitly contained in its proof. O. Staffans adopted these into [Sbook] and expanded them to early variants of Theorem 6.7.10 (including (c)(ii)&(iii) and weaker variants of (a)(ii) and (d)(ii)&(iii)) and of Lemma 6.7.11. We then adopted these and expanded them to Theorem 6.7.10 by adding further results and (d)(iii) from Theorem 7.4 of [WR00] (which extended Corollary 1.8 of [Rebarber]), and the H^∞ parts of (d)(ii)&(iii) from Propositions 6.1&6.2 of [WR00].

The claims on optimizability and estimatability in Lemma 6.7.11(c) are from Theorem 7.3 of [WR00]; most of the rest is from Lemma 8.2.7 of [Sbook]. Much of Lemma 6.7.11(a)&(b) and Lemma 6.7.12 is based on Lemma 4.5 of [S98a].

The r.c. parts of (a1) and (a2) and the (r.c.) I/O part of (b1) of Theorem 6.7.15 are from Theorem 8.4.11 of [Sbook].

Several of the above results can be found in [Sbook] (probably even more in its final version) with a more detailed proof; see [Sbook] also for further results. The article [WR00] includes further results on optimizability and estimatability.

6.8 Systems with $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,1];H)$

I must Create a System, or be enslav'd by another Man's;
I will not Reason and Compare: my business is to Create.
— William Blake (1757–1827), "The Words of Los"

In this section we shall study the properties of systems whose semigroup is smoothing in the sense described below. In Section 9.2, we shall establish a rather complete Riccati equation theory for such systems.

If B is bounded ($B \in \mathcal{B}(U,H)$) or A is smoothing, then we may have $\mathbb{A}(t)Bu_0 \in H$ for a.e. $t > 0$ whenever $u_0 \in U$ (this is typical for parabolic-type systems), in which case actually $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$, as shown in (b1) below. However, unless $Bu_0 \in H$, we have $\|\mathbb{A}(t)Bu_0\|_H \to \infty$ as $t \to 0+$. Nevertheless, one often has $\mathbb{A}Bu_0 \in \mathrm{L}^p((0,\varepsilon);H)$ for some (hence all) $\varepsilon > 0$; in that case we actually have $\mathbb{A}Bu_0 \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$ for any $\omega > \omega_A$. If the above condition is satisfied for all $u_0 \in U$, then $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$ for all $\omega > \omega_A$ and we have a number of additional tools and regularity properties at hands, as one observes from the results of this section. Naturally, an analogous claim applies to the dual property $\mathbb{A}^*C^*y_0 \in \mathrm{L}^p((0,\varepsilon);H)$ ($y_0 \in Y$).

As above, by "$\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);*)$" we mean just that $\pi_{[0,T)}\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);*)$ (i.e., this expression does not say anything about $\pi_{[T,\infty)}\mathbb{A}Bu_0$). The reader might wish to recall Proposition 6.3.4 before going on.

Lemma 6.8.1 ($\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p)$) *Let $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega > \omega_A$, $T > 0$ and $p \in [1,\infty]$.*

(a) We have $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$ iff $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);H)$ for all $u_0 \in U$.

This holds for $p = 2$ iff $(s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,H))$.

(b1) For any $t > 0$, the following are equivalent:

(i) $\mathbb{A}^tB[U] \subset H$;
(ii) $\mathbb{A}^{t*}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$;
(iii) $\mathbb{A}^{t*}[H] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})$;
(iv) $B^*\mathbb{A}^{t*}$ extends to $\mathcal{B}(H,U)$.

If (i) holds, then $\mathbb{A}^sB \in \mathcal{B}(U,H)$, $\mathbb{A}^{s*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ and $(\mathbb{A}^sB)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{s*}$ for all $s \geq t$, $\mathbb{A}B \in \mathcal{C}([t,\infty);\mathcal{B}(U,H))$ and $\mathbb{A}^* \in \mathcal{C}([t,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$.

(b2) $\mathbb{A}^tBu_0 \in H$ for a.e. $t \in [0,T)$, for all $u_0 \in U$ iff $B^*\mathbb{A}^{t*}$ extends to $\mathcal{B}(H,U)$ for a.e. $t \in [0,T)$.

Assume, in addition, that $\mathbb{A}Bu_0 \in H$ a.e. on $[0,T)$ for all $u_0 \in U$, and that $q \in [1,2]$, $\alpha \in \mathbf{R}$.

(c) $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$, $\mathbb{A}^* \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$ and $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q_{\mathrm{loc}}((0,\infty);Y) \cap \mathrm{L}^q_\omega([T,\infty);Y)$ (in particular, $\mathbb{A}Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e. on $\mathbf{R}_+$) for all $u_0 \in U$.

(d1) We have $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathrm{L}^q_\omega(\mathbf{R}_+;Y))$ iff $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ for all $u_0 \in U$.

This holds for $q = 2$ iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(d2) We have $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^q_\omega(\mathbf{R}_+)))$ iff $\langle C_{\mathrm{L,s}}\mathbb{A}Bu_0, y_0\rangle_Y \in \mathrm{L}^q([0,T))$ for all $u_0 \in U$, $y_0 \in Y$.

This holds for $q = 2$ iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(e1) If $Fu_0 := C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ for all $u_0 \in U$, then $\mathbb{D} \in \mathrm{SLR} \cap \mathrm{SVR}$ (and $\mathbb{D} \in \mathrm{ULR}$ if $q > 1$), $\widehat{\mathbb{D}} = \widehat{F} + D$ and $\mathbb{D}u = F * u + Du$ for all finite-dimensional $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ (for all $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ if $F \in \mathrm{L}^p(\mathbf{R}_+;\mathcal{B}(U,Y))$).

(e2) Conversely, if $\widehat{\mathbb{D}} = \widehat{F} + D$ for some $F \in \mathcal{B}(U, \mathrm{L}^p_\alpha(\mathbf{R}_+;Y))$, then $F = C_{\mathrm{L,s}}\mathbb{A}B$.

If, in addition, $F \in \mathrm{L}(\mathbf{R}_+;\mathcal{B}(U,Y))$, then $\mathbb{A}B \in \mathcal{B}(U, \mathrm{Dom}(C_{\mathrm{L,s}}))$ a.e. and $C_{\mathrm{L,s}}\mathbb{A}B = F$ a.e., hence then $C_{\mathrm{L,s}}\mathbb{A}B \in e^{\alpha\cdot}\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y)) \cap \mathrm{L}(\mathbf{R}_+;\mathcal{B}(U,Y))$.

(e3) Claims (e1)–(e2) also hold with replacements $\mathcal{B}(U, \mathrm{L}^p_\alpha(\mathbf{R}_+;Y)) \mapsto \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\alpha(\mathbf{R}_+)))$, $\mathrm{L}^p_\alpha \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{L}^q \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{L}^p_{\mathrm{strong}} \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{SLR} \mapsto \mathrm{WLR}$, $\mathrm{SVR} \mapsto \mathrm{WVR}$ (and $C_{\mathrm{L,s}} \mapsto C_{\mathrm{L,w}}$ in the "in addition" paragraph).

(f) We have $\pi_+\mathbb{D}\pi_- u = \pi_+(C_{\mathrm{L,s}}\mathbb{A}B * \pi_- u) \in \mathcal{C}((0,\infty);Y)$ for any finite-dimensional $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$.

Actually, in (a) we have $\mathbb{A}B \in \mathrm{L}^p_{\mathrm{strong},\omega}(\mathbf{R}_+;\mathcal{B}(U,H))$, by (c). Note also that $\mathrm{L}^p([0,T);H) \subset \mathrm{L}^{p'}([0,T);H)$ for $p' \in [1,p]$, hence the case $p = 1$ is the weakest one (this applies to Lemma 6.8.3(a) too).

We rephrase the most important results of (d1), (e1) and (e2) as follows:

Corollary 6.8.2 ($\mathbb{D} = D + C_{\mathrm{L,s}}\mathbb{A}B*$) *Let $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ be WR and s.t. $\mathbb{A}Bu_0 \in H$ a.e. for all $u_0 \in U$. Then $\mathbb{D} - D$ is a strong convolution iff $C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ for some $q \in [1,2]$ and all $u_0 \in U$.*

If this is the case, then $C_{\mathrm{w}}\mathbb{A}B = C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathrm{L}^q_\omega(\mathbf{R}_+;Y))$, and $(\mathbb{D} - D)u = C_{\mathrm{w}}\mathbb{A}B * u = C_{\mathrm{L,s}}\mathbb{A}B * u$ for each $\omega > \omega_A$ and each finite-dimensional $u \in \mathrm{L}^2_\omega$. □

(Note that "$C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$" includes the assumption that $\mathbb{A}^t Bu_0 \in \mathrm{Dom}(C_{\mathrm{w}})$ for a.e. $t \in [0,T)$. Since $\mathrm{L}^q([0,T);Y) \subset \mathrm{L}^2([0,T);Y)$ for $q \in (2,\infty]$, we could allow for any $q \in [1,\infty]$ in the above equivalence (but possibly not in the latter paragraph).)

In deriving (b)–(f), we take advantage of the fact that $\mathbb{C}$ is "almost $\mathrm{L}^2_{\mathrm{strong},\omega}(\mathbf{R}_+;\mathcal{B}(U,H))$", i.e., $\mathbb{C} \in \mathcal{B}(U, \mathrm{L}^2_\omega(\mathbf{R}_+;H))$. Since $\mathbb{C}$ need not satisfy corresponding "uniform" condition, we cannot present complete "uniform" analogies of (b)–(f) in Lemma 6.8.3.

The reader might wish to consult Lemmas F.2.2–F.2.4 (resp. Lemma D.1.7) for convolutions corresponding to (a) (resp. to Lemma 6.8.3(a); the former ("strong") convolutions coincide with these standard ("uniform") convolutions for L^p_ω functions).

Proof of Lemma 6.8.1: (The logical order of the proof goes as follows: (a)–(b2), (c), (f), (d1)&(d2)1°, (e1)–(e3), (d1)&(d2)2°.)

(a) 1° *The first equivalence:* "Only if" is obvious, so assume that $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);H)$ for all $u_0 \in U$.

Let $u_0 \in U$ and choose $t \in (0,T)$ s.t. $x_t := \mathbb{A}^t Bu_0 \in H$ (recall that $\mathbb{A}Bu_0 \in \mathcal{C}(\mathbf{R}_+;H_{-1})$, so that $\mathbb{A}^t Bu_0$ is well-defined for each $t \geq 0$).

Then $\mathbb{A}^{t+\cdot} Bu_0 = \mathbb{A}x_t \in \mathrm{L}^p_\omega(\mathbf{R}_+;H) \cap \mathcal{C}(\mathbf{R}_+;H)$, by Lemma A.4.5. Consequently, $\mathbb{A}^{t+\cdot} Bu_0 \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$. Because $u_0 \in U$ was arbitrary, we have $\mathbb{A}B[U] \subset \mathrm{L}^p_\omega(\mathbf{R}_+;H)$. But, $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H_{-1}))$, hence $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$, by Lemma A.3.6.

2° *The claim on* $\mathrm{H}^2_{\mathrm{strong}}$*:* We have $\|(s-A)^{-1}Bu_0\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;H)} = \sqrt{2\pi}\|\mathbb{A}Bu_0\|_{\mathrm{L}^2_\omega}$, by (D.36).

(b1) Note first that $\mathbb{A}^t B \in \mathcal{B}(U,H)$ follows from (i) and $\mathbb{A}^{t*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ follows from (iii), by Lemma A.3.6. Because $\mathbb{A}^{\cdot - t} \in \mathcal{C}([t,\infty);\mathcal{B}(H))$, these imply that $\mathbb{A}B \in \mathcal{C}([t,\infty);\mathcal{B}(U,H))$ and $\mathbb{A}^* \in \mathcal{C}([t,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$, and we have $\mathbb{A}^s B \in \mathcal{B}(U,H)$, $\mathbb{A}^{s*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ for all $s \geq t$.

1° *(i)⇔(ii),* $(\mathbb{A}^t B)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{t*}$*:* For each $x_0 \in H_1$, we have

$$\langle x_0, \mathbb{A}^t Bu_0\rangle_{\langle H_1,H_{-1}\rangle} = \lim_{r\to 0+} \langle r(r-A^*)^{-1}x_0, \mathbb{A}^t Bu_0\rangle \tag{6.198}$$

$$= \lim_{r\to 0+} \langle B^* r(r-A^*)^{-1}\mathbb{A}^{t*}x_0, u_0\rangle_U = \langle B^*_{\mathrm{w}}\mathbb{A}^{t*}x_0, u_0\rangle_U. \tag{6.199}$$

If (i) holds, then the above limit exists for all $x_0 \in H$, hence then (ii) holds and $(\mathbb{A}^t B)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{t*}$.

Conversely, if (ii) holds, then (6.198) is bounded w.r.t. $\|x_0\|_H$ for all $u_0 \in H$, i.e., $\mathbb{A}^t Bu_0 \in H$ for all $u_0 \in H$ (see Definition A.3.23).

2° *The rest:* Because $\mathrm{Dom}(B^*_{\mathrm{L,s}}) \subset \mathrm{Dom}(B^*_{\mathrm{w}})$, we have (iii)⇒(ii). Obviously, (ii)⇒(iv). Finally, assume (iv), i.e., that $B^*\mathbb{A}^{t*} \in \mathcal{B}(H_1^*,U)$ has an extension $R \in \mathcal{B}(H,U)$. Then

$$B^*\frac{1}{r}\int_0^r \mathbb{A}^{s*}\mathbb{A}^{t*}x_0\,ds = R\mathbb{A}^{t*}\frac{1}{r}\int_0^r \mathbb{A}^{s*}x_0\,ds \to R\mathbb{A}^{t*}x_0, \tag{6.200}$$

by continuity, hence $B^*_{\mathrm{L,s}}\mathbb{A}^t x_0 = R\mathbb{A}^t$ exists, for any $x_0 \in H$, i.e., (iii) holds.

(b2) "Only if" follows from (b1). Assume then that $\mathbb{A}^t Bu_0 \in H$ for a.e. $t \in [0,T)$, for all $u_0 \in U$. Let $u_0 \in U$ be arbitrary. Then $\mathbb{A}^t Bu_0 \in H$ for arbitrarily small $t > 0$, and for such t we have $\mathbb{A}^{t+\cdot} Bu_0 = \mathbb{A}^{\cdot}\mathbb{A}^t Bu_0 \in \mathcal{C}(\mathbf{R}_+;H)$. Thus, $\mathbb{A}^s Bu_0 \in H$ for all $s > 0$.

Because $u_0 \in U$ was arbitrary, we have $\mathbb{A}^s B[U] \subset H$ for all $s > 0$, hence $B^*\mathbb{A}^{s*}$ extends to $\mathcal{B}(H,Y)$ for all $s > 0$, by (b1)(i)&(iv).

(c) Let $u_0 \in U$. For arbitrarily small $t > 0$, we have $\mathbb{A}^t Bu_0 \in H$, hence $\mathbb{A}^{t+\cdot} Bu_0 = \mathbb{A}^{\cdot}\mathbb{A}^t Bu_0 \in \mathcal{C}(\mathbf{R}_+;H)$, i.e., $\pi_{[t,\infty)}\mathbb{A}Bu_0 \in \mathcal{C}([t,+\infty);H)$, $\mathbb{A}^{t+r}Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ for a.e. $r > 0$, and $C_{\mathrm{L,s}}\mathbb{A}^{t+\cdot}Bu_0 = \mathbb{C}(\mathbb{A}^t Bu_0)(\cdot) \in \mathrm{L}^2_\omega(\mathbf{R}_+;Y)$, by Lemma 6.2.12(a).

In particular, $\mathbb{A}^t B[U] \subset H$ for any $t > 0$, hence $\mathbb{A}B \in \mathcal{C}((0,+\infty);\mathcal{B}(U,H))$, and $\mathbb{A}^* \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$, by (b1).

Now we have established (c) for $q = 2$. If $q < 2$, replace ω by some $\alpha \in (\omega_A, \omega)$ and recall that $\mathrm{L}^2_\alpha([T,\infty);Y) \subset \mathrm{L}^q_\omega([T,\infty);Y)$.

(d1) 1° The first claim follows from (c).

2° We have $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ iff $\widehat{\mathbb{D}} = \widehat{F}$ for some $F \in \mathcal{B}(U,\mathrm{L}^2_\omega(\mathbf{R}^+;Y))$, by Lemma F.3.4(d). Thus, "only if" follows from (e1) and 1°, and "if" from (e2).

(d2) The proof of (d1) applies mutatis mutandis.

(e1) We have $F \in \mathcal{B}(U,\mathrm{L}^q_\alpha(\mathbf{R}_+;Y))$ for any $\alpha > \omega_A$, by (d1). Thus, $\widehat{\widetilde{\mathbb{D}}} := \widehat{F}$ defines an operator $\widetilde{\mathbb{D}} \in \mathrm{TIC}_\omega(U,Y)$ with the properties claimed in (e1), by Proposition 6.3.4(a3)(&(a1)). By (f), density and continuity, we have $\pi_+\mathbb{D}\pi_- = \pi_+\widetilde{\mathbb{D}}\pi_-$, hence $\mathbb{D} = \widetilde{\mathbb{D}} + D$, by Corollary 2.1.8.

(e2) 1° By Proposition 6.3.4(a3), we have $(\mathbb{D} - D)u = F * u$ for finite-dimensional $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$. Let $u_0 \in U$. Substitute $f := \frac{1}{r}\chi_{[-r,0)}$ to (6.204) to observe that

$$\frac{1}{r}(\mathbb{D} - D)\chi_{[-r,0)}u_0 \to C_{\mathrm{L,s}}\mathbb{A}^t Bu_0 \tag{6.201}$$

for each t s.t. $\mathbb{A}^t Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$, hence a.e. By combining this with Proposition 6.3.4(a3), we obtain that $Fu_0 = C_{\mathrm{L,s}}\mathbb{A}^t Bu_0$ a.e. Because $u_0 \in U$ was arbitrary, we have $F = C_{\mathrm{L,s}}\mathbb{A}^t B$ as elements of $\mathcal{B}(U,\mathrm{L}^q_\alpha)$.

2° Assume that, in addition, $F \in \mathrm{L}(R_+;\mathcal{B}(U,Y))$. Then the limit $C_{\mathrm{L,s}}\mathbb{A}^t Bu_0 = F(t)u_0$ exists for all $u_0 \in U$ at each Lebesgue point t of F, by the computations in 1°. Therefore, $\mathbb{A}^t Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ (hence $C_{\mathrm{L,s}}\mathbb{A}^t B \in \mathcal{B}(U,Y)$) and $C_{\mathrm{L,s}}\mathbb{A}^t B = F(t)$ for such t, hence for a.e. $t \in \mathbf{R}_+$.

(e3) The proofs of (e1)–(e2) apply mutatis mutandis: add $\Lambda \in Y^{\mathrm{B}}$ to the left of suitable terms (i.e., use $\not\!\!\int$ instead of $\int$ etc.).

By (b1) (applied to Σ^{d}), we may use $C_{\mathrm{L,s}}$ instead of $C_{\mathrm{L,w}}$ everywhere except possibly in the "in addition" claim of (e2) (we only know that $\mathbb{A}B \in \mathcal{B}(U,\mathrm{Dom}(C_{\mathrm{L,w}}))$ a.e., hence $C_{\mathrm{L,w}}\mathbb{A}B \in \mathcal{B}(U,Y)$ a.e.; we do not know whether $C_{\mathrm{L,s}}\mathbb{A}^t B$ is defined for all $u_0 \in U$ at any $t \in \mathbf{R}_+$).

(f) Let $u = fu_0$, $f \in \mathrm{L}^2_\omega(\mathbf{R}_-)$, $u_0 \in U$ (the general case follows by linearity). For a.e. $t > 0$, we have (use 2.&4. of Definition 6.1.1, (6.24) and Lemma 6.2.12(c1)&(c4), and note that $x_0 := \mathbb{A}^t Bu_0 \in H$))

$$(\mathbb{D}\pi_- u)(t) = (\mathbb{C}\mathbb{B}u)(t) = C_{\mathrm{L,s}}(\mathbb{A}^t\mathbb{B}u) = C_{\mathrm{L,s}}(\mathbb{B}\tau^t u) = C_{\mathrm{L,s}}(\mathbb{A}B * u) \tag{6.202}$$

$$= C_{\mathrm{L,s}}\int_{-\infty}^0 \mathbb{A}^{t-s}Bu(s)\,ds = C_{\mathrm{L,s}}\int_0^\infty \mathbb{A}^s\mathbb{A}^t Bu_0 f(-s)\,ds \tag{6.203}$$

$$= \int_0^\infty C_{\mathrm{L,s}}\mathbb{A}^s\mathbb{A}^t Bu_0 f(-s)\,ds = (C_{\mathrm{L,s}}\mathbb{A}B * u)(t). \tag{6.204}$$

(We did not have to write $\lim_{T\to+\infty}\int_{-T}^0$ in (6.203), since we had $\mathbb{A}^{-\cdot}x_0 f(\cdot) \in \mathrm{L}^1(\mathbf{R}_-;H)$, because $\mathbb{A}^{-\cdot}x_0 \in \boldsymbol{Я}\mathrm{L}^2_\omega = \mathrm{L}^2_{-\omega}$.)

Since $\mathbb{A}^t Bu_0 \in \mathcal{C}((0,\infty);H)$, by (c), $C_{\mathrm{L,s}}\mathbb{A}^\cdot = \mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$ and $f(-\cdot) \in \mathrm{L}^2_{-\omega}$, we have $C_{\mathrm{L,s}}\mathbb{A}^\cdot\mathbb{A}^t Bu_0 f(-\cdot) \in \mathcal{C}((0,\infty);\mathrm{L}^1(\mathbf{R}_+;Y))$, hence (6.204)$\in \mathcal{C}((0,\infty);Y)$. $\square$

Next we present "uniform" counterparts of the "strong" claims presented in the above lemma:

Lemma 6.8.3 ($\mathbb{A}B \in \mathrm{L}^p$) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega > \omega_A$, $T > 0$ and $p \in [1,\infty]$.*

(a) The following are equivalent:

(i) $\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$;
(ii) $\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$;
(iii) $\mathbb{A}Bu_0 = Fu_0$ a.e. on $[0,T)$ for all $u_0 \in U$ and some $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$;
(iv) $\mathbb{B}\tau^T \phi u_0 = (Fu_0 * \phi)(T)$ for all $u_0 \in U$ and $\phi \in C_c^\infty((0,T))$, and some $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H)) + \mathrm{L}^1_\infty(\mathbf{R}_+;\mathcal{B}(U,H_{-1}))$.
(v) $B_w^*\mathbb{A}^* \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(H,U))$;
(vi) $B_w^*\mathbb{A}^* \in \mathrm{L}^p([0,T);\mathcal{B}(H,U))$;
(vii) $B^*\mathbb{A}^*x_0 = Fx_0$ a.e. on $[0,T)$ for all $x_0 \in H_1^*$ and some $F \in \mathrm{L}^p([0,T);\mathcal{B}(H,U))$;

If (i) holds, then $\mathbb{A}^* \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$, hence then we may above replace B_w^* by $B^*_{\mathrm{L,s}}$, $B^*_{\mathrm{L,w}}$ or B^*_{s}.

(b) If $\mathbb{C}x_0 = Fx_0$ a.e. on $[0,T)$ for all $x_0 \in H_1$ and some $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$, then $\mathbb{A} \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(C_{\mathrm{L,s}}))$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$.

(c) If $C_w\mathbb{A} \in \mathcal{B}(H,Y)$ and $\mathbb{A}B \in \mathcal{B}(U,H)$ a.e. on $[0,T)$, and $C_w\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,Y))$, then $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)) \cap C((0,\infty);\mathcal{B}(U,Y))$ and $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$.

Naturally, if we apply (a) to Σ^{d}, then (v)–(vii) turn to results on C and $\mathbb{A}$. Thus, we may use $C_{\mathrm{L,w}}$, C_{s} or C_{w} instead of $C_{\mathrm{L,s}}$ in (d) and (e). Sometimes one may also wish to use the fact that $B^*_{\mathrm{L,s}}$ is the dual of B (see Proposition 6.2.8(e)).

Note that the assumptions in (a)–(c) are satisfied by parabolic systems of the type described in Hypothesis 9.5.1.

Proof: (a) 1° *(i)⇒(iv):* This follows from (6.23).

2° *(ii)⇔(i):* Fix $t > 0$ s.t. $\mathbb{A}^t B \in \mathcal{B}(U,H)$ and work as in the proof of Lemma 6.8.1(a).

3° *(iii)⇒(ii):* For any $t > 0$, we have $\mathbb{A}^t B[U] \in H$, by Lemma 6.8.1(b), hence $\mathbb{A}B \in C((0,\infty);\mathcal{B}(U,H))$, by (b). But $F(t)u_0 = \mathbb{A}^t Bu_0$ for all u_0 at every Lebesgue point t of F, hence $\mathbb{A}B = F$ a.e. on $[0,T)$, hence $\pi_{[0,T)}\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B})$.

4° *(iv)⇒(iii):* By (6.23), we have $\int_0^T (\mathbb{A}Bu_0 - Fu_0)\phi\, dm = 0$ (the integral is taken in H_{-1}) for all ϕ, hence $\mathbb{A}Bu_0 = Fu_0$ as elements of $\mathrm{L}^1([0,T);H_{-1})$, by Theorem B.4.12(d), hence a.e. on $[0,T)$, i.e., (iii) holds.

5° *The rest:* By Lemma 6.8.1(b)&(c), any of (i)–(ii) and (v)–(vii) implies that $B^*\mathbb{A}^{t*}$ extends to $\mathcal{B}(H,U)$ a.e., $\mathbb{A}^* \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{L,s}})))$, and $(B^*_{\mathrm{L,s}}\mathbb{A}^*)^* = \mathbb{A}B \in C((0,\infty);\mathcal{B}(U,H))$.

Therefore, B_w^* and $B^*_{\mathrm{L,s}}$ are interchangeable everywhere in (a), and we have the equivalencies "(i)⇔(v)", "(ii)⇔(vi)", and "(vi)⇔(vii)" (because the unique extension $F(t)$ of $B^*\mathbb{A}^{t*}$ must be $B_{\mathrm{w}}^*\mathbb{A}^{t*}$ wherever the equality holds in (vii), hence a.e.).

(b) Because $C\mathbb{A} \in \mathcal{C}(\mathbf{R}_+;\mathcal{B}(H_1,Y))$, we have $C\mathbb{A}^t x_0 = F(t)x_0$ for all $x_0 \in H_1$ at each Lebesgue point t of F. Thus, $C\mathbb{A}^t$ extends to $F(t) \in \mathcal{B}(H,Y)$ at those points, so that $\mathbb{A} \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(C_{\mathrm{L,s}})))$, by Lemma 6.8.1(b), and $C\mathbb{A}^t = F(t)$ at those points; in particular, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$.

Let $\alpha \in (\omega_A,\omega)$. Set $M := \|\mathrm{e}^{-\omega\cdot}C_{\mathrm{L,s}}\mathbb{A}\|_{\mathrm{L}^p([0,T);\mathcal{B})}$, $M_\alpha := \|\mathrm{e}^{-\alpha\cdot}\mathbb{A}\|_\infty$. If $p=\infty$, then

$$\|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|_{\mathcal{B}} \le \|\mathrm{e}^{-\omega T}C_{\mathrm{L,s}}\mathbb{A}^T\|\|\mathrm{e}^{-\omega(t-T)}\mathbb{A}^{t-T}\|, \tag{6.205}$$

which is bounded for $t>T$. Thus, $\|\mathrm{e}^{-\omega\cdot}C_{\mathrm{L,s}}\mathbb{A}\|_{\mathcal{B}}$ is then bounded. Assume then that $p<\infty$. Then

$$\int_0^\infty \|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|_{\mathcal{B}}^p\,dt = \sum_{n\in\mathbf{N}}\int_0^T \|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|_{\mathcal{B}}^p\,dt\|\mathrm{e}^{-\omega Tn}\mathbb{A}^{Tn}\|_{\mathcal{B}}^p \tag{6.206}$$

$$\le M^p\sum_{n\in\mathbf{N}} M_\alpha^p \mathrm{e}^{-p(\omega-\alpha)Tn} < \infty. \tag{6.207}$$

(c) By Lemma 6.8.1(b1) (applied to Σ and Σ^{d}), we have $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathcal{C}((0,\infty);\mathcal{B}(H,Y))$. In particular, $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,Y))$, $C_{\mathrm{L,s}}\mathbb{A}^{T/2} \in \mathcal{B}(H,Y)$ and $\mathbb{A}^{T/2}B \in \mathcal{B}(U,H)$. Therefore, $\|\mathrm{e}^{-\alpha t}C_{\mathrm{L,s}}\mathbb{A}^{T/2}\mathbb{A}^t\mathbb{A}^{T/2}B\|_{\mathcal{B}(U,Y)}$ is bounded for each $\alpha > \omega_A$. Consequently,

$$C_{\mathrm{L,s}}\mathbb{A}^{T/2}\mathbb{A}^{\cdot}\mathbb{A}^{T/2}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)), \tag{6.208}$$

hence $C_{\mathrm{L,s}}\mathbb{A}^{\cdot}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,Y)) \cap \tau^{-T}\mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)) = \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$. Consequently, $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$, by Lemma 6.8.1(e1) and density (see Theorem B.3.11). $\square$

The $\mathrm{L}^2_{\mathrm{strong}}$ properties and all L^p properties described above are unaffected by bounded state feedback operators:

Lemma 6.8.4 (Bounded *K*) *Assume that* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $K \in \mathcal{B}(H,U)$*, and let* $\Sigma_\flat$ *be the corresponding closed-loop system, so that* $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\end{array}\right]$ *is generated by* $\left[\begin{array}{c|c}A+BK & B\end{array}\right]$*. Let* $p\in[1,\infty]$*.*

(a1) If $\mathbb{A}B \in \mathrm{L}^p([0,1];\mathcal{B}(U,H))$*, then* $\mathbb{A}_\flat B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *for all* $\omega > \omega_{A_\flat}$*.*

(a2) If $\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *and* $\mathbb{M} \in \mathrm{TIC}_\omega(U)$ *for some* $\omega \in \mathbf{R}$*, then* $\mathbb{A}_\flat B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *and* $\mathbb{M} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$*.*

(b) If $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ *for all* $u_0\in U$*, then* $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$ *for all* $\omega > \omega_{A_\flat}$ *and* $u_0 \in U$*.*

(c1) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$*,* $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Y))$ *and* $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,Y))$*, then* $\mathbb{D}_\flat \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$ *for any* $\omega > \omega_{\mathbb{A}_\flat}$*.*

(c2) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$ *and* $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);Y)$ *for all* $u_0\in U$*, then* $\mathbb{D}_\flat - D \in \mathcal{B}(U,\mathrm{L}^1_\omega(\mathbf{R}_+;Y))*$ *(i.e.,* $(C_\flat)_{\mathrm{L,s}}\mathbb{A}_\flat Bu_0 \in \mathrm{L}^1_\omega(\mathbf{R}_+;Y)$ *for all* $u_0\in U$*) for all* $\omega > \omega_{A_\flat}$*.*

(c3) If $\mathbb{A}Bu_0, C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];*)$ for all $u_0 \in U$, then $\widehat{\mathbb{D}_\flat} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ for all $\omega > \omega_{A_\flat}$.

(d) Assume that $\omega_{A_\flat} < 0$. Then $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{ULR} \cap \mathrm{SHPR}$ in (a1)–(c3); in (c1)–(c3) we also have $\mathbb{D}_\flat \in \mathrm{SHPR}$ (and $\mathbb{D}_\flat \in \mathrm{ULR}$ in (c1) and (c3)).

Note that the assumption in (c2) holds iff $\mathbb{B}\tau \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty$ and $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^1_\alpha(\mathbf{R}_+;Y))*$ for some $\alpha \in \mathbf{R}$. As usual, $\mathbb{M} := (I - \mathbb{F})^{-1}$, where $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ is generated by K.

Proof: (As in Definition 6.6.10, we have set $\mathbb{M} := I - \mathbb{F}$, where $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ is the state feedback pair corresponding to K. See Proposition 6.6.18(d3) for the generators of $\Sigma_\flat$.)

(a2) By Lemma 6.1.16(b), we have $\mathbb{B}\tau u = \mathbb{A}B * u$; by (6.46), we have $\mathbb{F} = K\mathbb{B}\tau = K\mathbb{A}B*$. But $K\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, hence $\mathbb{F} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U) \subset \mathrm{TIC}_\omega(U)$.

Thus, $\mathbb{X} := I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\omega(U) \cap \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$, hence $\mathbb{M}^{-1} = \mathbb{X} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$, by Theorem 4.1.1(b)(i)&(ii) and Remark 6.1.9.

But then $\mathbb{B}_\flat\tau = \mathbb{B}\tau\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,H)$ (by Lemma D.1.7), i.e., $\mathbb{A}_\flat B_\flat = \mathbb{A}_\flat B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ (see Lemma 6.8.3(a)(iv)&(i)).

(a1) Obviously, we may replace L^1 by $\mathrm{L}^1 \cap \mathrm{L}^p$ in (a2) and its proof.

Choose some $\alpha > \max\{\omega, \omega_A\}$. By Lemma 6.8.3(a), we have $\mathbb{A}B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,H))$. But $\Sigma_\flat$ is ω-stable, hence $\mathbb{M} \in \mathrm{TIC}_\omega$. Consequently, $\mathbb{A}_\flat B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,H))$, by (a2) (modified, as noted above). By Lemma 6.8.3(a), $\mathbb{A}_\flat B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$.

(b) By Lemma 6.3.3(b1), we have $\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$, hence $\widehat{\mathbb{B}_\flat\tau} = \widehat{\mathbb{B}\tau}\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$, by Lemma 6.3.3(b1). But $\widehat{\mathbb{B}_\flat\tau}(+\infty) = 0$, hence $\widehat{\mathbb{B}_\flat\tau} \in \mathrm{H}^2_{\mathrm{strong},\infty}$. By Lemma 6.8.1(a), we have $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$ for all $\omega > \omega_{A_\flat}$ and $u_0 \in U$.

(c1) (As one observes from the proof, we have $C_{\mathrm{L,s}}\mathbb{A}_\flat B \in \mathrm{L}^p_\omega$ if $\mathbb{A}B, C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1$ on $[0,1)$.)

We have

$$\mathbb{C}_\flat = \mathbb{C} + \mathbb{D}K\mathbb{A}_\flat \in \mathrm{L}^1 + \mathrm{L}^p * \mathrm{L}^\infty \subset \mathrm{L}^1 \tag{6.209}$$

on $[0,1)$ (because $\pi_{[0,1)}\mathrm{L}^1 * \pi_{[0,1)}\mathrm{L}^1 \subset \pi_{[0,1)}\mathrm{L}^1$, by Lemma D.1.7, and $K\mathbb{A}_\flat \in \mathcal{C} \subset \mathrm{L}^\infty_{\mathrm{loc}}$). Therefore, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1_\infty$, by Lemma 6.8.3(b). But $\mathbb{A}_\flat B \in \mathrm{L}^1_\infty$ and $\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y)$, by (a1), hence $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y)$. Consequently, $\mathbb{D}_\flat \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$, by Lemma 6.8.3(c).

(c2) By (a1), we have $K\mathbb{A}_\flat B \in \mathrm{L}^1([0,1];\mathcal{B}(U))$, hence $\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}{}_\infty(U)$. Since $\mathbb{D} \in \mathrm{SMTIC}^{\mathrm{L}^1}{}_\infty$, by the assumption, hence $\mathbb{D}_\flat := \mathbb{D}\mathbb{M} \in \mathrm{SMTIC}^{\mathrm{L}^1}{}_\infty$ and $D_\flat = D$, by Theorem 2.6.4(a1)&(h1)&(d). Thus, we obtain (c2) from this and Lemma 6.8.1(e2)&(d1)

(c3) This follows from (b) and Lemma 6.8.1(d1).

(d) Choose some $\omega \in (\omega_{A_\flat}, 0)$. In (a1) and (a2), we have $\mathbb{A}_\flat B, K\mathbb{A}_\flat B \in \mathrm{L}^1_\omega$, hence $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{ULR} \cap \mathrm{UHPR}$. In (b), we have $\mathbb{A}_\flat B, K\mathbb{A}_\flat B \in \mathrm{L}^2_\omega$, hence $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{ULR} \cap \mathrm{SHPR}$, by Proposition 6.3.4(a3).

The claim on $\mathbb{D}_\flat$ follows analogously from (c1)–(c3): use Proposition 6.3.4(a1)&(a3) for (c1)&(c2), respectively, and Proposition 6.3.3(a) for (c3). □

The systems described in the following lemma are in certain sense the most general class of systems to which we can extend the full connection between optimal control and exponentially stabilizing solutions Riccati equations without any a priori factorization or stability assumptions (see, e.g., Theorem 9.2.18):

Lemma 6.8.5 ($\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$) *Assume that* $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $p,q \in [1,\infty]$, $\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^q([0,t);\mathcal{B}(H,Y))$, *and* $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U,Y))$ *for some* $t>0$.

(a) Then $\mathbb{A}B \in \mathrm{L}^p_\omega$, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^q_\omega$ *and* $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega$ *on* $\mathbf{R}_+$
(in particular, $\mathbb{B}\tau, \mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,*) \subset \mathrm{ULR}\cap\mathrm{UVR}$*) for any* $\omega > \omega_A$.

(b) If $L \in \mathcal{B}(Y,U)$ *is an admissible output feedback operator for* Σ*, then* $\mathbb{A}_L B_L \in \mathrm{L}^p_\omega$, $(C_L)_{\mathrm{L,s}}\mathbb{A}_L \in \mathrm{L}^q_\omega$ *and* $(C_L)_{\mathrm{L,s}}\mathbb{A}_L B_L \in \mathrm{L}^p_\omega$ *on* $\mathbf{R}_+$ *for any* $\omega > \omega_{A_L}$.

(c) Let $K = SC + T$, $S \in \mathcal{B}(Y,U)$, $T \in \mathcal{B}(H,U)$*. Then* K *is a ULR admissible state feedback operator for* Σ*, and also the corresponding extended system* Σ_{ext} *and hence the closed-loop system* $\Sigma_\flat$ *satisfy the assumptions of this lemma.*

(d) If $p=q$*, then also* Σ^{d} *satisfies the assumptions of this lemma.*

(Naturally, we can replace the exponents p and q in the conclusion parts by smaller ones.)

We conclude that the above type of systems are closed w.r.t. static output feedback and w.r.t. state feedback of kind described in (c) (which often appears in connection with Riccati equations and optimal control). We observe from Lemma 9.5.4 and Proposition 6.6.18(b3) that an analogous claim holds for systems of the (parabolic) type of Hypothesis 9.5.1 as well as for those of the type of Hypothesis 9.5.7(3.).

Proof: (We shall use the facts that $\mathrm{L}^p([0,t);*) \subset \mathrm{L}^1([0,t);*)$ and that $\mathbb{A} \in \mathcal{C}\cap\mathrm{L}^s_\omega(\mathbf{R}_+;\mathcal{B}(H))$ for any $s\in[1,\infty]$ and $\omega>\omega_A$. See Lemma D.1.7 for convolutions. We observe from the proof that we could use a third exponent $r\in[1,\infty]$ (instead of p) for $C_{\mathrm{w}}\mathbb{A}B$ and $C_{\mathrm{L,s}}\mathbb{A}B$, but then (b) and (c) would no longer hold.)

(a) Let $\omega > \omega_A$. By Lemma 6.8.3(a) (applied to Σ^{d} and Σ), we can replace C_{w} by $C_{\mathrm{L,s}}$ and we have $\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^q_\omega(\mathbf{R}_+;\mathcal{B}(H,Y))$. By Lemma 6.8.3(c), it follows that $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ (in particular, $\mathbb{A}(t)B \in \mathcal{B}(U,\mathrm{Dom}(C_{\mathrm{L,s}}))$ for a.e. $t\in\mathbf{R}_+$). We conclude that $\mathbb{B}\tau,\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega$, by Corollary 6.8.2. Recall from Proposition 6.3.4(a1) that $\mathrm{MTIC}^{\mathrm{L}^1}_\infty \subset \mathrm{ULR}\cap\mathrm{UVR}$.

(b) Let $\alpha > \max\{\omega_A, \omega_{A_\flat}\}$. Set $\mathbb{X} := I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. Then $\mathbb{D} \in \mathcal{B}(U,Y) + (\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*$, by (a), hence $\mathbb{X} \in \mathcal{B}(U) + (\mathrm{L}^1_\alpha\cap\mathrm{L}^p_\alpha)*$.

We have $\mathbb{D},\mathbb{X},\mathbb{D}_L \in \mathrm{TIC}_\alpha$. Since $\mathbb{M} := \mathbb{X}^{-1} = I + L\mathbb{D}_L \in \mathrm{TIC}_\alpha$, we obtain from Proposition 6.3.4(a1) that $\mathbb{M} \in \mathcal{B}(U) + (\mathrm{L}^1_\alpha\cap\mathrm{L}^p_\alpha)*$. Analogously, we conclude that $(I-\mathbb{D}L)^{\pm 1} \in \mathcal{B}(Y)+(\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*$. Consequently, (see Proposition 6.6.18)

$$\mathbb{B}_L\tau = \mathbb{B}_L\mathbb{M}\tau = \mathbb{B}\tau\mathbb{M} \in \mathrm{L}^p_\alpha * (I+\mathrm{L}^p_\alpha *) = \mathrm{L}^p_\alpha, \tag{6.210}$$

$$\mathbb{D}_L = \mathbb{D}\mathbb{M} \in (D+(\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*)(M+\mathrm{L}^p_\alpha *) \subset DM+\mathrm{L}^p_\alpha *, \text{ and} \tag{6.211}$$

$$\mathbb{C}_L = (I-\mathbb{D}L)^{-1}\mathbb{C} \in (\mathcal{B}+\mathrm{L}^1_\alpha *)\mathrm{L}^p_\alpha = \mathrm{L}^p_\alpha. \tag{6.212}$$

We conclude from Corollary 6.8.2 that $\mathbb{A}_L B_L, (C_L)_{\mathrm{L,s}}\mathbb{A}_L B_L \in \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,*))$. Apply (a) to Σ_L to obtain the rest of (b).

(c) By Lemma 6.3.17, K is admissible and ULR for Σ. Obviously, $SC_{\mathrm{L,s}} + T \subset K_{\mathrm{L,s}}$ (in particular, $\mathrm{Dom}(C_{\mathrm{L,s}}) \subset \mathrm{Dom}(K_{\mathrm{L,s}})$), hence $K_{\mathrm{L,s}}\mathbb{A} = (SC_{\mathrm{L,s}} + T)\mathbb{A} \in \mathrm{L}^p([0,t);\mathcal{B}(H,U))$ (where the equality holds a.e.). Analogously, $K_{\mathrm{L,s}}\mathbb{A}B = (SC_{\mathrm{L,s}}+T)\mathbb{A}B \in \mathrm{L}^s([0,t);\mathcal{B}(U))$ (since $SC_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^r([0,t);\mathcal{B}(U))$ and $T\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U))$), where $s := \min\{p,r\}$. Thus, Σ_{ext} (see (6.132)) satisfies the assumptions of this lemma with s in place of r, hence so does $\Sigma_\flat$, by (b).

(d) By Lemma 6.8.3(a)(ii)&(vi), we have $B^*_{\mathrm{w}}\mathbb{A}^* \in \mathrm{L}^p([0,t);\mathcal{B}(H,U))$ and $\mathbb{A}^*C^* \in \mathrm{L}^p([0,t);\mathcal{B}(Y,H))$. By Lemma 6.8.3(c), $F := C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$, hence $F^* \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(Y,U))$, by Lemma B.3.6.

By Corollary 6.8.2, $\mathbb{D}-D = F*$, hence $\mathbb{D}^{\mathrm{d}} - D^* = (\mathbb{D}-D)^{\mathrm{d}} = F^**$, by Proposition 6.3.4(a1), hence $F^* = B^*_{\mathrm{w}}\mathbb{A}^*C^*$, by Corollary 6.8.2. Thus, $B^*_{\mathrm{w}}\mathbb{A}^*C^* \in \mathrm{L}^p([0,t);\mathcal{B}(Y,U)$. □

Notes

Pritchard–Salamon (PS) systems often satisfy the assumptions of Lemma 6.8.5 for $p = 2 = q$ (cf. Lemma 9.5.2), and in a sense the lemma allows for twice as much unboundedness as the axioms of PS-systems, but in general a PS-system might violate the assumptions of the lemma (but not those of Hypothesis 9.2.2, hence our complete "smooth Riccati equation theory", which uses the properties established in this section, covers also PS-systems).

In the control theory of optimal control of partial differential equations, one often makes similar assumptions on $C\mathbb{A}B$ with C bounded. E.g., in Section 8 of [LT00b], I. Lasiecka and R. Triggiany do not pose the assumption on $\mathbb{A}Bu_0 \in \mathrm{L}^p_{\mathrm{loc}}$ ($u_0 \in U$) and compensated this by a stronger assumption on C. They require the original system to be a WPLS but do not study the well-posedness of closed-loop systems.

The well-posedness of a closed-loop system means that under any error, disturbance or other external input to the feedback loop (the signal u_L of (6.124), the state, effective control and output of the system remain well defined and their dependence on this external input is continuous (from $\mathrm{L}^2_{\mathrm{loc}}$ to H, $\mathrm{L}^2_{\mathrm{loc}}$ and $\mathrm{L}^2_{\mathrm{loc}}$, respectively). In particular, finite input energy cannot lead to infinite output energy (or to undefined state and output) under a finite period of time. If, in addition, the closed-loop system is strongly stable, then the effect of any external L^2 signal vanishes asymptotically with time.

6.9 Bounded B, bounded C, PS-systems

'And death', said Thingol, 'thou shouldst taste,
had I not sworn an oath in haste
that blade nor chain thy flesh should mar.
Yet captive bound by never a bar
unchained, unfettered shalt thou be
in lightless labyrinths, endlessly.
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section we shall show that a transfer function $\widehat{\mathbb{D}}$ has a realization with a bounded input operator B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\text{strong}}$ over some right half-plane. We also establish analogous results for realizations with a bounded C and for Pritchard–Salamon realizations. In addition to U, H and Y, also $\mathcal{V}$ and $\mathcal{W}$ denote (arbitrary) Hilbert spaces in this section.

By Proposition 6.3.3(a), any $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\text{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ is the transfer function of some $\mathbb{D} \in \mathrm{TIC}_{\omega+\varepsilon}(U,Y) \cap \mathrm{ULR}$ with $D = 0$ (for any $\varepsilon > 0$). By strengthening the assumption slightly, we get a necessary and sufficient condition for $\widehat{\mathbb{D}}$ being the transfer function of a WPLS with a bounded B or C:

Theorem 6.9.1 ($\widehat{\mathbb{D}} \in \mathrm{H}^2_{\text{strong}} \Leftrightarrow B$ bounded) *Let $\omega \in \mathbf{R}$ and $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.*

(a) $\widehat{\mathbb{D}}$ is the transfer function of the I/O map of some $\Sigma \in \mathrm{WPLS}_\omega$ with a bounded B iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y)) \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(b) $\widehat{\mathbb{D}}$ is the transfer function of the I/O map of some $\Sigma \in \mathrm{WPLS}_\omega$ with a bounded C iff $\widehat{\mathbb{D}}(\bar{\cdot})^* - D^* \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(Y,U)) \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$.

(c) The corresponding realizations can be chosen so that they are minimal and they satisfy $\|B\|_{\mathcal{B}(U,H)} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2_{\text{strong}}}$ in (a), (or $\|C\|_{\mathcal{B}(H,Y)} \le \|\widehat{\mathbb{D}}(\bar{\cdot})\|_{\mathrm{H}^2_{\text{strong}}}$ in (b)), where H is the state space of the corresponding realization.

(d1) If we drop the assumption $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty_\omega$, then (a)–(c) still hold except that $\mathbb{B}$ and $\mathbb{D}$ in (a) (or $\mathbb{C}$ and $\mathbb{D}$ in (b)) are only known to be ω'-stable for any $\omega' > \omega$.

(d2) We can replace "$\Sigma \in \mathrm{WPLS}_\omega$" by "$\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$ s.t. $\mathbb{C}$ and $\mathbb{D}$ are ω-stable" in (a).

Analogously, in (b) it suffices to require $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$ be s.t. $\mathbb{B}$ and $\mathbb{D}$ are ω-stable. We may require Σ to be strongly ω-stable in (b).

Thus, $\mathbb{D}$ has a realization with a bounded B and $D = 0$ iff $\widehat{\mathbb{D}}(\cdot - \omega) \in \mathrm{H}^2_{\text{strong}}$ for some $\omega \in \mathbf{R}$ (see Lemma F.3.2(a)); a dual claim holds for C.

Proof: (a) 1° *"Only if":* Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ and $B \in \mathcal{B}(U,H)$ (in fact, $\Sigma \in \mathrm{SOS}_\omega$ is enough). W.l.o.g., we assume that $D = 0$ (see also Lemma 6.3.16(b)). We have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, by Theorem 6.2.1. Moreover, $\widehat{\mathbb{C}} \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(H,Y))$, by Theorem 6.2.11(c2), hence $\widehat{\mathbb{D}} = C(\cdot - A)^{-1}B \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ (and $\|\widehat{\mathbb{D}}\|_{\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))} \le \sqrt{2\pi}\|\mathbb{C}\|\|B\| < \infty$).

2° *"If":* Apply "2°" of (b) to $\mathbb{D}^{\mathrm{d}}$. In fact, by applying Lemma 6.7.17 with $E = \mathbf{Я}$ for the resulting realization of $\mathbb{D}$, we see that the input operator of the strongly ω-stable WPLS

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\mathbb{D}\pi_- \\ \hline I & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y) \tag{6.213}$$

is bounded.

(b) 1° *"Only if":* Apply "1°" of (a) to $\mathbb{D}^{\mathrm{d}}$.

2° *"If":* W.l.o.g. we assume that $\omega = 0$. It suffices to show the stable (and exactly reachable) realization

$$\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] := \left[\begin{array}{c|c} \tau\pi_- & \pi_- \\ \hline \pi_+\mathbb{D}\pi_- & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(U, H, Y) \tag{6.214}$$

on $H := \mathrm{L}^2(\mathbf{R}_-;U)$ has a bounded output operator C. By Lemma F.3.7(b2), the map

$$\bar{C}x_0 := (\mathbb{C}x_0)(0) = (\pi_+\mathbb{D}\pi_- x_0)(0) = (\mathbb{D}x_0)(0) \tag{6.215}$$

satisfies $\bar{C} \in \mathcal{B}(H,Y)$, $\|\bar{C}\| \le \|\widehat{\mathbb{D}}(\cdot)\|_{\mathrm{H}^2_{\mathrm{strong}}}$ (recall from (6.26) that $\bar{C}$ is an extension of the output operator $C \in \mathcal{B}(H_1, Y)$ of (6.214), and that C is called "bounded" iff it has an extension to $\mathcal{B}(H,Y)$ (which is necessarily unique, by density), and that C is identified with this extension (i.e., we write $C = \bar{C} \in \mathcal{B}(H,Y)$)).

(c) We prove this for (a); use duality for (b): Let $H_{\mathbb{B}}$ be the closure of $\pi_+\mathbb{D}\pi_-[\mathrm{L}^2_{\mathrm{c}}]$ in $\mathrm{L}^2_\omega(\mathbf{R}_+;Y)$ (i.e., the reachability subspace). Let P be the orthogonal projection $\mathrm{L}^2_\omega(\mathbf{R}_+;Y) \to H_{\mathbb{B}}$. By Lemma 6.3.26(e), $\Sigma' := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\mathbb{D}\pi_- \\ \hline I & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U, H_{\mathbb{B}}, Y)$ (note that $\pi_+\tau[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$) is reachable and also Σ' has bounded input operator. Because Σ' is (exactly ω-)observable, it is minimal.

(d1) We prove this for (b); use duality for (a):

Observe from the proof that C is bounded whenever $\widehat{\mathbb{D}}(\cdot) \in \mathrm{H}^2_{\mathrm{strong}}$, but if we do not assume that $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$, then we only know that $\mathbb{C}$ (by Lemma F.3.7(b2)) and $\mathbb{D}$ (by Lemma F.3.2(a)) are ω-bounded for any $\omega > 0$.

(d2) One observes from part 1° of the proof of (a) that Σ need not be ω-stable, it suffices that $\mathbb{C}$ and $\mathbb{D}$ are ω-stable. Part 2° of the proof of (a) shows that Σ can be required to be strongly ω-stable in (a) (and strongly* ω-stable in (b)). □

We shall soon show that a transfer function can be realized as a PS-system iff it has a (WPLS) realization with bounded B and one with bounded C. Before this we must define PS-systems:

Definition 6.9.2 (PS-systems) *A system* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U, \mathcal{V}, Y)$ *is a* Pritchard–Salamon system (PS-system) *iff* $B \in \mathcal{B}(U, \mathcal{V})$ *and there is a Hilbert space* $\mathcal{W} \subset \mathcal{V}$ *s.t. 1. the embedding* $\mathcal{W} \to \mathcal{V}$ *is dense and continuous, 2.* $\mathbb{A}|_{\mathcal{W}}$

is an ω*-stable* C_0*-semigroup on* $\mathcal{W}$*, 3.* $\mathbb{B}^t \in \mathcal{B}(\mathrm{L}^2([0,t);U),\mathcal{W})$ *for some (hence all)* $t>0$*, and 4. some* $C' \in \mathcal{B}(\mathcal{W},Y)$ *satisfies* $C'\mathbb{A} = \mathbb{C}$ *on* $\mathcal{W}$*.*

If, in addition, $\omega \in \mathbf{R}$ *is s.t.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y) \cap \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$*, then* Σ *is called* an ω-stable PS-system. *If* $\mathrm{Dom}(A) \subset \mathcal{W}$*, then* Σ *is called a* smooth PS-system.

(We also say that Σ is a *PS-system w.r.t.* U*,* $\mathcal{W}$*,* $\mathcal{V}$ *and* Y.)

Remark 6.9.3 *Definition 6.9.2 is equivalent to the standard definition of PS-systems (see, e.g., [KMR] or Definition 2.3 of [Keu]).*

(Note that it is not a limitation that [Keu] assumes the Hilbert spaces to be real — any complex Hilbert space is also a real Hilbert space.)

The above definition of ω-stability makes also "exponential stability" (i.e., being ω-stable for some $\omega < 0$) equivalent to the standard one, by Lemma 6.9.4.

Proof: Sufficiency is rather obvious. Conversely, if Σ is a PS-system, then the axioms of the above definition are satisfied as follows: The maps in (2.4) and (2.5) of [Keu] define maps $\mathbb{B}$ and $\mathbb{C}$. By (2.6) of [Keu], we have $\left[\frac{\mathbb{A}}{\mathbb{C}}\right] \in \mathrm{WPLS}$; let $\left[\frac{A}{C''}\right]$ be its generators. Then $\left[\begin{array}{c|c} A & B \\ \hline C'' & D \end{array}\right]$ generate a WPLS $\Sigma \in \mathrm{WPLS}(U,\mathcal{V},Y)$, by Lemma 6.3.13. Obviously, Σ is the original (PS-)system. $\square$

Thus, given a PS-system and a minimization problem, we "have a bounded B" (w.r.t. $\mathcal{V}$, i.e., $B \in \mathcal{B}(U,\mathcal{V})$). However, we do not necessarily "have a bounded C": if, for example, we are given a function such as $\mathcal{J}(x_0,u) := \|x\|^2_{\mathrm{L}^2(\mathbf{R}_+;\mathcal{V})} + \|u\|_2^2$ to be minimized, we would more be interested in $C \in \mathcal{B}(\mathcal{V},Y)$ rather than in $C \in \mathcal{B}(\mathcal{W},Y)$ in order to use the tools corresponding to a "bounded C", and C need not belong to $\mathcal{B}(\mathcal{V},Y)$. (To minimize, instead, $\|x\|^2_{\mathrm{L}^2(\mathbf{R}_+;\mathcal{W})} + \|u\|_2^2$, one could use the system $\Sigma \in \mathrm{WPLS}(U,\mathcal{W},Y)$ which does have a "bounded C".)

Sometimes the following characterization of PS-systems is more useful:

Lemma 6.9.4 (PS-systems) *A system* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,\mathcal{V},Y)$ *is an* ω*-stable PS-system iff 1.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y) \cap \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$*, where* $\mathcal{W} \subset \mathcal{V}$ *densely and continuously, 2.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)$ *has a bounded input operator, and 3.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$ *has a bounded output operator.*

Any PS-system is ULR and ω*-stable for some* $\omega \in \mathbf{R}$*; in particular, it is exponentially stable (i.e.,* ω*-stable for some* $\omega < 0$*) iff* $\mathbb{A}$ *is exponentially stable on* $\mathcal{W}$ *and on* $\mathcal{V}$*.*

The system in 3. is obtained from that in 2. by just replacing $\mathcal{V}$ by $\mathcal{W}$ (as the domain and/or range space of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$).

Proof: The first claim is obviously true (the operator $C' \in \mathcal{B}(\mathcal{W},Y)$ is the output operator of $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$). By Lemma 6.3.16(b) (or (c)), the I/O map of a PS-system ULR.

Assume that Σ is a PS-system. Choose $\omega \in \mathbf{R}$ s.t. $\mathbb{A}$ is ω'-stable on $\mathcal{W}$ and $\mathcal{V}$ for some $\omega' < \omega$. One easily verifies (cf. the reasoning on p. 158) that $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega,\mathcal{W})$; it follows that $\Sigma \in \mathrm{WPLS}(U,\mathcal{W},Y)$. By Lemma 6.1.10, both systems are ω-stable WPLSs, hence Σ is an ω-stable PS-system w.r.t. U, $\mathcal{W}$, $\mathcal{V}$ and Y. $\square$

PS-systems are a strict subset of WPLSs; e.g., any H^∞_∞ transfer function has a realization as a WPLS (see Definition 6.1.6), but the same does not hold for PS-systems, by Theorem 6.9.6. For a PS-system, the unboundedness of C severely limits the possible unboundedness of B, and vice versa:

Remark 6.9.5 (*B* and *C* of PS-systems may be as unbounded as those of WPLSs)

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$*, then* $\left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$ *(or* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *for any* $\begin{bmatrix} C & D \end{bmatrix} \in \mathcal{B}(H_{-1} \times U, Y)$*) is a PS-system w.r.t.* $\mathcal{V} := H_{-1}$ *and* $\mathcal{W} := H$*, by Lemmas 6.9.4 and 6.3.16. An analogous claim holds for any* $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,\{0\})$.

Thus, the input and output operators in a PS-system may be exactly as unbounded as in a WPLS, but not simultaneously*, since B must be "compatible" with* $\mathcal{W}$ *and C with* $\mathcal{V}$. $\square$

For example, any parabolic system of Hypothesis 9.5.1 with "unboundedness distance of C and B" $\gamma-\beta<1$ is a WPLS, but we must require that $\gamma-\beta<1/2$ in order to make sure that it is a PS-system. Note also the while B can be as unbounded w.r.t. $\mathcal{W}$ as the input operator of any WPLS, the operator B must be bounded w.r.t. $\mathcal{V}$.

From Theorem 6.9.1 we observe that $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{D}}^{\mathrm{d}}$ must be $\mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane for $\mathbb{D}$ to have a PS-realization. This condition is also sufficient:

Theorem 6.9.6 ($\widehat{\mathbb{D}},\widehat{\mathbb{D}}^{\mathrm{d}} \in \mathrm{H}^2_{\mathrm{strong}} \Leftrightarrow$ PS-realization) *Let* $\omega \in \mathbf{R}$ *and* $\widehat{\mathbb{D}} : \mathbf{C}^+_\omega \to \mathcal{B}(U,Y)$*. Then* $\widehat{\mathbb{D}}$ *is the transfer function of the I/O map of some* ω*-stable PS-system with* $D=0$ *iff* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) \cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))$.

We note two sufficient conditions:

1. if $\widehat{\mathbb{D}}(\cdot-\omega) \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))$ for some $\omega \in \mathbf{R}$, then $\widehat{\mathbb{D}}$ has a ($\omega+\varepsilon$-stable) PS-realization (with $D=0$).

2. By Lemma F.3.3(c2), $\mathcal{L}\mathrm{L}^2_{\mathrm{strong}} \subset \mathrm{H}^2_{\mathrm{strong}}$, hence $\mathbb{D}$ has a PS-realization (with $D=0$) whenever $\mathbb{D}u = F*u$ (for all u), where $\mathrm{e}^{-\omega\cdot}F, \mathrm{e}^{-\omega\cdot}F^* \in \mathrm{L}^2_{\mathrm{strong}}$ for some $\omega \in \mathbf{R}$ (this has already been shown in [KMR]).

Most PS theory (e.g., [Keu]) cover only smooth PS-systems, for which the above theorem gives only necessary conditions. In this monograph, a PS-system satisfies the regularity assumptions of almost any result (see Theorem 8.4.9(γ)), hence the above conditions are more than sufficient for our results.

The requirement $D=0$ simplifies the theorem but does not restrict generality: given an arbitrary $\widehat{\mathbb{D}}$, set $D := \widehat{\mathbb{D}}(+\infty)$ (regularity is a necessary condition) and apply Theorem 6.9.6 to $\widehat{\mathbb{D}}-D$.

Proof of Theorem 6.9.6:

Part I — "only if": This follows from Lemma 6.9.4 and Theorem 6.9.1.

Part II — "if": We shall show that conditions 1.–3. of Lemma 6.9.4 are satisfied. Set $H := \mathrm{L}^2_\omega(\mathbf{R}_+;Y)$, and let $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ be the strongly ω-stable realization (6.11). Let $\mathcal{W}$ be the space "H":$= \mathbb{B}[\mathcal{X}]$ of Definition 6.1.6.

1° $\Sigma \in \mathrm{WPLS}_\omega(U, \mathcal{W}, Y)$*:* This is shown in Lemma 6.1.7.

2° *There is* $C' \in \mathcal{B}(\mathcal{W}, Y)$ *s.t.* $C'\mathbb{A} = \mathbb{C}$*:* As noted in the proof of Lemma 6.1.7, $T := \mathbb{B}_{|\mathcal{X}} : \mathcal{X} \to \mathcal{W}$ satisfies $T \in \mathcal{GB}(\mathcal{X}, \mathcal{W})$, where $\mathcal{X} := \mathrm{Ran}(\mathbb{B})^\perp$. Consequently, $C' \in \mathrm{Hom}(\mathcal{W}, Y)$ becomes well-defined by setting $C'\mathbb{B}u := (\mathbb{D}u)(0)$ $(u \in \mathcal{X})$, because then

$$\|C'\mathbb{B}u\|_Y \leq \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{\mathrm{L}^2_\omega} \quad (u \in \mathcal{X}), \tag{6.216}$$

by Lemma F.3.7(b2), so that actually $C' \in \mathcal{B}(\mathcal{W}, Y)$ (note that $C' = (\mathbb{D}T^{-1}\cdot)(0)$). Now, for $u \in \mathcal{X}$, we have

$$C\mathbb{A}^t\mathbb{B}u = C\mathbb{B}\tau^t u = (\mathbb{D}\tau^t u)(0) = (\mathbb{D}u)(t) = (\pi_+\mathbb{D}\pi_- u)(t) = (\mathbb{C}\mathbb{B}u)(t) \tag{6.217}$$

for $t \geq 0$, i.e., $C'\mathbb{A}x_0 = \mathbb{C}x_0$ $(= x_0 \in \mathcal{W})$ for all $x_0 \in \mathbb{B}[\mathcal{X}] = \mathcal{W}$,

(Now we have established all claims of Lemma 6.9.4 that do not include $\mathcal{V}$, and we can complete the proof by establishing also the part concerning $\mathcal{V}$.)

3° $\Sigma \in \mathrm{WPLS}_\omega(U, \mathcal{V}, Y)$ *(with a bounded input operator):* Let $\mathcal{V}$ be the closure of $\mathcal{W}$ in H (note also that $\mathcal{W} \subset H$ continuously, hence $\mathcal{W} \subset V$ continuously). By 1°, $\mathbb{A}^t x_0 \in \mathcal{W}$ for all $x_0 \in \mathcal{W}$; since $\mathbb{A}^t \in \mathcal{B}(H)$, it follows that $\mathbb{A}^t[\bar{\mathcal{W}}] \subset \bar{\mathcal{W}}$, i.e., that $\mathbb{A}^t x_0 \in \mathcal{V}$ for all $x_0 \in \mathcal{V}$, for any $t \geq 0$. Therefore, $\mathbb{A}$ is a (strongly) ω-stable C_0-semigroup on $\mathcal{V}$ (since $\mathbb{A}$ is a (strongly) ω-stable C_0-semigroup on H, as noted in Definition 6.1.6).

Moreover, $\mathrm{Ran}(\mathbb{B}) = \mathcal{W} \subset \mathcal{V}$. Therefore, $\Sigma \in \mathrm{WPLS}_\omega(U, \mathcal{V}, Y)$ (the properties 1.–4. of Definition 6.1.1 are inherited from $\Sigma \in \mathrm{WPLS}_\omega(U, H, Y)$).

By (the proof of) Theorem 6.9.1(a), the input operator B of $\Sigma \in \mathrm{WPLS}_\omega(U, H, Y)$ is bounded ($B \in \mathcal{B}(U, H)$); by uniqueness (see Lemma 6.1.16(b)) the input operator of $\Sigma \in \mathrm{WPLS}_\omega(U, \mathcal{V}, Y)$ is again B ($\in \mathcal{B}(U, \mathcal{V})$, i.e., we restrict its range space to $\mathcal{V}$). □

Thus, if $\dim Y < \infty$, then a system with a bounded C has a realization with a bounded B:

Corollary 6.9.7 *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U, H, Y)$*. If C is bounded and* $\dim Y < \infty$*, or B is bounded and* $\dim U < \infty$*, then $\mathbb{D}$ has an ω-stable PS-realization.*

In fact, it suffices that C is bounded, $\mathbb{B}$ and $\mathbb{D}$ are ω-stable and $\dim Y < \infty$*.*

Proof: As the proof of Theorem 6.9.1 shows, we only need the ω-stability of $\mathbb{C}$ and $\mathbb{D}$ for bounded B, and the ω-stability of $\mathbb{B}$ and $\mathbb{D}$ for bounded C.

By Theorem 6.9.1, we have $\widehat{\mathbb{D}}(\cdot - \omega) - D \in \mathrm{H}^\infty \cap \mathrm{H}^2$, hence $\mathbb{D}$ has an ω-stable PS-realization, by Theorem 6.9.6. □

Finally, we present the simplest case:

Corollary 6.9.8 *Let* $\dim U, \dim Y < \infty$*,* $\omega \in \mathbf{R}$ *and* $\mathbb{D} \in \mathrm{TIC}_\omega(U, Y)$*. Then the following are equivalent:*

(i) There is $f \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U, Y))$ *s.t.* $\mathbb{D}u = f * u + Du$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$*;*

(ii) $\mathbb{D}$ *has an* ω*-stable realization with bounded B;*

(iii) $\mathbb{D}$ *has an* ω*-stable realization with bounded C;*

(iv) $\mathbb{D}$ *has an* ω*-stable PS-realization;*

(v) $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(Without the assumption $\mathbb{D} \in \mathrm{TIC}_\omega$ we would lose an ε of stability in (i) and (iv). E.g., for an arbitrary $f \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$ we would only know that $u \mapsto f * u$ has an ω'-stable PS-realization for any $\omega' > \omega$.)

By analogous arguments one observes that if $m := \dim U < \infty$ (so that $\mathcal{B}(U,Y) \cong Y^m$), then we have (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv); if $\dim Y < \infty$, then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv).

The condition "$\mathbb{D}u = f * u + Du$ for some $f : \mathbf{R}_+ \to \mathcal{B}(U,Y)$ s.t. $fu_0, f^* y_0 \in \mathrm{L}^2_\omega$ for all $u_0 \in U$ and $y_0 \in Y$" is not necessary for general U and Y (see the notes below).

Proof of Corollary 6.9.8: (Naturally, in (i) it would suffice to assume $\mathbb{D}u = f * u$ for all $u \in \mathcal{C}_c([0,T]; U)$ for some $T > 0$.) We take $D = 0$ w.l.o.g.

By Corollary 6.9.7 and its proof, (ii)–(iv) are equivalent. Set $m := \dim U$, $n := \dim Y$. By Theorem 6.9.1, condition (ii) holds iff (v) holds, i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^2(\mathbf{C}^+_\omega; \mathbf{C}^{n\times m})$. But the latter holds iff $\widehat{\mathbb{D}} = \widehat{f}$ for some $f \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathbf{C}^{n\times m})$, by Theorem 3.3.1(b). On the other hand, $\widehat{f}\widehat{u} = \widehat{\mathbb{D}}\widehat{u}$ on $\mathbf{C}^+_{\omega+1}$ (hence on $\overline{\mathbf{C}^+_\omega}$) iff $\mathbb{D}u = f * u$, by Lemma D.1.11(c') (because $f \in \mathrm{L}^1_{\omega+1}$). $\square$

Notes

The equivalence of (i)–(iii) of Corollary 6.9.8 is essentially contained in Theorem 5.2 of [Sal89], and his proof covered the cases where B was bounded and $\dim Y < \infty$ or C was bounded and $\dim U < \infty$, as noted in [WW].

In [KMR] it was stated and "proved" that $\mathbb{D}$ has a PS-realization iff $\mathbb{D}u = f * u$ ($u \in \mathrm{L}^2$), where $f, f^* \in \mathrm{L}^2_{\text{strong}}$, the convolution existing as a weak (i.e., Pettis) integral. However, that condition is only sufficient but not necessary.

The fault was in the claim that the term $f := \mathbb{C}B$ would be well defined (a.e.) as a function $\mathbf{R}_+ \to \mathcal{B}(U,Y)$ (we do have $f \in \mathcal{B}(U, \mathrm{L}^2_{\text{loc}})$ since B is bounded). Indeed, let $U = \ell^2(\mathbf{N})$ and $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+; \mathcal{B}(U))$ be as in Example F.3.6, so that $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+; \mathcal{B}(U))$. By Theorem 6.9.6, there is an ω-stable PS-realization $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ of $\mathbb{D}$ for any $\omega > 0$. Moreover, $f := \mathbb{C}B \in \mathcal{B}(U, \mathrm{L}^2_\omega)$ satisfies $\widehat{f} = \widehat{\mathbb{D}}$, by (e1) (or (e2)) of Lemma 6.8.1, hence f equals the operator F of Example F.1.10, hence f does not have a representation of form $\mathbf{R}_+ \to \mathcal{B}(U,Y)$ (as shown in the example, this would lead to the contradiction $\|f(t)\|_{\mathcal{B}(U,Y)} = \infty$ for a.e. $t \geq 0$).

Nevertheless, the paper [KMR] is an elegant introduction to PS-systems, and it inspired us to write Theorem 6.9.6. Moreover, the definition of $\mathcal{W}$ in the proof Theorem 6.9.6 is from [KMR] (the rest of our techniques are different and keep an exact track on stability).

The existence parts of the results of [Sal89], [KMR] and ours are based on the shift semigroup system (see Definition 6.1.6). We do not know corresponding conditions for smooth PS-realizations (this refers to the additional condition that $\mathrm{Dom}(A_{\mathcal{V}}) \subset \mathcal{W}$).